Honam Mathematical J. **46** (2024), No. 1, pp. 1–11
https://doi.org/10.5831/HMJ.2024.46.1.1

σ-JORDAN AMENABILITY OF BANACH ALGEBRAS

Jun Li, Lin Chen*, and Mohammad Javad Mehdipour

Abstract. In this paper, we introduce the notion of σ-Jordan amenability of Banach algebras and some hereditary are investigated. Similar to Johnson's classic result, we give the notions of σ-Jordan approximate and σ-Jordan virtual diagonals, and find some relations between the existence of them and σ-Jordan amenability.

1. Introduction and preliminaries

Let A be a Banach algebra and X be a Banach A-bimodule. A linear mapping $D : A \to X$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b)$$

holds for all $a, b \in A$. Also, D is called a *Jordan derivation* if

$$D(a^2) = D(a) \cdot a + a \cdot D(a).$$

A derivation D is called an *inner derivation* if there exists $x \in X$ such that

$$D(a) = \delta_x(a) = x \cdot a - a \cdot x$$

for all $a \in A$. If X is a Banach A-bimodule, then X^* becomes a Banach A-bimodule via the following module actions

$$\langle x, f \cdot a\rangle = \langle a \cdot x, f\rangle \quad \text{and} \quad \langle x, a \cdot f\rangle = \langle x \cdot a, f\rangle$$

for all $a \in A$, $x \in X$ and $f \in X^*$. Johnson [14] first introduced the concept of amenability of Banach algebras; a Banach algebra A is *amenable* if every bounded derivation from A into dual Banach A-bimodule X^* is inner. He gave characterizations of amenability in terms of nets in the projective tensor product $A\widehat{\otimes}A$ (bounded approximate diagonal) and through the existence of a special type of elements in the second dual $(A\widehat{\otimes}A)^{**}$ (virtual diagonal). Let us recall that a *bounded approximate diagonal* for a Banach algebra A is a bounded net $\{\mathbf{m}_\alpha\}_\alpha$ in $A\widehat{\otimes}A$ satisfying

$$a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a \to 0 \quad \text{and} \qquad a\pi(\mathbf{m}_\alpha) = a$$

Received April 25, 2023. Accepted July 13, 2023.
2020 Mathematics Subject Classification. 46H25, 47A30.
Key words and phrases. σ-Jordan derivation, σ-Jordan amenability, Banach algebra.
*Corresponding author

for all $a \in A$. A *virtual diagonal* for A is an element $\mathbf{M} \in (A\widehat{\otimes}A)^{**}$ with the properties

$$a \cdot \mathbf{M} = \mathbf{M} \cdot a \quad \text{and} \qquad a \cdot \pi^{**}(\mathbf{M}) = a$$

for all $a \in A$, where $\pi : A\widehat{\otimes}A \to A$ is defined by $\pi(a \otimes b) = ab$ and

$$c \cdot (a \otimes b) = ca \otimes b \quad \text{and} \quad (a \otimes b) \cdot c = a \otimes bc$$

for all $a, b, c \in A$. Consider the *opposite algebra* A° that is the Banach space A with the product $a \circ b = ba$. Let us remark that a bounded approximate diagonal for Banach algebra A° is a bounded net $\{\mathbf{m}_\alpha\}_\alpha$ in $A\widehat{\otimes}A$ with

$$a \circ \mathbf{m}_\alpha - \mathbf{m}_\alpha \circ a \to 0 \quad \text{and} \quad a \circ \pi_\circ(\mathbf{m}_\alpha) = a,$$

where $\pi_\circ : A\widehat{\otimes}A \to A$ is defined by $\pi_\circ(a \otimes b) = ba$ and

$$c \circ (a \otimes b) = a \otimes cb \quad \text{and} \quad (a \otimes b) \circ c = ac \otimes b$$

for all $a, b, c \in A$. For a comprehensive account of the amenability of Banach algebras the reader is referred to the books [8, 9, 26]. An element $\mathbf{t} \in A\widehat{\otimes}A$ is called *symmetric* if $\mathbf{t}^\circ = \mathbf{t}$, where "$\circ$" is defined by $(a \otimes b)^\circ = b \otimes a$. A Banach algebra A is called *symmetric amenable* if A has a symmetric bounded approximate diagonal.

Let σ be a continuous homomorphism on a Banach algebra A. Naturally, we can define the notions of σ-derivations, σ-Jordan derivations and σ-inner derivations. Purely algebraic results in the framework of ring or semi-prime ring about σ-derivations, σ-Jordan derivations can be found in [4, 3, 19, 17, 5], and other results in the framework of operator algebras can be found in [13, 23, 24, 7, 25] and reference therein. Moslehian and Motlagh in [20] first introduced and studied the (σ, τ)-amenability of Banach algebras. In [7] the authors introduced the (φ, ψ)-weak amenability of Banach algebras and investigated the relations between weak amenability and (φ, ψ)-weak amenability. About symmetric amenability, Alaminos, Mathieu and Villena [2] proved that every Lie derivation on symmetric amenable semisimple Banach algebras can be uniquely decomposed into the sum of a derivation and a centre-valued trace. Recently, Valaei, Zivari-Kazempour and Bodaghi [27] introduced and studied the Jordan amenability of Banach algebras. The authors in [6] studied amenability and weak amenability of triangular Banach algebras T_{σ_A, σ_B}, where σ_A and σ_B are continuous homomorphisms on A and B, respectively. These considerations motivate us to study σ-Jordan amenability of Banach algebras.

This paper is organized as follows. In Section 2, we first introduce the concept of σ-Jordan amenability of Banach algebras and investigate the hereditary of it. In Section 3, we investigate some relations between the existence of σ-Jordan approximate, σ-Jordan virtual diagonals and σ-Jordan amenability.

2. Some properties of σ-Jordan amenable Banach algebras

Let A be a Banach algebra and X be a Banach A-bimodule. Let us recall that X is called *pseudo-unital*, if $X = \{a \cdot x \cdot b : a, b \in A, x \in X\}$. In the case where,

$$X = \overline{\text{lin}\{a \cdot x \cdot b : a, b \in A, x \in X\}},$$

then X is called *essential*. Clearly, if X is pseudo-unital, then X is essential. By Cohen's factorization theorem [12], if A has a bounded approximate identity and X is essential, then X is pseudo-unital. Let us remark that by Johnson's classic results [26, Proposition 1.1.5], if A is a Banach algebra with a bounded approximate identity, then A is amenable if and only if for any pseudo-unital Banach A-bimodule X, every bounded derivation from A into X^* is inner; or equivalently, for any essential Banach A-bimodule X, every bounded derivation from A into X^* is inner. Hence, it is natural to give the following definition.

Definition 2.1. *Let A be a Banach algebra, σ be a continuous homomorphism on A and X be a Banach A-bimodule. A linear mapping $D : A \to X$ is called a* σ-Jordan derivation *if for every $a \in A$,*

$$D(a^2) = D(a) \cdot \sigma(a) + \sigma(a) \cdot D(a).$$

Also, D is called σ-inner *if there exists $x \in X$ such that*

$$D(a) = \delta_x(\sigma(a))$$

for all $a \in A$. Banach algebra A is called σ-Jordan amenable *if for any essential Banach A-bimodule X, every bounded σ-Jordan derivation from A into X^* is σ-inner.*

It is obvious that if σ is the identity map, then every σ-Jordan amenable Banach algebra is Jordan amenable [27].

Example 2.2. *(i) Let A be an essential Banach algebra, i.e., $\overline{A^2} = A$. Let σ_0 be the zero map on A. It is easy to see that the zero map is the only σ_0-Jordan derivation from A into any essential Banach A-bimodule X. Thus A is σ_0-Jordan amenable. Note that every Banach algebra with a bounded approximate identity is essential* [12] *and so it is σ_0-Jordan amenable. This shows that every amenable Banach algebra is σ_0-Jordan amenable. Note that weakly amenable Banach algebras are essential* [9]. *Hence weakly amenable Banach algebras are σ_0-Jordan amenable.*

(ii) Let G be a locally compact group. Let $L_0^\infty(G)^*$ and $M_*(G)^*$ be as defined in [16] *and* [18, 20], *respectively; see also* [1, 11, 21, 22]. *Then $L_0^\infty(G)^*$ is a Banach algebra with a right identity and $M_*(G)^*$ is a Banach algebra with identity. Thus $L_0^\infty(G)^*$ and $M_*(G)^*$ are essential. Therefore, $L_0^\infty(G)^*$ and $M_*(G)^*$ are σ_0-Jordan amenable.*

Theorem 2.3. *Let σ be an epimorphism on Banach algebra A. Then A is σ-Jordan amenable if and only if A is Jordan amenable.*

Proof. Let A be σ-Jordan amenable. Let X be an essential Banach A-module and $D : A \to X^*$ be a Jordan derivation. Then

$$d \circ \sigma : A \to X^*$$

is a σ-Jordan derivation. Thus there exists $f \in X^*$ such that for every $a \in A$,

$$D \circ \sigma(a) = \sigma(a) \cdot f - f \cdot \sigma(a).$$

Since σ is onto, it follows that $D(a) = a \cdot f - f \cdot a$ for all $a \in A$. Thus A is Jordan amenable. Conversely, define the module actions $\star$ on essential Banach A-module X by

$$a \star x = \sigma(a) \cdot x \quad \text{and} \quad x \star a = x \cdot \sigma(a)$$

for all $a \in A$ and $x \in X$. From the surjectivity of σ we see that

$$\begin{aligned} \text{lin}\{a \star x \star b, a, b \in A, x \in X\} &= \text{lin}\{\sigma(a) \cdot x \cdot \sigma(b), a, b \in A, x \in X\} \\ &= \text{lin}\{a \cdot x \cdot b, a, b \in A, x \in X\}. \end{aligned}$$

Thus X with the module actions $\star$ is an essential Banach A-module. Assume now that $D : A \to A^*$ is a σ-Jordan derivation. Then

$$D(a^2) = D(a) \cdot \sigma(a) + \sigma(a) \cdot D(a) = D(a) \star a - a \star D(a)$$

for all $a \in A$. Hence, $D : A \to (X^*, \star)$ is a Jordan derivation. So there exists $f \in X^*$ such that

$$D(a) = a \star f - f \star a = \sigma(a) \cdot f - f \cdot \sigma(a)$$

for all $a \in A$. Thus D is σ-inner. Therefore A is σ-Jordan amenable. □

The next example shows that the assumption of surjectivity of σ in Theorem 2.3 is necessary.

Example 2.4. *Let $\mathbb{R}$ be the additive group of real numbers endowed with the usual topology. Let $M(\mathbb{R})$ be the measure algebra of $\mathbb{R}$. Then $M(\mathbb{R})$ is a unital commutative Banach algebra* [12]. *Since $\mathbb{R}$ is non-discrete, $M(\mathbb{R})$ is not amenable* [10]. *So $M(\mathbb{R})$ is not Jordan amenable by Corollary 3.1.2 of* [27]. *In view of Example 2.2, $M(\mathbb{R})$ is σ_0-Jordan amenable, where σ_0 is the zero map on $M(\mathbb{R})$.*

Let A be a Banach algebra and I be a closed two sided ideal of A. Let σ be a continuous homomorphism on A with $\sigma(I) \subseteq I$. Then $\hat{\sigma} : A/I \to A/I$ defined by $\hat{\sigma}(a + I) = \sigma(a) + I$ is a continuous homomorphism on A/I. Now, let $A^\sharp$ be the unitization of A. Then one can define $\sigma^\sharp : A^\sharp \to A^\sharp$ by

$$\sigma^\sharp((a, \lambda)) = (\sigma(a), \lambda)$$

for all $a \in A$ and $\lambda \in \mathbb{C}$. Clearly, $\sigma^\sharp$ is a homomorphism on $A^\sharp$. The following theorem summarizes the main hereditary properties of σ-Jordan amenable Banach algebras.

Theorem 2.5. *Let σ be a continuous homomorphism on a Banach algebra A and τ be a continuous homomorphism on a Banach algebra B. If I is a closed two-sided ideal of A, then the following statements hold.*

(i) *If A is σ-Jordan amenable and $u : A \to B$ is a continuous homomorphism with dense range satisfying $\tau u = u\sigma$, then B is τ-Jordan amenable. In particular, if A is σ-Jordan amenable, then A/I is $\hat{\sigma}$-Jordan amenable for all closed ideals I with $\sigma(I) \subseteq I$.*

(ii) *Let σ be one to one and I be a closed two-sided ideal of A with $\sigma(I) \subseteq I$. If I is σ-Jordan amenable, A/I is $\hat{\sigma}$-Jordan amenable and has an identity in $\hat{\sigma}(A/I)$ for any Banach A-module, then A is σ-Jordan amenable.*

(iii) *$A^\sharp$ is $\sigma^\sharp$-Jordan amenable if and only if A is σ-Jordan amenable.*

Proof. (i) Assume that X is a Banach B-module. It suffices to show that every τ-Jordan derivation from B into X^* is τ-inner. Note that X can be considered as a Banach A-module with the following module action

$$a \bullet x = u(a) \cdot x \quad \text{and} \quad x \bullet a = x \cdot u(a) \quad (a \in A, x \in X).$$

Let $d : B \to X^*$ be a τ-Jordan derivation. It is routine to check that $d \circ u$ is a σ-Jordan derivation on A. In fact, for $a, b \in A$,

$$\begin{aligned} d \circ u(ab + ba) &= d(u(a)u(b) + u(b)u(a)) \\ &= d(u(a)) \cdot \tau(u(b)) + \tau(u(a)) \cdot d(u(b)) \\ &+ d(u(b)) \cdot \tau(u(a)) + \tau(u(b)) \cdot d(u(a)) \\ &= d(u(a)) \cdot u(\sigma(b)) + u(\sigma(a)) \cdot d(u(b)) \\ &+ d(u(b)) \cdot u(\sigma(a)) + u(\sigma(b)) \cdot d(u(a)) \\ &= d \circ u(a) \bullet \sigma(b) + \sigma(a) \bullet d \circ u(b) \\ &+ d \circ u(b) \bullet \sigma(a) + \sigma(a) \bullet d \circ u(a). \end{aligned}$$

Since A is σ-Jordan amenable, it follows that there exists $f \in X^*$ such that for every $a \in A$,

$$\begin{aligned} d \circ u(a) = \delta_f(a) &= \sigma(a) \bullet f - f \bullet \sigma(a) \\ &= u(\sigma(a)) \cdot f - f \cdot u(\sigma(a)) = \tau(u(a)) \cdot f - f \cdot \tau(u(a)). \end{aligned}$$

Since $\overline{u(A)} = B$, for every $b \in B$, there exists a sequence $\{a_n\}$ in A such that $u(a_n) \to b$. Using the continuity of τ and module actions, we have

$$\begin{aligned} d(b) = d(\lim_n u(a)) &= \lim_n d \circ u(a_n) = \lim_n \delta_f(a_n) \\ &= \lim_n \big(\tau(u(a_n)) \cdot f - f \cdot \tau(u(a_n))\big) = \tau(b) \cdot f - f \cdot \tau(b). \end{aligned}$$

This shows that d is τ-inner. Hence B is τ-Jordan amenable.

(ii) Let X be a Banach A-module and $D' : A \to X^*$ be a σ-Jordan derivation. Then $D'|_I$ is a σ-Jordan derivation. Note $\sigma(I) \subseteq I$. Hence σ is a continuous homomorphism on I. By σ-Jordan amenability of I, there exists $f \in X^*$ such that $D'|_I = \delta'_f$, where δ'_f is the inner σ-Jordan derivation from I into X^*.

If we denote δ_f to be the natural extension of δ'_f on A, then $D := D' - \delta_f$ is also a σ-Jordan derivation which vanishes on I. Suppose that Y is the subspace of X generated by

$$\hat{\sigma}(I) \cdot X \cup X \cdot \hat{\sigma}(I).$$

Then X/Y is a Banach A/I-bimodule via

$$(a + I) \cdot (x + Y) = \sigma(a) \cdot x + Y \text{ and } (x + Y) \cdot (a + I) = x \cdot \sigma(a) + Y$$

for all $a \in A$ and $x \in X$. Since $D(I) = 0$, we can define $\hat{\sigma}$-Jordan derivation $\tilde{D} : A/I \to X^*$ by

$$\tilde{D}(a + I) = D(a).$$

Take $e \in A$ such that $\hat{\sigma}(\bar{e})$ is an identity for all Banach A-module X, where $\bar{e} = e + I$. Hence, for all $a \in A$, we have

$$\tilde{D}(a) \cdot \hat{\sigma}(\bar{e}) = \hat{\sigma}(\bar{e}) \cdot \tilde{D}(a) = \tilde{D}(a). \tag{1}$$

Note that A/I is also a Banach A-module. Hence $\hat{\sigma}(\bar{e})$ is an identity for A/I. Thus

$$\hat{\sigma}(\bar{t}) = \hat{\sigma}(\bar{t})\hat{\sigma}(\bar{e}) = \hat{\sigma}(\bar{t}\bar{e})$$

for all $t \in A$. Since $\hat{\sigma}$ is one-to-one, it follows that $\bar{t} = \bar{t}\bar{e}$. This implies that $\bar{e}$ is an identity for A/I. So

$$\tilde{D}(\bar{e}) = \tilde{D}(\bar{e}\bar{e}) = \hat{\sigma}(\bar{e}) \cdot \tilde{D}(\bar{e}) + \tilde{D}(\bar{e}) \cdot \hat{\sigma}(\bar{e}) = 2\tilde{D}(\bar{e}).$$

This shows that

$$\tilde{D}(\bar{e}) = 0. \tag{2}$$

From (1) and (2) we conclude that

$$\begin{aligned} 2\hat{\sigma}(\bar{b}) \cdot \tilde{D}(\bar{a}) &= \hat{\sigma}(b) \cdot \tilde{D}(\bar{a}) \cdot \hat{\sigma}(\bar{e}) + \hat{\sigma}(\bar{b}) \cdot \hat{\sigma}(\bar{e}) \cdot \tilde{D}(\bar{a}) \\ &= \hat{\sigma}(b) \cdot \tilde{D}(\bar{a}) \cdot \hat{\sigma}(\bar{e}) + \hat{\sigma}(\bar{b}) \cdot \hat{\sigma}(\bar{e}) \cdot \tilde{D}(\bar{a}) \\ &+ \hat{\sigma}(\bar{b}) \cdot \hat{\sigma}(\bar{a}) \cdot \tilde{D}(\bar{e}) + \hat{\sigma}(\bar{b}) \cdot \tilde{D}(\bar{e}) \cdot \hat{\sigma}(\bar{a}) \\ &= \hat{\sigma}(\bar{b}) \cdot \tilde{D}(\bar{e}\bar{a} + \bar{a}\bar{e}) = 0. \end{aligned}$$

for all $a \in A$ and $b \in I$. Therefore, $\hat{\sigma}(\bar{b}) \cdot \tilde{D}(\bar{a}) = 0$. Similarly $\tilde{D}(\bar{a}) \cdot \hat{\sigma}(\bar{b}) = 0$. By the definition of Y, for every $x \in X$, $a \in A$ and $b \in I$, we have

$$\langle \tilde{D}(\bar{a}), x \cdot \hat{\sigma}(\bar{b}) \rangle = \langle \hat{\sigma}(\bar{b}) \cdot \tilde{D}(\bar{a}), x \rangle = 0$$

and

$$\langle \tilde{D}(\bar{a}), \hat{\sigma}(\bar{b}) \cdot x \rangle = \langle \tilde{D}(\bar{a}) \cdot \hat{\sigma}(\bar{b}), x \rangle = 0.$$

Hence $\tilde{D}(A/I) \subseteq Y^{\perp} \cong (X/Y)^*$. Consequently, the $\hat{\sigma}$-Jordan amenability of A/I implies that $\tilde{D} = \delta_g$, for some $g \in X^*$. Thus, $D = \delta_f + \delta_g = \delta_{f+g}$. This shows that D' is σ-inner.

(iii) If A is σ-Jordan amenable, then since A is a closed ideal of $A^\sharp = A \otimes \mathbb{C}e$, and $\mathbb{C}$ is $\hat{\sigma}$-Jordan amenable, it follows that $A^\sharp$ is $\sigma^\sharp$-Jordan amenable by (ii).

Suppose that X is a Banach A-bimodule and $d : A \to X^*$ is a σ-Jordan derivation. Then X is $A^\sharp$-bimodule by module actions

$$x \cdot (a, \lambda) = x \cdot a + \lambda x \quad \text{and} \quad (a, \lambda) \cdot x = a \cdot x + \lambda x$$

for all $a \in A, x \in X$ and $\lambda \in \mathbb{C}$. Define $d^\sharp : A^\sharp \to X^*$ by $d^\sharp(a, \lambda) = d(a)$ for all $(a, \lambda) \in A^\sharp$. It is simple to verify that $d^\sharp$ is $\sigma^\sharp$-Jordan derivation. Thus, there exists $f \in X^*$ such that

$$d^\sharp(a, \lambda) = \sigma^\sharp(a, \lambda) \cdot f - f \cdot \sigma^\sharp(a, \lambda).$$

Therefore, for all $a \in A, x \in X$, we have

$$\begin{aligned}\langle d(a), x\rangle &= \langle d^\sharp(a, \lambda), x\rangle = \langle \sigma^\sharp(a, \lambda) \cdot f - f \cdot \sigma^\sharp(a, \lambda), x\rangle \\ &= \langle x \cdot (\sigma(a), \lambda), f\rangle - \langle (\sigma(a), \lambda) \cdot x, f\rangle \\ &= \langle x \cdot \sigma(a) + \lambda x, f\rangle - \langle \sigma(a) \cdot x + \lambda x, f\rangle \\ &= \langle x \cdot \sigma(a), f\rangle - \langle \sigma(a) \cdot x, f\rangle = \langle \sigma(a) \cdot f - f \cdot \sigma(a), x\rangle.\end{aligned}$$

Hence, A is σ-Jordan amenable. □

3. Characterization of σ-Jordan amenable Banach algebras

In this section, we will give a characterization of σ-Jordan amenable Banach algebras in terms of asymptotic versions of a projective diagonal in a similar fashion as [15]. We begin with the following definition.

Definition 3.1. *Let A be a Banach algebra and σ be a continuous homomorphism on A.*
(i) An element $\mathbf{M} \in (A\widehat{\otimes}A)^{**}$ is called a σ-Jordan virtual diagonal *for A, if*

$$\sigma(a) \cdot \pi^{**}(\mathbf{M}) = \sigma(a) \cdot \pi_\circ^{**}(\mathbf{M}) = \sigma(a)$$

and

$$\sigma(a) \cdot \mathbf{M} = \sigma(a) \circ \mathbf{M} = \mathbf{M} \cdot \sigma(a) = \mathbf{M} \circ \sigma(a).$$

(ii) A bounded net $\{\mathbf{m}_\alpha\}_\alpha$ in $A\widehat{\otimes}A$ is called a σ-Jordan approximate diagonal *for A, if*

$$\sigma(a)\pi(\mathbf{m}_\alpha) \to \sigma(a), \quad \sigma(a)\pi_\circ(\mathbf{m}_\alpha) \to \sigma(a)$$

and

$$\lim_\alpha \sigma(a) \cdot \mathbf{m}_\alpha = \lim_\alpha \sigma(a) \circ \mathbf{m}_\alpha = \lim_\alpha \mathbf{m}_\alpha \cdot \sigma(a) = \lim_\alpha \mathbf{m}_\alpha \circ \sigma(a).$$

Theorem 3.2. *Let σ be a continuous homomorphism on a Banach algebra A. Then A has a σ-Jordan approximate diagonal if and only if it has a σ-Jordan virtual diagonal.*

Proof. Let $\{\mathbf{m}_\alpha\}_\alpha$ be a σ-Jordan approximate diagonal of A. Since $\{\mathbf{m}_\alpha\}_\alpha$ is a bounded net in $A\widehat{\otimes}A$, by Alaoglu's theorem there exists a w^*-accumulation point, $\mathbf{M} \in (A\widehat{\otimes}A)^{**}$, of $\{\hat{\mathbf{m}}_\alpha\}_\alpha$. By passing subnet, we may assume that w^*-$\lim\limits_\alpha \hat{\mathbf{m}}_\alpha = \mathbf{M}$. Then by the weak continuity of $c \mapsto c \circ (a \otimes b)$ from A into $A\widehat{\otimes}A$, we have

$$\begin{aligned}\sigma(a) \circ \mathbf{M} - \mathbf{M} \circ \sigma(a) &= w^* - \lim_\alpha(\sigma(a) \circ \hat{\mathbf{m}}_\alpha - \hat{\mathbf{m}}_\alpha \circ \sigma(a)) \\ &= w - \lim_\alpha(\sigma(a) \circ \mathbf{m}_\alpha - \mathbf{m}_\alpha \circ \sigma(a)) = 0\end{aligned}$$

for all $a \in A$. Similarly, we can prove $\sigma(a) \cdot \mathbf{M} = \mathbf{M} \cdot \sigma(a)$ and $\sigma(a) \cdot \mathbf{M} = \mathbf{M} \circ \sigma(a)$. Furthermore, by the weak* continuity of $\pi_\circ^{**}$, we get

$$\begin{aligned}\sigma(a) \circ \pi_\circ^{**}(\mathbf{M}) &= w^* - \lim_\alpha(\sigma(a) \circ \pi_\circ^{**}(\hat{\mathbf{m}}_\alpha)) \\ &= w - \lim_\alpha(\sigma(a)\pi_\circ(\mathbf{m}_\alpha)) = \sigma(a)\end{aligned}$$

for all $a \in A$. Similarly, $\sigma(a) \cdot \pi^{**}(\mathbf{M}) = \sigma(a)$.

Conversely, let $\mathbf{M}$ be a virtual diagonal in $(A\widehat{\otimes}A)^{**}$. By Goldstine's theorem there exists a bounded net $\{\mathbf{m}_\alpha\}_\alpha$ in $A\widehat{\otimes}A$ such that $M = w^* - \lim\limits_\alpha(\hat{\mathbf{m}}_\alpha)$. It is easy to see that module actions on a dual module are weak* continuous for a fixed element of A. Also, for any Banach space X, the w^*-topology of X^{**} restricted to X is the w-topology. These facts enable us to get the following statements.

$$\sigma(a)\cdot\mathbf{m}_\alpha - \mathbf{m}_\alpha\cdot\sigma(a) \to 0, \sigma(a)\circ\mathbf{m}_\alpha - \mathbf{m}_\alpha\circ\sigma(a) \to 0, \sigma(a)\cdot\mathbf{m}_\alpha - \mathbf{m}_\alpha\circ\sigma(a) \to 0$$

and

$$\sigma(a) \cdot \pi^{**}(\hat{\mathbf{m}}_\alpha) \to \sigma(a), \quad \sigma(a) \circ \pi_\circ^{**}(\hat{\mathbf{m}}_\alpha) \to \sigma(a)$$

in the w-topology of $(A\widehat{\otimes}A)^{**}$ and A^{**}, respectively. Following the argument given in the proof of [9, Lemma 2.9.64], we can show that there exists a net $\{\mathbf{m}_\beta\}_\beta$ in $A\widehat{\otimes}A$ such that each $\mathbf{m}_\beta$ is a convex combination of $\mathbf{m}_\alpha$'s satisfying conditions (ii) in definition 3.1. □

We close this section with the following theorem for commutative Banach algebras.

Theorem 3.3. *Let σ be a continuous homomorphism on a commutative Banach algebra A. If A is σ-Jordan amenable and has a bounded approximate identity, then A has a σ-Jordan virtual diagonal.*

Proof. Let $\{e_\alpha\}_\alpha$ be a bounded approximate identity for A and consider the bounded net $\{e_\alpha \otimes e_\alpha\}_\alpha$ in $(A\widehat{\otimes}A)^{**}$. Let E be a w^*-accumulation point of $\{e_\alpha \otimes e_\alpha\}_\alpha$. We may assume, by passing subnet if necessary, that

$$w^* - \lim_\alpha e_\alpha \otimes e_\alpha = E.$$

Consider the σ-inner derivation $d_E : A \to (A\widehat{\otimes}A)^{**}$ by

$$d_E(a) = \sigma(a) \cdot E - E \cdot \sigma(a).$$

Then we have

$$\begin{aligned}\pi^{**}(d_E(a)) &= w^* - \lim_\alpha \pi^{**}\big(\sigma(a)\cdot(e_\alpha\otimes e_\alpha) - (e_\alpha\otimes e_\alpha)\cdot\sigma(a)\big)\\ &= w - \lim_\alpha \pi\big(\sigma(a)\cdot(e_\alpha\otimes e_\alpha) - (e_\alpha\otimes e_\alpha)\cdot\sigma(a)\big)\\ &= w - \lim_\alpha \big(\sigma(a)e_\alpha^2 - e_\alpha^2\sigma(a)\big) = 0,\end{aligned}$$

because $\{e_\alpha^2\}_\alpha$ is also a bounded approximate identity for A. Therefore $d_E(A)\subset \ker\pi^{**}$. Since $\ker\pi$ is a closed submodule of $A\widehat{\otimes}A$, we know $(\ker\pi)^{**}$ is a dual Banach A-module. It is know that $\ker\pi^{**} = (\ker\pi)^{**}$ [26]. Thus d_E is a σ-Jordan derivation from A into $\ker\pi^{**}$. The σ-Jordan amenability of A implies that there exists an $N\in\ker\pi^{**}$ such that $d_E = d_N$. Put $\mathbf{M} = E - N$. Then

$$\sigma(a)\cdot\mathbf{M} - \mathbf{M}\cdot\sigma(a) = d_{\mathbf{M}}(a) = d_E(a) - d_N(a) = 0$$

and

$$\begin{aligned}\sigma(a)\cdot\pi^{**}(\mathbf{M}) &= \sigma(a)\cdot\big(\pi^{**}(E) - \pi^{**}(N)\big) = \sigma(a)\cdot\pi^{**}(E)\\ &= w^* - \lim_\alpha \sigma(a)\cdot\pi^{**}(e_\alpha\otimes e_\alpha)\\ &= w - \lim_\alpha \sigma(a)\cdot\pi(e_\alpha\otimes e_\alpha) = w - \lim_\alpha \sigma(a)\cdot e_\alpha^2) = \sigma(a).\end{aligned}$$

Since A is commutative, we have

$$\sigma(a)\cdot\mathbf{M} = \sigma(a)\circ\mathbf{M} \quad\text{and}\quad \mathbf{M}\cdot\sigma(a) = \mathbf{M}\circ\sigma(a).$$

Thus M is a σ-Jordan diagonal for A. □

Acknowledgments. The authors wish to thank anonymous reviewers for their constructive and valuable suggestions which have considerably improved the presentation of the paper. This work is supported by the National Natural Science Foundation of China (No. 12061018).

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Jun Li
Department of Mathematics and Statistics,
Changshu Institute of Technology,
Changshu 215500, P. R. China.
E-mail: lijunlijun2005@163.com

Lin Chen
Department of Mathematics and Statistics,
Changshu Institute of Technology,
Changshu 215500, P. R. China.
E-mail: linchen198112@cslg.edu.cn

Mohammad Javad Mehdipour
Department of Mathematics,
Shiraz University of Technology,
Shiraz 71555-313, Iran.
E-mail: mehdipour@sutech.ac.ir