Nonlinear Functional Analysis and Applications
Vol. 29, No. 1 (2024), pp. 295-306
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.01.18
http://nfaa.kyungnam.ac.kr/journal-nfaa

GENERALIZED HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL INEQUALITY IN MODULAR SPACES AND β-HOMOGENEOUS BANACH SPACES

Abderrahman Baza[1] **and Mohamed Rossafi**[2]

[1]Laboratory of Analysis, Geometry and Application,
Departement of Mathematics, Ibn Tofail University, B.P. 133, Kenitra, Morocco
e-mail: abderrahmane.baza@gmail.com

[2]Departement of Mathematics, Faculty of Sciences Dhar El Mahraz,
Sidi Mohamed Ben Abdellah University, Fes, Morocco
e-mail: rossafimohamed@gmail.com

Abstract. In this work, we investigate the generalized Hyers-Ulam stability of quadratic functional inequality in modular spaces satisfying Δ_2-conditions and Fatou property, and in β-homogeneous Banach spaces.

1. Introduction

The issue of stability for a general functional equation was first raised in 1940 by Ulam [11]. Ulam asked the following question regarding a group homomorphism: "How likely to an automorphism should a function behave in order to ensure the existence of an automorphism near such functions?" The following year, Hyers [3] was the first to respond positively to Ulam's query in the context of Banach spaces. By Aoki [1] and Rassias [8], the latter of whom has had a significant impact on numerous advancements in the stability theory, it was expanded to the situations of additive mappings and linear mappings.

A functional equation is typically algebraic in character, while the stability is more metrical. Therefore, working in a normed linear space is a good

[0]Received August 18, 2023. Revised September 9, 2023. Accepted September 10, 2023.
[0]2020 Mathematics Subject Classification: 39B82, 39B52.
[0]Keywords: Hyers-Ulam stability, quadratic functional inequality, modular spaces, Δ_2-condition, β-homogeneous Banach spaces, Fatou property.
[0]Corresponding author: A. Baza(abderrahmane.baza@gmail.com).

option. However, various discoveries in the literature have shown that there exist many linear topological spaces, particularly for function spaces, whose proper topologies fail to be normable. Successfully examining the possibility of substituting a norm with a so-called modular system. A modular produces fewer propieties than a norm does, yet it is more appropriate in many unique circumstances. The assumption of some extra characteristics, such as some relaxed continuities or some Δ_2-related conditions on a modular, is still reasonable.

Therefore, it makes sense to expand the framework for determining the stability of functional equations into a broader context of modular spaces, as Sadeghi [9] did in the case of Cauchy and Jensen functional equations.

Definition 1.1. ([6]) Let Y be an arbitrary vector space. A functional $\rho : Y \to [0,\infty)$ is called a modular if for arbitrary $x, y \in Y$;

(1) $\rho(x) = 0$ if and only if $x = 0$.
(2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$.
(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.
If (1) is replaced by:
(4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, that is, the vector space Y_ρ given by:

$$Y_\rho = \{x \in Y : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Definition 1.2. ([4]) Let $\{x_n\}$ and x be in Y_ρ. Then:

(1) The sequence $\{x_n\}$ with $x_n \in Y_\rho$ is ρ-convergent to x and write: $x_n \to x$ if $\rho\,(x_n - x) \to 0$ as $n \to \infty$.
(2) The sequence $\{x_n\}$ with $x_n \in Y_\rho$ is called ρ-Cauchy if $\rho\,(x_n - x_m) \to 0$ as $n, m \to \infty$.
(3) Y_ρ is called ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.
(4) It is said that the modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$, whenever $x = \rho - \lim_{n\to\infty} x_n$.
(5) A function modular is said to satisfy the Δ_2-condition if there exist $\tau > 0$ such that $\rho(2x) \leq \tau\rho(x)$ for all $x \in Y_\rho$.

Proposition 1.3. ([6]) *In modular space,*

(1) *If $x_n \overset{\rho}{\to} x$ and a is a constant vector, then $x_n + a \overset{\rho}{\to} x + a$.*
(2) *If $x_n \overset{\rho}{\to} x$ and $y_n \overset{\rho}{\to} y$, then $\alpha x_n + \beta y_n \overset{\rho}{\to} \alpha x + \beta y$, where $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 0$.*

Remark 1.4. Note that $\rho(x)$ is an increasing function, for all $x \in X$. Suppose $0 < a < b$, then property (3) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in Y$. Moreover, if ρ is a convex modular on X and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$.

In general, if $\lambda_i \geq 0$, $i = 1, \ldots, n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ then $\rho(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \lambda_1\rho(x_1) + \lambda_2\rho(x_2) + \cdots + \lambda_n\rho(x_n)$.

If $\{x_n\}$ is ρ-convergent to x, then $\{cx_n\}$ is ρ-convergent to cx, where $|c| \leq 1$. But the ρ-convergent of a sequence $\{x_n\}$ to x does not imply that $\{\alpha x_n\}$ is ρ-convergent to αx_n for scalars α with $|\alpha| > 1$.

If ρ is a convex modular satisfying Δ_2-condition with $0 < \tau < 2$, then $\rho(x) \leq \tau\rho(\frac{1}{2}x) \leq \frac{\tau}{2}\rho(x)$ for all x.
Hence $\rho = 0$. Consequently, we must have $\tau \geq 2$ if ρ is convex modular.

Skof [10] has demonstrated that quadratic mappings are generalized Ulam-Hyers-Rassias stable under the condition that X and Y are respectively normed and Banach spaces. Later, it was discovered [2] that the same pattern holds true even when X is an Abelian group.

In the present paper consisting of 3 sections, we consider the case where V is a linear space and X_ρ is a ρ-complete modular space, with arbitrary scalar fields. In section 2, we show the stability of the following inequality in modular space satisfying Δ_2-condition with $\tau = 2$:

$$\begin{aligned}&\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\\&\quad\leq \rho\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}(y)\right).\end{aligned} \tag{1.1}$$

In section 3, we obtains a like result in homogeneous Banach space of the following inequality, using the Gavruta control

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\\&\quad\leq \left\|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\|.\end{aligned} \tag{1.2}$$

2. Quadratic functional inequalities in modular space without Δ_2-condition

Throughout this section, assume that ρ is a convex modular satisfying Δ_2-condition with $\tau = 2$ and X_ρ is a ρ-complete modular space.

Lemma 2.1. *Let f be a mapping $f : V \to X_\rho$ satisfies:*

$$\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}(y)\right) \tag{2.1}$$

for all $x, y \in V$. Then f is quadratic.

Proof. Letting $x = y = 0$ in (2.1), we get

$$\rho(2f(0)) \leq \rho(f(0)).$$

Hence

$$\rho(f(0)) \leq \frac{1}{2}\rho(f(0)).$$

So $f(0) = 0$. Letting $y = x$ in (2.1), we get:

$$\rho(f(2x) - 4f(x)) \leq 0.$$

And so $f(2x) = 4f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \dfrac{1}{4}f(x)$ for all $x \in X$. It follows from (2.1) that:

$$\begin{aligned}\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) &\leq \rho\left(\frac{1}{4}f(x+y)+\frac{1}{4}f(x-y)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right)\\ &\leq \frac{1}{4}\rho(f(x+y)+f(x-y)-2f(x)-2f(y)).\end{aligned}$$

And so, $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for all $x, y \in V$. Then f is quadratic. □

Theorem 2.2. *Let $\varphi : V^2 \to [0, \infty)$ be a function with $\varphi(0,0) = 0$ and let $f : V \to X_\rho$ be a mapping such that:*

$$\psi(x,y) = \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi\left(2^{j-1}x, 2^{j-1}y\right) < \infty \tag{2.2}$$

and

$$\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \leq \rho\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right)+\varphi(x,y) \tag{2.3}$$

for all $x, y \in V$. Then there exists a unique quadratic mapping $h : V \to X_\rho$ such that:

$$\rho(f(x) - h(x)) \leq \psi(x,x) \tag{2.4}$$

for all $x \in V$.

Proof. Letting $x = y = 0$ in (2.3), we get $\rho(2f(0)) \leq \rho(f(0))$. So $f(0) = 0$. Letting $y = x$ in (2.2), we get $\rho(f(2x) - 4f(x)) \leq \varphi(x,x)$ for all $x \in V$. So

$$\rho\left(\frac{1}{4}f(2x) - f(x)\right) \leq \frac{1}{4}\varphi(x,x). \tag{2.5}$$

Then by induction, we write

$$\rho\left(\frac{f(2^k x)}{4^k} - f(x)\right) \leq \sum_{j=1}^{k} \frac{1}{4^j}\varphi\left(2^{j-1}x, 2^{j-1}x\right) \tag{2.6}$$

for all $x \in V$ and all positive integer k. Indeed, the case $k = 1$ follows from (2.5). Assume that (2.6) holds for $k \in \mathbb{N}$. Then we have the following inequality:

$$\begin{aligned}
\rho\left(\frac{f\left(2^{k+1}x\right)}{4^{k+1}} - f(x)\right) &= \rho\left(\frac{1}{4}\frac{f\left(2^k \cdot 2x\right)}{4^k} - f(2x) + \frac{1}{4}f(2x) - f(x)\right) \\
&= \rho\left(\frac{1}{4}\left(\frac{f\left(2^k \cdot 2x\right)}{4^k} - f(2x)\right) + \frac{1}{4}(f(2x) - 4f(x))\right) \\
&\leq \frac{1}{4}\rho\left(\frac{f\left(2^k \cdot 2x\right)}{4^k} - f(2x)\right) + \frac{1}{4}\rho(f(2x) - 4f(x)) \\
&\leq \frac{1}{4}\sum_{j=1}^{k}\frac{1}{4^j}\varphi\left(2^j x, 2^j x\right) + \frac{1}{4}\varphi(x,x) \\
&= \sum_{j=1}^{k+1}\frac{1}{4^j}\varphi\left(2^{j-1}x, 2^{j-1}x\right).
\end{aligned}$$

Hence (2.6) hlods for every $k \in \mathbb{N}$.

Let m and n be nonnegative integers with $n > m$. By (2.6), we have

$$\begin{aligned}
\rho\left(\frac{f(2^n x)}{4^n} - \frac{f\left(2^m x\right)}{4^m}\right) &= \rho\left(\frac{1}{4^m}\left(\frac{f\left(2^{n-m} \cdot 2^m x\right)}{4^{n-m}}\right) - f(2^m x)\right) \\
&\leq \frac{1}{4^m}\sum_{j=1}^{n-m}\frac{1}{4^j}\varphi\left(2^{j+m-1}x, 2^{j+m-1}x\right) \\
&= \sum_{k=m+1}^{n}\frac{1}{4^k}\varphi\left(2^{k-1}x, 2^{k-1}x\right).
\end{aligned} \tag{2.7}$$

Then (2.2) and (2.7) yield that $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a ρ-Cauchy sequence in X_ρ. The ρ-completeness of X_ρ guarantees its ρ-convergence. Hence, there exists a mapping: $h : V \to X_\rho$ defined by: $h(x) = \rho - \lim \frac{f(2^n x)}{4^n}$, $x \in V$. Moreover

$$\begin{aligned}\rho(f(x) - h(x)) &\leq \liminf_{n\to\infty} \rho\left(f(x) - \frac{f(2^n x)}{4^n}\right)\\ &\leq \sum_{k=1}^{+\infty} \frac{1}{4^k}\varphi(2^{k-1}x, 2^{k-1}x)\\ &= \psi(x, x).\end{aligned}$$

Then, we get the estimation (2.4).

Now, we prove that h is quadratic. We note that

$$\begin{aligned}&\rho\left(\frac{1}{16}h\left(x+y\right) + \frac{1}{16}h\left(x-y\right) - \frac{1}{8}f(x) - \frac{1}{8}f\left(y\right)\right)\\
&\leq \frac{1}{16}\rho\left(h(x+y) - \frac{f\left(2^n(x+y)\right)}{4^n}\right) + \frac{1}{16}\rho\left(h(x-y) - \frac{f\left(2^n(x-y)\right)}{4^n}\right)\\
&\quad + \frac{1}{8}\rho\left(f(x) - \frac{f(2^n x)}{4^n}\right) + \frac{1}{8}\rho\left(f(y) - \frac{f\left(2^n y\right)}{4^n}\right)\\
&\quad + \frac{1}{16}\rho\left(\frac{f\left(2^n(x+y)\right)}{4^n} + \frac{f\left(2^n(x-y)\right)}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f\left(2^n y\right)}{4^n}\right)\\
&\leq \frac{1}{16}\rho\left(h(x+y) - \frac{f\left(2^n(x+y)\right)}{4^n}\right) + \frac{1}{16}\rho\left(h(x-y) - \frac{f\left(2^n(x-y)\right)}{4^n}\right)\\
&\quad + \frac{1}{8}\rho\left(f(x) - \frac{f(2^n x)}{4^n}\right) + \frac{1}{8}\rho\left(f(y) - \frac{f\left(2^n y\right)}{4^n}\right)\\
&\quad + \frac{1}{4^{n+2}}\rho\left(f\left(2^n(x+y)\right) + f\left(2^n(x-y)\right) - 2f(2^n x) - 2f\left(2^n y\right)\right)\\
&\leq \frac{1}{16}\rho\left(h(x+y) - \frac{f\left(2^n(x+y)\right)}{4^n}\right) + \frac{1}{16}\rho\left(h(x-y) - \frac{f\left(2^n(x-y)\right)}{4^n}\right)\\
&\quad + \frac{1}{8}\rho\left(f(x) - \frac{f(2^n x)}{4^n}\right) + \frac{1}{8}\rho\left(f(y) - \frac{f\left(2^n y\right)}{4^n}\right)\\
&\quad + \frac{1}{4^{n+2}}\rho\left(f\left(\frac{2^n(x+y)}{2}\right) + f\left(\frac{2^n(x-y)}{2}\right) - \frac{1}{2}f(2^n x) - \frac{1}{2}f\left(2^n y\right)\right)\\
&\quad + \frac{1}{4^{n+2}}\varphi(2^n x, 2^n y)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{16}\rho\left(h(x+y)-\frac{f\left(2^n(x+y)\right)}{4^n}\right)+\frac{1}{16}\rho\left(h(x-y)-\frac{f\left(2^n(x-y)\right)}{4^n}\right)\\
&\quad+\frac{1}{8}\rho\left(f(x)-\frac{f(2^nx)}{4^n}\right)+\frac{1}{8}\rho\left(f(y)-\frac{f\left(2^ny\right)}{4^n}\right)\\
&\quad+\frac{1}{16}\rho\left(\frac{f\left(\frac{2^n(x+y)}{2}\right)}{4^n}-h\left(\frac{x+y}{2}\right)\right)+\frac{1}{16}\rho\left(\frac{f\left(\frac{2^n(x-y)}{2}\right)}{4^n}-h\left(\frac{x-y}{2}\right)\right)\\
&\quad+\frac{1}{32}\rho\left(\frac{f\left(2^nx\right)}{4^n}-h\left(x\right)\right)+\frac{1}{32}\rho\left(\frac{f\left(2^ny\right)}{4^n}-h\left(y\right)\right)\\
&\quad+\frac{1}{16}\rho\left(h\left(\frac{(x+y)}{2}\right)+h\left(\frac{x-y}{2}\right)-\frac{1}{2}h(x)-\frac{1}{2}h\left(y\right)\right)+\frac{1}{4^{n+2}}\varphi(2^nx,2^ny).
\end{aligned}
$$

Letting $n\to\infty$, we get

$$
\begin{aligned}
&\rho\left(\frac{1}{16}h\left((x+y)\right)+\frac{1}{16}h\left((x-y)\right)-\frac{1}{8}h(x)-\frac{1}{8}h\left(y\right)\right)\\
&\qquad\le\frac{1}{16}\rho\left(h\left(\frac{x+y}{2}\right)+h\left(\frac{x-y}{2}\right)-\frac{1}{2}h\left(x\right)-\frac{1}{2}h\left(y\right)\right)
\end{aligned}
$$

and we have

$$
\begin{aligned}
&\rho(h(x+y)+h(x-y)-2h(x)-2h(y))\\
&\qquad\le 16\rho\left(\frac{1}{16}h(x+y)+\frac{1}{16}h(x-y)-\frac{1}{8}h(x)-\frac{1}{8}h(y)\right)\\
&\qquad\le\rho\left(h\left(\frac{x+y}{2}\right)+h\left(\frac{x-y}{2}\right)-\frac{1}{2}h(x)-\frac{1}{2}h(y)\right).
\end{aligned}
$$

Then by Lemma 2.1, h is quadratic. Now, we see that

$$
\begin{aligned}
\rho\left(\frac{h(2x)-4h(x)}{4^2}\right)&=\rho\left(\frac{1}{4^2}\left(h(2x)-\frac{f\left(2^{n+1}x\right)}{4^n}\right)\right)+\frac{1}{4}\left(\frac{f\left(2^{n+1}x\right)}{4^{n+1}}-h(x)\right)\\
&\le\frac{1}{4^2}\rho\left(h(2x)-\frac{f\left(2^{n+1}x\right)}{4^n}\right)+\frac{1}{4}\rho\left(\frac{f\left(2^{n+1}x\right)}{4^{n+1}}-h(x)\right)
\end{aligned}
\tag{2.8}
$$

for all $x\in V$. By definition of h, the right hand side of (2.8) tends to 0 as $n\to\infty$. Therefore, it follows that $h(2x)=4h(x)$.

Finally, to show the uniqueness of h, assume that h_1 and h_2 are quadratic mappings satisfying (2.4). Then, we write:

$$\begin{aligned}
\rho\left(\frac{h_1(x)-h_2(x)}{2}\right) &= \rho\Bigg(\frac{1}{2}\left(\frac{h_1\left(2^k x\right)}{4^k}-\frac{f\left(2^k x\right)}{4^k}\right)+\frac{1}{2}\left(\frac{f\left(2^k x\right)}{4^k}-\frac{h_2\left(2^k x\right)}{4^k}\right)\Bigg) \\
&\leq \frac{1}{2}\rho\left(\frac{h_1\left(2^k x\right)}{4^k}-\frac{f(2^k x)}{4^k}\right)+\frac{1}{2}\rho\left(\frac{f\left(2x^k\right)}{4^k}-\frac{h_2(2^k x)}{4^k}\right) \\
&\leq \frac{1}{2}\frac{1}{4^k}\left\{\rho\left(h_1\left(2^k x\right)-f\left(2^k x\right)\right)+\rho\left(h_2\left(2^k x\right)-f\left(2^k x\right)\right)\right\} \\
&\leq \frac{1}{4^k}\psi\left(2^k x, 2^k x\right) \longrightarrow 0 \text{ as } k\to\infty.
\end{aligned}$$

This implies that $h_1 = h_2$. □

Corollary 2.3. *Let V be a normed linear space, ρ be a convex modular and X_ρ be a ρ-complete modular space satisfying Δ_2-condition with $\tau = 2$. Let $\theta > 0$ and $0 < p < 2$ real numbers. Assume that $f : V \to X_\rho$ is a mapping satisfying*

$$\begin{aligned}
&\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \\
&\leq \rho\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right)+\theta\left(\|x\|^p+\|y\|^p\right) \quad (2.9)
\end{aligned}$$

for all $x, y \in V$. Then there exists a unique quadratic mapping $T : V \to X_\rho$ such that:

$$\rho(f(x)-T(x)) \leq \frac{2\theta\|x\|^p}{4-2^p}.$$

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in V$. □

Corollary 2.4. *Let V be a linear space, ρ be a convex modular and X_ρ be a ρ-complete modular space. Assume $f : V \to X_\rho$ is a mapping such that $f(0) = 0$ and*

$$\begin{aligned}
&\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \\
&\leq \rho\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right)+\epsilon
\end{aligned}$$

for all $x, y \in V$. Then there exists a unique quadratic mapping $T : V \to X_\rho$ such that

$$\rho(f(x)-T(x)) \leq \frac{\epsilon}{3}.$$

Proof. The proof is a result of Theorem 2.2 by putting $\varphi = \epsilon > 0$. □

3. Stability of (2.1) in β-homogeneous spaces.

In 2016, Park [7] proved the generalized Hyer-Ulam-Rassias stability of additive ρ-functional inequalities in β-homogeneous complex Banach space. In this section, we prove the generalized Hyers-Ulam stability of (3.2) from linear space to β-homogeneous Banach complex space, using the Gavruta control.

Theorem 3.1. *Let V be a linear space, X be a β-homogenerous complex Banach space $(0 < \beta \leq 1)$ and $\varphi : V^2 \to [0, \infty)$ be a function with*

$$\psi(x, y) = \frac{1}{4^\beta} \sum_{j=1}^{\infty} \frac{1}{4^{\beta(j-1)}} \varphi\left(2^{j-1}x, 2^{j-1}y\right) < \infty \tag{3.1}$$

for all $x, y \in V$. Assume that $f : V \to X$ is a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| + \varphi(x, y) \end{aligned} \tag{3.2}$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leq \psi(x, x). \tag{3.3}$$

Proof. Putting $y = x$ in (3.2), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x)$$

and so

$$\left\| \frac{1}{4}f(2x) - f(x) \right\| \leq \frac{1}{4^\beta}\varphi(x, x).$$

By induction on $k \in \mathbb{N}$, it is easy to see that

$$\left\| \frac{f\left(2^k x\right)}{4^k} - f(x) \right\| \leq \frac{1}{4^\beta} \sum_{j=1}^{k} \frac{1}{4^{(j-1)\beta}} \varphi\left(2^{j-1}x, 2^{j-1}x\right) \tag{3.4}$$

for all $k \in \mathbb{N}$. Let m and n be nonnegative integers with $n > m$.
Then by (3.4), we have

$$\begin{aligned}\left\|\frac{f\left(2^n x\right)}{4^n}-\frac{f\left(2^m x\right)}{4^m}\right\| &= \left\|\frac{1}{4^m}\left(\frac{f\left(2^{n-m}\cdot 2^m x\right)}{4^{n-m}}-f\left(2^m x\right)\right)\right\| \\ &\leq \frac{1}{4^{m\beta}}\frac{1}{4^\beta}\sum_{j=1}^{n-m}\frac{1}{4^{(j-1)\beta}}\varphi\left(2^{j+m-1}x,2^{j+m-1}x\right) \\ &= \frac{1}{4^\beta}\sum_{j=1}^{n-m}\frac{1}{4^{(j+m-1)\beta}}\varphi\left(2^{j+m-1}x,2^{j+m-1}x\right) \\ &= \frac{1}{4^\beta}\sum_{k=m+1}^{n}\frac{1}{4^{(k-1)\beta}}\varphi\left(2^{k-1}x,2^{k-1}x\right). \qquad (3.5)\end{aligned}$$

Since the last expression of (3.5) goes to 0 by (3.1), it follows that, for every $x \in V$, the sequence $\left\{\frac{f(2^k x)}{4^k}\right\}$ is a Cauchy sequence in X. Since X is complete, we know that the sequence is convergent. Hence, there exists a mapping $T : V \to X$ defined by

$$T(x) = \lim_{n\to\infty}\frac{f(2^n x)}{4^n}, \quad x \in V.$$

Letting $m = 0$ and passing the limit $n \to \infty$ in (3.5), we obtain the estimate (3.3).

In order to show that T is quadratique, we write

$$\begin{aligned}&\left\|\frac{f\left(2^n(x+y)\right)}{4^n}+\frac{f\left(2^n(x-y)\right)}{4^n}-2\frac{f\left(2^n x\right)}{4^n}-2\frac{f\left(2^n y\right)}{4^n}\right\| \\ &= \frac{1}{4^{n\beta}}\left\|f\left(2^n(x+y)\right)+f\left(2^n(x-y)\right)-2f\left(2^n x\right)-2f\left(2^n y\right)\right\| \\ &\leq \frac{1}{4^n\beta}\left\|f\left(2^n\left(\frac{x+y}{2}\right)\right)+f\left(2^n\left(\frac{x-y}{2}\right)\right)+\frac{1}{2}f\left(2^n x\right)+\frac{1}{2}f\left(2^n y\right)\right\| \\ &\quad+\frac{1}{4^{\beta n}}\varphi\left(2^n x,2^n y\right) \\ &= \left\|\frac{f\left(2^n\left(\frac{x+y}{2}\right)\right)}{4^n}+\frac{f\left(2^n\left(\frac{x-y}{2}\right)\right)}{4^n}-\frac{1}{2}\frac{f(2^n x)}{4^n}-\frac{1}{2}\frac{f(2^n y)}{4^n}\right\|+\frac{1}{4^{\beta n}}\varphi\left(2^n x,2^n y\right).\end{aligned}$$

Hence

$$\begin{aligned}&\|T(x+y)+T(x-y)-2T(x)-2T(y)\| \\ &\leq \left\|T\left(\frac{x+y}{2}\right)+T\left(\frac{x-y}{2}\right)-\frac{1}{2}T(x)-\frac{1}{2}T(y)\right\|\end{aligned}$$

for all $x, y \in V$. Then by Lemma [7, Lemma 2.1], it follows that T is quadratic.

Next, assume that $S : V \to X$ is another quadratic mapping satisfying (3.3). Then, we have

$$\begin{aligned} \|T(x) - S(x)\| &\leq \left\| \frac{T\left(2^k x\right) - f\left(2^k x\right)}{4^k} \right\| + \left\| \frac{S(2^k x) - f(2^k x)}{4^k} \right\| \\ &\leq \frac{2}{4^{k\beta}} \cdot \frac{1}{4^\beta} \sum_{j=1}^{\infty} \frac{1}{4^{\beta(j-1)}} \varphi\left(2^{k+j-1}x, 2^{k+j-1}x\right) \\ &= \frac{2}{4^\beta} \cdot \sum_{l=k+1}^{\infty} \frac{2}{4^{\beta(l-1)}} \varphi\left(2^{l-1}x, 2^{l-1}x\right) \\ &\longrightarrow 0 \quad \text{as} \quad k \to \infty, \end{aligned}$$

for all $x \in V$, from which it follows that $T = S$. □

Letting $\varphi = \epsilon > 0$ in Theorem 3.1, we obtain a result on classical Ulam stability of the quadratic functional inequality.

Corollary 3.2. *Let V ba a linear space, and X be a β-homogeneous complex Banach space with $0 < \beta \leq 1$. If $f : V \to X$ is a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| + \varepsilon \end{aligned}$$

for all $x, y \in V$, then there exists a unique quadratic mapping $T : V \to X$ such that

$$\|f(x) - T(x)\| \leq \frac{\epsilon}{4^\beta - 1}, \quad \forall\, x \in V.$$

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2**(1–2) (1950), 64–66.

[2] P.W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math., **27**(1) (1984), 76–86.

[3] D.H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.

[4] M.A. Khamsi, *Quasicontraction mappings in modular spaces without Δ_2-condition,* Fixed Point Theory Appl., **2008** (2008).

[5] J. Musielak and W. Orlicz, *On modular spaces,* Stud. Math., **18** (1959), 591–597.

[6] H. Nakano, *Modular semi-ordered spaces*, Tokyo, Japan, 1959.

[7] C. Park, S.O. Kim, J.R. Lee and Y. Shin, *Quadratic ρ-functional inequalities in β-homogeneous normed spaces,* Int. J. Nonlinear Anal. Appl., **6**(2) (2015), 21–26.

[8] T.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., **72**(2) (1978), 297–300.
[9] G. Sadeghi, *A fixed point approach to stability of functional equations in modular spaces*, Bull. Malay. Math. Sci. Soc., **37**(2) (2014), 333–344.
[10] F. Skof, *Proprieta locali e approssimazione di operatori,* Rend. del Semin. Mat. e Fis. di Milano, **53**(1) (1983), 113–129.
[11] S.M. Ulam, *Problems in Modern Mathematics*, Probl. Mod. Math., 1964.