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A NOTE ON SOBOLEV TYPE TRACE INEQUALITIES FOR s-HARMONIC EXTENSIONS

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Abstract. In this paper, apply the regularities of the fractional Poisson kernels, we establish the Sobolev type trace inequalities of homogeneous Besov spaces, which are invariant under the conformal transforms. Also, by the aid of the Carleson measure characterizations of Q type spaces, the local version of Sobolev trace inequalities are obtained.

1. Introduction

In the study of many fields, such as analysis, geometry and partial differential equations, Sobolev type trace inequalities play an important role. Essentially, there exists a profound connection between Sobolev trace inequalities and the boundary problem of differential operators. Also, in the theory of function spaces, Sobolev type trace inequalities provide the characterization of boundary behaviour of functions with sufficient smoothness. Let φ be any real-valued function on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$ which is sufficiently smooth up to boundary and decaying fast at infinity. In [9], Escobar proved the following Sobolev type trace inequality: there exists a constant C such that

(1)
$$\left(\int_{\mathbb{R}^n} |\varphi(x,0)|^{2(n-1)/(n-2)} dx\right)^{(n-2)/(2n-2)} \leq C\left(\int_{\mathbb{R}^{n+1}_+} |\nabla_{x,t}\varphi(x,t)|^2 dxdt\right)^{1/2},$$

where the symbol $\nabla_{x,t}$ denotes the gradient operator $\nabla_{x,t} = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_t)$, see [9, Theorem 1]. Using different technology, Beckner in [1] obtained the above trace inequality (1) independently. A direct consequence of (1) is as follows, with $p_t(\cdot)$ being the Poisson kernel on $\mathbb{R}^{n+1}_+$, i.e.,

$$p_t(x) := \frac{t}{(t^2 + |x|^2)^{(n+1)/2}},$$

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there holds

$$\left(\int_{\mathbb{R}^n}|f(x)|^{2(n-1)/(n-2)}dx\right)^{(n-2)/(2n-2)}\leq C\left(\int_{\mathbb{R}^{n+1}_+}|\nabla_{x,t}p_t*f(x)|^2dxdt\right)^{1/2}. \tag{2}$$

Applying a weighted integral of the Fourier transform of the given function and to E. H. Lieb's sharp estimate for the Hardy-Littlewood-Sobolev inequality, Xiao [16] established an analogue of (2) for the fractional-order derivatives. For further information, we refer to [12], [13], [15], [17] and the references therein.

In this note our aim is to establish the Sobolev type trace inequalities via the higher order regularity of the s-harmonic functions which are the solutions to the following equations:

$$\begin{cases}\operatorname{div}(t^{1-s}\nabla u)=0,\ (x,t)\in\mathbb{R}^{n+1}_+;\\ \qquad u(x,0)=f(x),\ \ x\in\mathbb{R}^n.\end{cases} \tag{3}$$

Let f be a regular function in $\mathbb{R}^n$. We say that $u(x,t)=P_sf(x,t)$ is the Caffarelli-Silvestre extension of f to the upper half-space $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,\infty)$, if u is a solution to equations (3). The Caffarelli-Silvestre extension is well defined for smooth functions through the fractional Poisson kernel

$$p^s_t(x)=\frac{c(n,s)t^s}{(|x|^2+t^2)^{(n+s)/2}},\ c(n,s)=\frac{\Gamma((n+s)/2)}{\pi^{n/2}\Gamma(s/2)}$$

as

$$P_sf(x,t)=p^s_t*f(x,t)=c(n,s)\int_{\mathbb{R}^n}\frac{f(y)t^s}{(|x-y|^2+t^2)^{(n+s)/2}}dy.$$

Here $f*g$ means the convolution of f and g. Caffarelli and Silvestre [4] proved that

$$(-\triangle)^{s/2}f(x)=-c_s\lim_{t\to0^+}t^{1-s}\partial_tu(x,t),\ c_s=\frac{\Gamma(s/2)}{2^{1-s}\Gamma(1-s/2)}. \tag{4}$$

This characterization has dramatically popularized the application of the fractional Laplace operators, see [3], [4], [5] and [6].

In Section 2, we prove an L^2-estimate for the Fourier transform $\widehat{p^s_t}(\cdot)$, see Proposition 2.3. In Theorem 3.1, by the aid of Proposition 2.3, we establish the following equivalent characterizations of homogeneous Sobolev spaces $H^{\nu/2}(\mathbb{R}^n)$, i.e., for $\gamma\geq0$ and $\nu\in(0,\min\{n,2s+2\gamma\})$,

$$\left(\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2}p^s_t*f(x)\right|^2t^{2\gamma+2m-1-\nu}dxdt\right)^{1/2}\approx\|(-\Delta)^{\nu/4}f\|_{L^2}, \tag{5}$$

see (15). This characterization enables us to obtain the Sobolev type trace inequality, with $\gamma \geq 0$,

(6)

$$\begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\, dx\right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2} u(x,t)\right|^2 t^{2\gamma+2m-1-\nu} dxdt; \\ \left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\, dx\right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\nabla^m_{x,t} u(x,t)\right|^2 t^{2m-1-\nu} dxdt. \end{cases}$$

The corresponding fractional logarithmic Sobolev inequality and the fractional Hardy inequality can be deduced, see (ii) and (iii) of Theorems 3.1 and 3.2, respectively. Specially, letting $m = 1$, Theorems 3.1 and 3.2 comes back to [13, Theorems 3.1-3.3]. Hence the Sobolev trace inequalities obtained in Section 3 are generalizations of those in [13].

A direct computation indicates that the inequality (6) is invariant under the transform $\phi(x) = \lambda x + x_0$ for $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, i.e.,

$$\left(\int_{\mathbb{R}^n} |(f\circ\phi)(x)|^{2n/(n-\nu)}\, dx\right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2}(u\circ\phi)(x,t)\right|^2 t^{2\gamma+2m-1-\nu} dxdt.$$

However, both the space $L^{2n/(n-\nu)}(\mathbb{R}^n)$ with

$$\|f\|_{L^{2n/(n-\nu)}} := \left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\, dx\right)^{(n-\nu)/2n} < \infty,$$

and the Sobolev space $H^{\nu/2}(\mathbb{R}^n)$ with

$$\left(\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2} p^s_t * f(x)\right|^2 t^{2\gamma+2m-1-\nu} dxdt\right)^{1/2} < \infty$$

are not invariant under the transform ϕ. In [16], using the characterization of Q type space $Q_\alpha(\mathbb{R}^n)$, Xiao obtained a revised conformal invariant Sobolev type trace inequality, see [16, Theorem 4.1]. In Theorem 3.11, following the idea of [16], we prove a local version of (6):

(7)

$$\begin{aligned} &\sup_I \left(\frac{1}{|I|}\int_I |f(x) - f_I|^{2n/(n-\nu)}\, dx\right)^{(n-\nu)/(2n)} \\ &\leq C\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r \int_{|y-x_0|<r} \left|\frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2} u(x,t)\right|^2 t^{2m-1-\nu} dxdt\right)^{1/2}. \end{aligned}$$

Notations. In this paper, $U \approx V$ indicates that there is a constant $c > 0$ such that $c^{-1}V \leq U \leq cV$, whose right inequality is also written as $U \lesssim V$. Similarly, we write $V \gtrsim U$ for $V \geq cU$. Throughout this paper, the symbol $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. The dual of $\mathscr{S}(\mathbb{R}^n)$ is denoted by $\mathscr{S}'(\mathbb{R}^n)$.

2. Basic lemmas

Lemma 2.1 ([13, Proposition 2.1]). *The fractional Poisson kernel $p_t^s(\cdot)$ satisfies the following properties.*

For a positive constant $C_{n,s}$, the Fourier transform $\widehat{p_t^s}(\cdot)$ can be represented as $\widehat{p_t^s}(\xi) = C_{n,s}G_s(t|\xi|)$, where

$$G_s(t) := \int_0^\infty w^{s/2-1} e^{-w-t^2/(4\lambda)} dw,$$

which satisfies

$$\int_0^\infty |G_s(t)|^2 t^\alpha dt < \infty, \quad -1-s < \alpha. \tag{8}$$

Denote $\mathbb{N}^+$ is a set includes all positive natural numbers.

Proposition 2.2. *Let $s \in (0,2)$, $(x,t) \in \mathbb{R}_+^{n+1}$ and i, $m \in \mathbb{N}^+$.*

$$\int_0^\infty \left| \frac{d^m}{dt^m} G_s(t) \right|^2 t^\alpha dt < \infty, \quad \alpha > 2m - 2s - 1.$$

Proof. By the higher-order derivative formula of composite functions

$$\begin{cases} \dfrac{d^m}{dx^m} f(a+cx^2) = \displaystyle\sum_{i=0}^{(m-1)/2} \frac{m(m-1)\cdots(m-2i+1)}{i!}(2x)^{m-2i}c^{m-i}\frac{\partial^{m-i}}{\partial u^{m-i}}f(u), \ m \text{ is odd;} \\ \dfrac{d^m}{dx^m} f(a+cx^2) = \displaystyle\sum_{i=0}^{m/2} \frac{m(m-1)\cdots(m-2i+1)}{i!}(2x)^{m-2i}c^{m-i}\frac{\partial^{m-i}}{\partial u^{m-i}}f(u), \ m \text{ is even.} \end{cases} \tag{9}$$

We can obtain $C_{i,m}$ depending on m and i such that

$$\begin{cases} \left|\dfrac{d^m}{dt^m}G_s(t)\right| = \left|\displaystyle\sum_{i=1}^{(m+1)/2} C_{i,m}t^{2i-1}\int_0^\infty \gamma^{(s-m+1)/2-i-1}e^{-\gamma-t^2/(4\gamma)}d\gamma\right|, \ m \text{ is odd;} \\ \left|\dfrac{d^m}{dt^m}G_s(t)\right| = \left|\displaystyle\sum_{i=1}^{m/2+1} C_{i,m}t^{2(i-1)}\int_0^\infty \gamma^{(s-m)/2-1-i}e^{-\gamma-t^2/(4\gamma)}d\gamma\right|, \ m \text{ is even.} \end{cases} \tag{10}$$

Below we assume m is odd.

$$\left|t^{m-s}\frac{d^m}{dt^m}G_s(t)\right| \lesssim \sum_{i=1}^{(m+1)/2} t^{m-s+2i-1}\left|\int_0^\infty \gamma^{(s-m+1)/2-1-i}e^{-t^2/(4\gamma)}\right| d\gamma.$$

For $t \to 0$, letting $\gamma = t^2/(4u)$, we have

$$\begin{aligned} \left|t^{m-s}\frac{d^m}{dt^m}G_s(t)\right| &\lesssim \sum_{i=1}^{(m+1)/2} t^{m-s+2i-1}\left|\int_0^\infty \gamma^{(s-m+1)/2-1-i}e^{-t^2/(4\gamma)}d\gamma\right| \\ &\lesssim \sum_{i=1}^{(m+1)/2} \int_0^\infty u^{i-1-(s-m+1)/2}e^{-u}du. \end{aligned}$$

For all $i \in [1,(m+1)/2] \cap \mathbb{N}^+$, $i-1-(s-m+1)/2 \geq (m-1-s)/2 > -1$. This indicates

$$\left|\frac{d^m}{dt^m}G_s(t)\right| \lesssim t^{s-m} \sum_{i=1}^{(m+1)/2} \int_0^\infty u^{i-1-(s-m+1)/2} e^{-u} du \lesssim t^{s-m}.$$

On the other hand, we consider $t \to \infty$. For any $\delta > 0$, we can obtain $e^{-\gamma} \lesssim \gamma^{-\delta}$ as $\gamma > 0$. Applying the change of variable $t^2/(4\gamma) = u$ to get

$$\begin{aligned}\left|\frac{d^m}{dt^m}G_s(t)\right| &\lesssim \left|\sum_{i=1}^{(m+1)/2} t^{2i-1} \int_0^\infty \gamma^{(s-m+1)/2-1-i-\delta} e^{-t^2/(4\gamma)} d\gamma\right| \\ &\lesssim t^{s-m-2\delta} \sum_{i=1}^{(m+1)/2} \int_0^\infty u^{\delta+i-1+(m-1-s)/2} e^{-u} du \\ &\lesssim t^{s-m-2\delta}.\end{aligned}$$

Similar to the proof of the case that m is odd, for m is even, we can obtain

$$\left|\frac{d^m}{dt^m}G_s(t)\right| \lesssim \begin{cases} t^{s-m}, & t \to 0; \\ t^{s-m-2\delta}, & t \to \infty \text{ and } \forall \delta > 0. \end{cases} \tag{11}$$

Summarizing, for $m \in \mathbb{N}^+$, we get

$$\int_0^\infty \left|\frac{d^m}{dt^m}G_s(t)\right|^2 t^\alpha dt < \infty, \quad h > 2m-2s-1.$$

□

Proposition 2.3. *Let $s \in (0,2)$ and $m \in \mathbb{N}^+$. If $\alpha > 2m-1-2s$, there exists a constant $C(m,n,s,\alpha)$ such that*

$$\int_0^\infty \left|\frac{d^m}{dt^m}\widehat{p}_t^s(\xi)\right|^2 t^\alpha dt = C(m,n,s,\alpha)|\xi|^{2m-1-\alpha}.$$

Proof. By Proposition 2.2 and Lemma 2.1, for $u = t|\xi|$, we can obtain

$$\begin{aligned}\int_0^\infty \left|\frac{d^m}{dt^m}\widehat{p}_t^s(\xi)\right|^2 t^\alpha dt &= C_{n,s} \int_0^\infty \left|\frac{d^m}{dt^m}G_s(t|\xi|)\right|^2 t^\alpha dt \\ &= C_{n,s}|\xi|^{2m-1-\alpha} \int_0^\infty \left|\frac{d^m}{du^m}G_s(u)\right|^2 u^\alpha du \\ &= C_{m,n,s,\alpha}|\xi|^{2m-1-\alpha}.\end{aligned}$$

□

3. Sobolev type trace inequalities via Caffarelli-Silvestre extensions

The homogeneous Besov spaces and the homogeneous Sobolev spaces are defined as follows.

Definition. (i) Let $\nu \in (0,1)$ and $p \in (1, n/\nu)$. The homogeneous Sobolev spaces $\dot{W}_p^{\nu}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\dot{W}_p^{\nu}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\nu}} dxdy \right)^{1/p}.$$

Specially, when $p=2$, $\dot{W}_2^{\nu}(\mathbb{R}^n)$ is also denoted as $\dot{H}^{\nu}(\mathbb{R}^n)$. Moreover, $\dot{W}_{p'}^{-\nu}(\mathbb{R}^n)$ is the dual of $\dot{W}_p^{\nu}(\mathbb{R}^n)$.

(ii) Let $(\beta, p, q) \in (0,\infty)\times(0,\infty)\times(0,\infty]$. The homogeneous Besov space $\dot{\Lambda}_{\nu}^{p,q}(\mathbb{R}^n)$ is defined as the completion of all $C_0^\infty(\mathbb{R}^n)$ functions with $\|f\|_{\dot{\Lambda}_{\nu}^{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{\Lambda}_{\nu}^{p,q}(\mathbb{R}^n)} := \begin{cases} \left(\int_{\mathbb{R}^n} \|\Delta_h^k f\|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|^{n+\nu q}} \right)^{1/q}, & q \in (0,\infty); \\ \sup_{h\in\mathbb{R}^n\setminus\{0\}} \|\Delta_h^k f\|_{L^p(\mathbb{R}^n)}^q |h|^{-\nu}, & q = \infty. \end{cases}$$

Theorem 3.1. *Denote by $u(x,t) = p_t^s * f(x)$ the Caffarelli-Silvestre extension of f. Let $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ and $m \in \mathbb{N}^+$. If the index γ and ν satisfy $\gamma > \max\{0, (\nu-2s)/2\}$ and $\nu \in (0,n)$, or $\gamma > 0$ and $\nu \in (0, \min\{n, 2s+2\gamma\})$.*

(i) *There holds*

(12)
$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\, dx \right)^{1-\nu/n} \lesssim \int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt.$$

(ii) *If $||f||_{L^2(\mathbb{R}^n)} = 1$, there holds*

(13)
$$\exp\left(\frac{\nu}{n} \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) dx \right) \lesssim \int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt.$$

(iii) *There holds*

$$\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^{\nu}} \lesssim \int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt. \tag{14}$$

Proof. In order to prove Theorem 3.1, we need to establish the following result:

$$\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt \approx \int_{\mathbb{R}^n} |\xi|^{\nu} |\widehat{f}(\xi)|^2 d\xi. \tag{15}$$

In fact, notice that $u(x,t) = p_t^s * f(x)$. It holds

$$\begin{aligned} &\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2m+2\gamma-1-\nu} dxdt \\ = &\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt \end{aligned}$$

$$= \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2\gamma} \left| \frac{\partial^m}{\partial t^m} \widehat{u}(\xi,t) \right|^2 t^{2\gamma+2m-1-\nu} d\xi dt$$
$$= \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2m+2\gamma} \left| \frac{\partial^m}{\partial t^m} G_s(t|\xi|) \right|^2 |\widehat{f}(\xi)|^2 t^{2\gamma+2m-1-\nu} d\xi dt$$
$$\approx \int_{\mathbb{R}^n} \left(\int_0^\infty \left(\frac{\partial^m}{\partial \omega^m} G_s(\omega) \right)^2 \omega^{2\gamma+2m-1-\nu} d\omega \right) |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi.$$

By Proposition 2.2, we get

$$\int_0^\infty \left| \frac{\partial^m}{\partial \omega^m} G_s(\omega) \right|^2 \omega^{2\gamma+2m-1-\nu} d\omega < \infty.$$

Hence

$$\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^m}{\partial t^m} \left(-\Delta^{\gamma/2}\right) u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dx dt$$
$$\approx \int_{\mathbb{R}^n} \left(\int_0^\infty \left| \frac{\partial^m}{\partial \omega^m} G_s(\omega) \right|^2 \omega^{2\gamma+2m-1-\nu} d\omega \right) |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi$$
$$\approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi.$$

We know that

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi &= || \widehat{f}(\cdot)| \cdot |^{\nu/2} ||^2_{L^2(\mathbb{R}^n)} \\ &= || \widehat{(-\Delta)^{\nu/4}} f ||^2_{L^2(\mathbb{R}^n)} \\ &= ||(-\Delta)^{\nu/4} f||^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

It follows from the well-known fractional Sobolev inequality:

$$||f||^2_{L^{2n/(n-\nu)}(\mathbb{R}^n)} \le B(n,\nu) ||(-\Delta)^{\nu/4} f||^2_{L^2(\mathbb{R}^n)}$$

for $\nu \in (0,n)$ and some constant $B(n,\nu)$ that (12) holds.

Now we are in a position to prove (13). Let $p = n(r-2)/\nu$, $2 < r < 2n/(n-\nu)$ and $\nu \in (0, \min(n, 2m))$. The Hölder inequality implies that

(16)

$$\|f\|^r_{L^r(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^p |f(x)|^{r-p} dx \le \|f\|^p_{L^{2n/(n-\nu)}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^2 dx \right)^{1-p(n-\nu)/2n}.$$

If $\|f\|_{L^2(\mathbb{R}^n)} = 1$, we can deduce from (16) that

$$\left(\int_{\mathbb{R}^n} |f(x)|^{r-2} |f(x)|^2 dx \right)^{1/(r-2)} = \left(\int_{\mathbb{R}^n} |f(x)|^r dx \right)^{1/(r-2)} \le ||f||^{n/\nu}_{L^{2n/(n-\nu)}(\mathbb{R}^n)}.$$

So, the inequality (12) implies that for a positive constant $A(n,s,\nu)$,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{r-2} |f(x)|^2 dx \right)^{1/(r-2)} \le \left(A(n,s,\nu) \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2m-1-\nu} dx dt \right)^{n/(2\nu)},$$

which yields

$$\exp\left(\frac{2\nu}{n(r-2)}\ln\left(\int_{\mathbb{R}^n}|f(x)|^{r-2}|f(x)|^2dx\right)\right)$$
$$\leq\left(A(n,s,\nu)\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2}u(x,t)\right|^2t^{2m-1-\nu}dxdt\right)^{n/(2\nu)}.$$

Since $\|f\|_{L^2(\mathbb{R}^n)}=1$, $d\mu(x):=|f(x)|^2dx$ can be treated as a probability measure on $\mathbb{R}^n$. Thus (13) can be obtained by letting $r\to 2$,

$$\lim_{r\to 2}\exp\left(\frac{2\nu}{n(r-2)}\ln\left(\int_{\mathbb{R}^n}|f(x)|^{r-2}|f(x)|^2dx\right)\right)$$
$$=\exp\left(\frac{\nu\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx}{n\int_{\mathbb{R}^n}|f(x)|^2dx}\right)$$
$$=\exp\left(\nu/n\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx\right),$$

which implies (13). At last, the inequality (14) follows from (15) and the fractional Hardy inequality

$$\left\|\frac{f(\cdot)}{|\cdot|^{\nu/2}}\right\|_{L^2(\mathbb{R}^n)}\lesssim\|(-\Delta)^{\nu/4}f\|_{L^2(\mathbb{R}^n)},$$

which is a special case of [17, (3.1) in Theorem 3.1]. □

Theorem 3.2. *Let $f\in\dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu\in(0,\min\{2s,n\})$ and $m\in\mathbb{N}^+$. Denote by $u(x,t)=p_t^s*f(x)$ the Caffarelli-Silvestre extension of f.*

(i) *There holds*

$$\left(\int_{\mathbb{R}^n}|f(x)|^{2n/(n-\nu)}dx\right)^{1-\nu/n}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2t^{2m-1-\nu}dxdt. \tag{17}$$

(ii) *If $||f||_{L^2(\mathbb{R}^n)}=1$, there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx\right)\lesssim\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2t^{2m-1-\nu}dxdt. \tag{18}$$

(iii) *There holds*

$$\int_{\mathbb{R}^n}|f(x)|^2\frac{dx}{|x|^\nu}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2t^{2m-1-\nu}dxdt. \tag{19}$$

Proof. In order to prove (3.2), we need to establish the following result. For $\nu\in(0,\min\{2\alpha,n\})$, there exists a constant $a(n,\alpha,\beta,\mu)$ such that

$$\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2t^{2m-1-\nu}dxdt=a(n,\alpha,\beta,\nu)\int_{\mathbb{R}^n}|\xi|^\nu|\widehat{f}(\xi)|^2d\xi.$$

In fact, nothing that $u(x,t) = p_t^s * f(x)$, we can apply Proposition 2.2 to deduce that

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \left|\frac{\partial^m}{\partial t^m} u(x,t)\right|^2 t^{2m-1-\nu} dxdt \\
&= \int_0^\infty \int_{\mathbb{R}^n} t^{2m-1-\nu} \left|\frac{\partial^m}{\partial t^m} \widehat{u}(x,t)\right|^2 d\xi dt \\
&= \int_{\mathbb{R}^n} \int_0^\infty t^{2m-1-\nu} \left|\frac{\partial^m}{\partial t^m} G_s(t|\xi|)\right|^2 |\widehat{f}(\xi)|^2 d\xi dt \\
&= \int_{\mathbb{R}^n} \left(\int_0^\infty u^{2m-1-\nu} \left|\frac{\partial^m}{\partial t^m} G_s(u)\right|^2 dt\right) |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi \\
&\approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi.
\end{aligned}$$

Then Theorem 3.2 can be proved in a way similar to that of Theorem 3.1. □

Theorem 3.3. *Let $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0, \min\{2s, n\})$ and $m \in \mathbb{N}^+$. Denote by $u(x,t) = p_t^s * f(x)$ the Caffarelli-Silvestre extension of f.*

(i) *There holds*

$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\, dx\right)^{1-\nu/n} \lesssim \int_{\mathbb{R}_+^{n+1}} |\nabla_x^m u(x,t)|^2 t^{2m-1-\nu} dxdt. \tag{20}$$

(ii) *If $||f||_{L^2(\mathbb{R}^n)} = 1$, there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) dx\right) \lesssim \int_{\mathbb{R}_+^{n+1}} |\nabla_x^m u(x,t)|^2 t^{2m-1-\nu} dxdt. \tag{21}$$

(iii) *There holds*

$$\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^\nu} \lesssim \int_{\mathbb{R}_+^{n+1}} |\nabla_x^m u(x,t)|^2 t^{2m-1-\nu} dxdt. \tag{22}$$

Proof. For $\gamma = m$, it can be deduced from [13, Theorem 3.3] that

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} |\nabla_x^m u(x,t)|^2 t^{2m-1-\nu} dxdt &\approx \int_{\mathbb{R}_+^{n+1}} |\xi|^{2m} \left|\widehat{u}(\xi,t)\right|^2 t^{2m-1-\nu} dxdt \\
&\approx \int_{\mathbb{R}_+^{n+1}} \left|(-\Delta)^{\gamma/2} u(x,t)\right|^2 t^{2m-1-\nu} dxdt \\
&\approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi.
\end{aligned}$$

Similarly to Theorem 3.1, Theorem 3.3 can be proved. □

Lemma 3.4 ([8, Theorem 6.5.]). *Let $\nu \in (0,2)$ and $p \in [1,\infty)$ with $\nu p/2 < n$. Then there exists a positive constant $C = C(n,p,\nu)$ such that for any compactly supported measurable function $f : \mathbb{R}^n \to \mathbb{R}$,*

$$||f||^p_{L^{p^*}(\mathbb{R}^n)} \le C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\nu p/2}} dxdy,$$

where $p^* = p^*(n,\nu)$ is the so-called "fractional critical exponent" and it equals to $np/(n - \nu p/2)$.

Theorem 3.5. *Let $1 < q < \infty$, $0 < \nu < 2n$, $1 < p < 2n/\nu$ and $f \in \dot{W}^{\nu/2}_p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $\|f\|_{L^q(\mathbb{R}^n)} > 0$. Then the inequality*

$$\exp\left(\left(\frac{1}{q} + \frac{\nu}{2n} - \frac{1}{p}\right) \int_{\mathbb{R}^n} \frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}} \ln\left(\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right) dx\right) \lesssim \frac{\|f\|_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)}}{\|f\|_{L^q(\mathbb{R}^n)}}.$$

Proof. Let

$$g(h) := h \ln\left(\int_{\mathbb{R}^n} |f(x)|^{1/h} dx\right),$$

where $g(h)$ is a convex function. For $h > h_1 \ge 0$, we can obtain

$$g'(h) = \ln\left(\int_{\mathbb{R}^n} |f(x)|^{1/h} dx\right) - h^{-1} \frac{\int_{\mathbb{R}^n} |u(x)|^{1/h} \ln|f(x)| dx}{\int_{\mathbb{R}^n} |f(x)|^{1/h} dx} \ge \frac{g(h_1) - g(h)}{h_1 - h}.$$

Taking $h = 1/q$, $h_1 = 1/p_1$ and $0 < q < p_1 \le \infty$, by [14, Lemma 1], we have

$$\exp\left(\int_{\mathbb{R}^n} \frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}} \ln\left(\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right) dx\right) \le \frac{p_1}{p_1 - q} \frac{\|f\|^q_{L^{p_1}(\mathbb{R}^n)}}{\|f\|^q_{L^q(\mathbb{R}^n)}}. \tag{23}$$

For $\gamma > 0$, Hölder's inequality implies

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |f(x)|^\gamma |f(x)|^{p_1-\gamma}\right)^{1/p_1} dx \\ &\le \|f\|^{\gamma/p_1}_{L^{\gamma p_2'}(\mathbb{R}^n)} \|f\|^{(p_1-\gamma)/p_1}_{L^{(p_1-\gamma)/p_3'}(\mathbb{R}^n)} \\ &= \|f\|^{\gamma/p_1}_{L^{p_2}(\mathbb{R}^n)} \|f\|^{(p_1-\gamma)/p_1}_{L^{p_3}(\mathbb{R}^n)}, \end{aligned}$$

where $1/p_2' + 1/p_3' = 1$, $p_2 := \gamma p_2'$ and $p_3 := (p_1 - \gamma)/p_3'$. For $1/p + (n - \nu/2)/n = 1 + 1/p_2$, we get

$$\|f\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \|f\|^{\gamma/p_1}_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)} \|f\|^{(p_1-\gamma)/p_1}_{L^{p_3}(\mathbb{R}^n)}.$$

Then we can choose $p_1 = nq/(n - q\nu/2) \in (q, \infty)$ for $p_3 = q$, γ and p satisfying $\gamma(1/p - \nu/(2n)) + (p_1 - \gamma)/p_3 = 1$. Hence,

$$\int_{\mathbb{R}^n} \frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}} \ln\left(\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right) dx \lesssim \frac{p_1}{p_1 - q} \ln\left(\frac{\left(\|f\|^{\gamma/p_1}_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)} \|f\|^{(p_1-\gamma)/p_1}_{L^{p_3}(\mathbb{R}^n)}\right)^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right)$$

$$\lesssim \frac{q\gamma}{p_1 - q} \ln \left(\frac{\|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}}{\|f\|_{L^q(\mathbb{R}^n)}} \right).$$

We can get $\frac{q\gamma}{p_1-q} = \frac{1}{\frac{1}{q}+\frac{\nu}{2n}-\frac{1}{p}}$. □

When $p = q$ and $\|f\|_{L^q(\mathbb{R}^n)} = 1$, there holds the L^p-logarithmic-type Sobolev inequality.

Corollary 3.6. *Let $0 < \nu < 2n$, $1 < p < 2n/\nu$, $f \in \dot{W}_p^{\nu/2}(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} = 1$. Then*

$$\exp\left(\frac{\nu}{2n} \int_{\mathbb{R}^n} |f(x)|^p \ln\left(|f(x)|^p\right) dx \right) \lesssim \|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}.$$

Lemma 3.7 ([10, Theorem 1.1])**.** *Let $n > 1$ and $0 < \nu < 2$. Then for all $f \in \dot{w}_P^{\nu/2}(\mathbb{R}^n)$ in case $1 \le p < 2n/\nu$, and for all $f \in \dot{W}_p^{\nu/2}(\mathbb{R}^n/\{0\})$ in case $p > 2n/\nu$*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{2p\nu}} dx \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n-2p\nu}} dxdy.$$

In Theorem 3.1, we consider the scope of (p, ν) when $p = 2$ and $\nu \in (0, \min\{n, 2\})$. Automatically, we can generalize inequalities in Theorem 3.1 to the general index $p \in [1, \infty)$ and $\nu \in (0, 2n/p)$. Let $u(x,t) = p_t^s * f(x)$. Following [2, Theorems 1.1 and 1.3], when $\mu \in (0, 2)$, we know

$$\left(\int_{\mathbb{R}_+^{n+1}} |\nabla_x^m u(x,t)|^p t^{pm - p\nu/2 - 1} dxdt \right)^{1/p} \approx ||f||_{\dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)}. \tag{24}$$

In Theorem 3.2, $p = 2$ and $\nu \in (0, \min\{2, n\})$, we have

$$\left(\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m}{\partial t^m} u(x,t) \right|^p t^{pm - p\nu/2 - 1} dxdt \right)^{1/p} \approx ||f||_{\dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)}, \tag{25}$$

where $\nu \in (0, 2)$.

Let $f \in \dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)$. Using Lemma 3.4, we can obtain

$$\left(\int_{\mathbb{R}^n} |f(x)|^{np/(n-p\nu/2)} dx \right)^{(n-p\nu/2)/np} \lesssim \|f\|_{\dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)},\ 1 \le p < 2n/\nu,$$

which implies that the following Sobolev trace-type inequalities: for $f \in \dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)$ with $1 \le p < 2n/\nu$

(26)

$$\|f\|_{L^{np/(n-p\nu/2)}(\mathbb{R}^n)} \lesssim \left(\int_{\mathbb{R}_+^{n+1}} |\nabla_x^m u(x,t)|^p \, t^{pm - p\nu/2 - 1} dxdt \right)^{1/p},\ \nu \in (0, \min\{2, 2n/p\}).$$

Applying the fractional Sobolev inequality to $f \in \dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)$ again, we can get for $1 \le p < 2n/\nu$,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{np/(n-p\nu/2)}dx\right)^{(n-p\nu/2)/np} \lesssim \|f\|_{\dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)}.$$

This indicates that

$$\|f\|_{L^{np/(n-p\nu/2)}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m}{\partial t^m}u(x,t)\right|^p t^{mp-1-p\nu/2}dxdt,\ \nu \in (0, \min\{2, 2n/p\}),$$

where $f \in \dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)$ with $1 \le p < 2n/\nu$.

Moreover, let $f \in \dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)$, there holds the logarithmic-type Sobolev inequality (Corollary 3.6), when $\|f\|_{L^p(\mathbb{R}^n)} = 1$, $\nu \in (0, 2n/p)$ and $p \in (1, 2n/\nu)$

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n} |f(x)|^p \ln(|f(x)|^p)dx\right) \lesssim \|f\|_{\dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)},$$

and the fractional Hary inequality with $\nu \in (0, 2)$ and $p \in [1, 2n/\beta]$ (Lemma 3.7):

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{p\nu/2}}dx\right)^{1/p} \lesssim ||f||_{\dot{\Lambda}^{p,p}_{\nu/2}(\mathbb{R}^n)}.$$

Applying (26), we can establish the logarithmic and Hardy trace-type inequalities for general p with ν in a similar range. Thus, Theorems 3.1 and 3.2 can be generalized to $p \ge 1$.

Definition. Assume that σ: $\mathbb{R}^{n+1}_+ \to \mathbb{R}_+$ is a positive measurable function. Denote by $L^p(\mathbb{R}^{n+1}_+, \sigma)$ the weighted Lebesgue space of all measurable functions $f : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ with

$$||f||_{L^p(\mathbb{R}^{n+1}_+,\sigma)} := \left(\int_{\mathbb{R}^{n+1}_+} |f(x,t)|^p\sigma(x,t)dxdt\right)^{1/p} < \infty.$$

Let

$$\Theta_p(f,\sigma) := A_{n,p}\left(\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|^{-n}_{L^p(\mathbb{R}^{n+1}_+,\sigma)}d\xi\right)^{-1/n},$$

where $A_{n,p}$ is a constant depending on n, p.

The following affine Sobolev type inequality was obtained by Haddad, Jiménez and Montenegro [11].

Theorem 3.8 ([11, Theorem 1.1])**.** *Define a function σ on $\mathbb{R}^{n+1}_+$ as $\sigma(x,t) := t^\gamma\,\forall (x,t) \in \mathbb{R}^{n+1}_+$. Let $\gamma \ge 0$, $1 \le p < n+\gamma+1$ and $p^*_\gamma = p(n+\gamma+1)/(n+\gamma+1-p)$. There exists a sharp constant $J(n,p,\gamma)$ such that*

(27)

$$||g(\cdot,\cdot)||_{L^{p^*_\gamma}(\mathbb{R}^{n+1}_+),\sigma} \le J(n,p,\gamma)(\Theta_p(g,\sigma))^{n/(n+\gamma+1)}\left|\left|\frac{\partial g}{\partial t}(\cdot,\cdot)\right|\right|^{(\gamma+1)/(n+\gamma+1)}_{L^p(\mathbb{R}^{n+1}_+,\sigma)}.$$

Moreover, in (27)*, the equality holds if*

$$(28)\quad g(x,t) := \begin{cases} \dfrac{c}{\left(1+|\delta t|^{(1+1/p)}+|A(x-x_0)|^{(1+1/p)}\right)^{(1+n+\gamma-p)/p}}, & p>1; \\ 1_{\mathbb{B}}^{n+1}(\delta t, A(x-x_0)), & p=1, \end{cases}$$

where $(c,|\delta|,x_0,A) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n\times GL_n$*, and* $1_{\mathbb{B}}^{n+1}$ *is the characteristic function of the unit ball in* $\mathbb{R}^{n+1}$ *and* GL_n *denotes the set of all invertible real* $n\times n$*-matrices.*

Theorem 3.9. *Let* $f\in C_0^\infty(\mathbb{R}^n)$ *and its time-space fractional extension* $u(x,t) = p_t^s * f(x)$ *when* $\nu \geq 1$*. For* $p=\frac{2(n+\nu+2m)}{n+\nu+2+2m}$*,* $m\in\mathbb{N}^+$*, there holds*

$$||f||_{\dot{H}^{-\nu/2}(\mathbb{R}^n)} \lesssim \left(\Theta_p\left(\frac{\partial^m}{\partial t^m}u, t^{\nu+2m-1}\right)\right)^{n/(n+\nu+2m)} \left|\left|\frac{\partial^{m+1}}{\partial t^{m+1}}u(\cdot,\cdot)\right|\right|^{\nu/(n+\nu+2m)}_{L^p(\mathbb{R}^{n+1}_+,t^{\nu+2m-1})}.$$

Proof. From Proposition 2.3, we know

$$\int_{\mathbb{R}^n} |\xi|^{-(\gamma+1-2m)}|\widehat{f}(\xi)|^2 d\xi \approx ||u(\cdot,\cdot)||^2_{L^2(\mathbb{R}^{n+1}_+,t^\gamma)},$$

when $\gamma > 2m-2s-1$. Then, let $\gamma=\nu+2m-1>2m-2s-1$. By Theorem 3.8, we can obtain $\sigma := t^\gamma = t^{\nu+2m-1}$ and

$$\begin{aligned} & ||f||_{\dot{H}^{-\nu/2}(\mathbb{R}^n)} \\ = & ||(-\Delta)^{-\nu/4}f||_{L^2(\mathbb{R}^n)} \\ \approx & \int_{\mathbb{R}^n}|\xi|^{-\nu}|\widehat{f}(\xi)|^2 d\xi \\ \approx & \left\|\frac{\partial^m}{\partial t^m}u(\cdot,\cdot)\right\|^2_{L^2(\mathbb{R}^{n+1}_+,t^\gamma)} \\ \lesssim & \left(\Theta_p\left(\frac{\partial^m}{\partial t^m}u, t^{\nu+2m-1}\right)\right)^{n/(n+\nu+2m)} \left|\left|\frac{\partial^{m+1}}{\partial t^{m+1}}u(\cdot,\cdot)\right|\right|^{\nu/(n+\nu+2m)}_{L^p(\mathbb{R}^{n+1}_+,t^{\nu+2m-1})}. \end{aligned}$$

□

Now, we prove (7) by use of the Caleson measure characterization of Q type spaces $Q_\alpha(\mathbb{R}^n)$ obtained in [7]. For $0\leq\alpha<1$, $Q_\alpha(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f such that

$$(29)\qquad \|f\|^2_{Q_\alpha(\mathbb{R}^n)} := \sup_I(\ell(I))^{2\alpha-n}\int_I\int_I\frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}dxdy<\infty,$$

where the symbol $\sup_I$ denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. By the aid of Hausdorff capacities and tent spaces, Dafni and Xiao [7] proved the following equivalent characterization of $Q_\alpha(\mathbb{R}^n)$.

Theorem 3.10. *Given a C^∞ real-valued function ψ on $\mathbb{R}^n$ with*

$$\psi \in L^1(\mathbb{R}^n),\ |\psi(x)| \lesssim (1+|x|)^{-(n+1)},\ \int_{\mathbb{R}^n} \psi(x)dx = 0.$$

Let $\psi_t(x) := t^{-n}\psi(x/t)$. Then $f \in Q_{v/2}(\mathbb{R}^n)$ if and only if

$$\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r \int_{|y-x_0|<r} |\psi_t * f(x)|^2 t^{-1-\nu} dxdt\right)^{1/2} < \infty.$$

Theorem 3.11. *Suppose that $\nu \in (0, \min\{n, 2s+2\gamma\})$ and $\gamma > \max\{1-s, 0\}$. Let $u(x,t) := p_t^s * f(x)$. Then there exists a constant $C > 0$ such that (7) holds.*

Proof. Take $\psi_t(x) := t^m \frac{\partial^m}{\partial t^m}(-\Delta)^{\gamma/2} p_t^s(x)$. We can obtain

$$\widehat{\psi_t}(\xi t) = \widehat{\psi}(\xi t) = (t|\xi|)^{\gamma+m}\frac{\partial^m}{\partial u^m}G_s(u)\big|_{u=t|\xi|}.$$

Hence we set

$$\psi(x) := \int_{\mathbb{R}^n} \widehat{\psi}(\xi)e^{ix\xi}d\xi,$$

by (11), where

$$\left|\widehat{\psi}(\xi)\right| \lesssim \begin{cases} |\xi|^{s+\gamma},\ t\to 0; \\ |\xi|^{s-2\delta+\gamma},\ t\to\infty \text{ and } \forall\delta>0. \end{cases}$$

Let $2\delta = -n-s-\gamma-3$, we can obtain

$$\begin{aligned} |\psi(x)| &\lesssim \int_{\mathbb{R}^n} |\widehat{\psi}(\xi)|d\xi \\ &\lesssim \int_0^1 |\xi|^{s+\gamma+n-1}d|\xi| + \int_1^\infty |\xi|^{-2}d|\xi| \lesssim 1. \end{aligned}$$

For $|x| \le 1$, we get

$$|\psi(x)| \lesssim 1/2^{n+1} \lesssim \frac{1}{(1+|x|)^{n+1}}.$$

Below we assume $|x| > 1$. For $j = \mathbb{N}^+ \cap [1, n]$

$$\begin{aligned} |x_j^{n+1}\psi(x)| &\lesssim \left|\int_{|\xi|\le 1} |\xi|^{s+\gamma}\frac{\partial^{n+1}}{\partial\xi_j^{n+1}}e^{ix\xi}d\xi\right| + \left|\int_{|\xi|>1} |\xi|^{s-2\delta+\gamma}\frac{\partial^{n+1}}{\partial\xi_j^{n+1}}e^{ix\xi}d\xi\right| \\ &\lesssim \left|\int_{|\xi|\le 1} e^{ix\xi}\frac{\partial^{n+1}}{\partial\xi_j^{n+1}}|\xi|^{s+\gamma}d\xi\right| + \left|\int_{|\xi|>1} e^{ix\xi}\frac{\partial^{n+1}}{\partial\xi_j^{n+1}}|\xi|^{s-2\delta+\gamma}d\xi\right| \\ &\lesssim \int_0^1 \sum_{k=0}^{\lfloor (n+1)/2\rfloor} \xi_j^{n+1-k}|\xi|^{s+\gamma-2(n+1)+2k+n-1}d|\xi| \\ &\quad + \int_1^\infty \sum_{k=0}^{\lfloor (n+1)/2\rfloor} \xi_j^{n+1-k}|\xi|^{s-2\delta+\gamma-2(n+1)+2k+n-1}d|\xi| \end{aligned}$$

$$\lesssim \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \int_0^1 |\xi|^{s+\gamma+k-2} + \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \int_1^\infty |\xi|^{s+\gamma-2\delta+k-2} d|\xi|.$$

Letting $2\delta = s+\gamma-n-1$ and $\gamma > \max\{0, 1-s\}$, we can obtain $|x_j^{n+1}\psi(x)| \lesssim 1$. For $|x| > 1$, we get

$$|\psi(x)| \lesssim \frac{1}{|x|^{n+1}} \lesssim \frac{1}{(1+|x|)^{n+1}}.$$

Then we can verify that ψ satisfies the conditions of Theorem 3.10. Hence it holds

$$\|f\|_{Q_\alpha} \simeq \left(\sup_{x_0 \in \mathbb{R}^n, r \in (0,\infty)} r^{\nu-n} \int_0^r \int_{|y-x_0|<r} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt \right)^{1/2}.$$

As $Q_\alpha(\mathbb{R}^n)$ is a subspace of $BMO(\mathbb{R}^n)$, it is obvious that $\|f\|_{BMO} \le \|f\|_{Q_\alpha}$. We can deduce from the equivalent norm:

$$\begin{aligned} \|f\|_{BMO} &:= \sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I| \, dx \right) \\ &\simeq \sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I|^{2n/(n-\nu)} \, dx \right)^{(n-\nu)/(2n)} \end{aligned}$$

that

$$\begin{aligned} &\sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I|^{2n/(n-\nu)} \, dx \right)^{(n-\nu)/(2n)} \\ &\lesssim \|f\|_{BMO} \le \|f\|_{Q_\alpha} \\ &\lesssim \left(\sup_{x_0 \in \mathbb{R}^n, r \in (0,\infty)} r^{\nu-n} \int_0^r \int_{|y-x_0|<r} \left| \frac{\partial^m}{\partial t^m} (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma+2m-1-\nu} dxdt \right)^{1/2}. \end{aligned}$$

□

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