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PROXIMAL TYPE CONVERGENCE RESULTS USING IMPLICIT RELATION AND APPLICATIONS

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Abstract. The goal of this study is to instigate various new and novel optimum proximity point theorems using the notion of implicit relation type $\beth$-proximal contraction for non-self mappings. An illustrated example is used to demonstrate the validity of the obtained results. Furthermore, some uniqueness results for proximal contractions are also furnished with partial order and graph. Various well-known discoveries in the present state-of-the-art are enhanced, extended, unified, and generalized by our findings. As an application, we generate some fixed point results fulfilling a modified contraction and a graph contraction, using the profundity of the established results.

1. Introduction and preliminaries

The use of fixed point equations in the form $\daleth \breve{a} = \breve{a}$ can simulate various topics in pure and applied mathematics, including difference and differential

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equations, discrete and continuous dynamic systems, and variational analysis. Consequently, fixed point theory is highly valuable in the fields of science and engineering, particularly for solving the aforementioned equation and optimizing structures, as proved by [1, 3, 5, 14, 23, 24].

If the operator $\daleth$ is not a self-map, it is unlikely that the equation $\daleth\breve{a} = \breve{a}$ has a solution because it requires an element in the domain and an element in the co-domain of the mapping to be equal. In such cases, it is beneficial to find an approximate value that minimizes the approximation error.

In other words, when $\daleth : A \to B$ is a nonself-mapping in the context of a metric space, the goal is to find an approximate solution $\breve{a}$ with the smallest error $\rho(\breve{a}, \daleth\breve{a})$. In situations where $\daleth$, is expected to be a nonself-mapping, the quantity $\min_{\breve{a}\in A} \rho(\breve{a}, \daleth\breve{a})$ represents the ideal optimal approximate solution to the equation $\daleth\breve{a} = \breve{a}$, which is unlikely to have a solution. If a solution $\breve{a}$ to the above non-linear programming problem satisfies $\rho(\breve{a}, \daleth\breve{a}) = \rho(A, B)$for every $\breve{a}$, it becomes a suitable estimation with the lowest feasible error for the associated equation $\daleth\breve{a} = \breve{a}$. This solution $\breve{a}$ is referred to as the mapping's best proximity point, and the research that investigates the possibility of best proximity points for non-self mappings is known as best proximity results. Notable papers such as [7, 8, 9, 11, 17, 18, 19, 21] offer a detailed discussion on best proximity point results.

On the flip side, Popa [15, 16] and Ali and Imdad [2] brought together various fixed point results by introducing generic contractive conditions through an implicit connection. Synthesis based on fixed point, common fixed point, and best proximity point using Popa's technique can be found in [4, 12, 14, 20] and related references. Additionally, Berinde established some fruitful fixed point theorems for almost contractions that preserve an implicit relation in [21]. Drawing inspiration from the aforementioned works, we first introduce the concept of implicit relation type $\beth$-proximal contraction and then present some novel best proximity point results based on implicit relation type $\beth$-proximal contraction in this paper. We provide a suitable example to validate our findings, which combine, expand, and broaden the previous multitude of results in the current state-of-the-art (see [3, 8, 13]).

The structure of this paper is as follows. In Section 1, we provide the necessary background concepts to support our work. In Section 2, we present some new results regarding implicit relation type $\beth$-proximal contractions. We support our findings with a suitable illustrative example. Section 3 is dedicated to presenting modified results in metric spaces with a graph. In Section 4, we showcase results in metric spaces that are equipped with a partial order.

Section 5 includes an application based on implicit relation type modified $\beth$-contraction. Finally, in Section 6, we summarize our article with a conclusion that outlines the proposed work's framework.

2. Main results

Let Θ be the set of non-decreasing functions $\theta : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \theta^n(t) < +\infty$ for all $t > 0$, where θ^n is the n-th iterate of θ.

Let $\mathcal{F}$ be the set of all continuous functions $\pounds : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following assertions:

(F1) if $\pounds(\partial, \eth, \eth, \partial, \partial + \eth, 0) \leq 0$ where $\partial, \eth > 0$, then $\partial \leq \theta(\eth)$;
(F2) $\pounds$ is decreasing in τ_5;
(F3) if $\pounds(\partial, \eth, 0, \partial + \eth, \partial, \eth) \leq 0$ where $\partial, \eth \geq 0$, then $\partial \leq \theta(\eth)$;
(F4) $\pounds(\partial, \partial, 0, 0, \partial, \partial) > 0$ for all $\partial > 0$.

Example 2.1. *Let*

$$\pounds(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - \theta\left(\max\left\{\tau_2, \tau_3, \tau_4, \frac{\tau_5 + \tau_6}{2}\right\}\right),$$

where $\theta \in \Theta$. Then $\pounds \in \mathcal{F}$.

Example 2.2. *Let*

$$\pounds(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - a\tau_2 - \frac{b[1 + \tau_3]\tau_4}{1 + \tau_2} - c[\tau_3 + \tau_4] - d[\tau_5 + \tau_6],$$

where $a + b + 2c + 2d < 1$. Then $\pounds \in \mathcal{F}$.

Definition 2.3. Let two nonempty subsets A and B of a metric space $(\mathfrak{X}, \rho)$ and $\beth : A \times A \to [0, \infty)$ be a function. Then implicit relation type $\beth$-proximal contraction define as $\daleth : A \to B$, if for all $\breve{a}, \tilde{a}, \partial, \eth \in A$,

$$\begin{cases} \beth(\breve{a}, \tilde{a}) \geq 1, \\ \rho(\partial, \daleth\breve{a}) = \rho(A, B), \\ \rho(\eth, \daleth\tilde{a}) = \rho(A, B) \end{cases}$$

then

$$\pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\breve{a}, \eth), \rho(\tilde{a}, \partial)\big) \leq 0, \tag{2.1}$$

where $\pounds \in \mathcal{F}$.

Definition 2.4. Let $(\mathfrak{X}, \rho)$ be a metric space and two nonempty subsets A and B of $\mathfrak{X}$. Then B is said to be approximatively compact with respect to A if every sequence $\{\tilde{a}_n\}$ in B, satisfying the condition $\rho(\breve{a}, \tilde{a}_n) \to \rho(\breve{a}, B)$ for some $\breve{a}$ in A, has a convergent subsequence.

Theorem 2.5. *Let two nonempty subsets A, B of a metric space $(\mathfrak{X}, \rho)$ such that A is complete and A_0 is nonempty. Let continuous implicit relation type $\beth$-proximal contraction as $\daleth : A \to B$ such that the following conditions hold:*

(i) *$\daleth$ is a an $\beth$-proximal admissible mapping and $\daleth(A_0) \subseteq B_0$,*
(ii) *there exist $\breve{a}_0, \breve{a}_1 \in A_0$ such that*

$$\rho(\breve{a}_1, \daleth\breve{a}_0) = \rho(A, B) \quad \text{and} \quad \beth(\breve{a}_0, \breve{a}_1) \geq 1.$$

Then $\daleth$ has a best proximity point. Further, the best proximity point is unique if,

(iii) *for every $\breve{a}, \tilde{a} \in A$ with $\rho(\breve{a}, \daleth\breve{a}) = \rho(A, B) = \rho(\tilde{a}, \daleth\tilde{a})$, we have $\beth(\breve{a}, \tilde{a}) \geq 1$.*

Proof. By (ii) there exist $\breve{a}_0, \breve{a}_1 \in A_0$ such that

$$\rho(\breve{a}_1, \daleth\breve{a}_0) = \rho(A, B) \quad \text{and} \quad \beth(\breve{a}_0, \breve{a}_1) \geq 1.$$

On the other hand, $\daleth(A_0) \subseteq B_0$, then there exists $\breve{a}_2 \in A_0$ such that

$$\rho(\breve{a}_2, \daleth\breve{a}_1) = \rho(A, B).$$

Now, since $\daleth$ is $\beth$-proximal admissible, we have $\beth(\breve{a}_1, \breve{a}_2) \geq 1$. Thus

$$\rho(\breve{a}_2, \daleth\breve{a}_1) = \rho(A, B) \quad \text{and} \quad \beth(\breve{a}_1, \breve{a}_2) \geq 1.$$

Since $\daleth(A_0) \subseteq B_0$, there exists $\breve{a}_3 \in A_0$ such that

$$\rho(\breve{a}_3, \daleth\breve{a}_2) = \rho(A, B).$$

Then, we have

$$\rho(\breve{a}_2, \daleth\breve{a}_1) = \rho(A, B), \quad \rho(\breve{a}_3, \daleth\breve{a}_2) = \rho(A, B), \quad \beth(\breve{a}_1, \breve{a}_2) \geq 1.$$

Again, since $\daleth$ is $\beth$-proximal admissible, we obtain that $\beth(\breve{a}_2, \breve{a}_3) \geq 1$ and hence

$$\rho(\breve{a}_3, \daleth\breve{a}_2) = \rho(A, B), \quad \beth(\breve{a}_2, \breve{a}_3) \geq 1.$$

By continuing this process, we construct a sequence $\{\breve{a}_n\}$ such that

$$\begin{cases} \beth(\breve{a}_{n-1}, \breve{a}_n) \geq 1, \\ \rho(\breve{a}_{n+1}, \daleth\breve{a}_n) = \rho(A, B), \\ \rho(\breve{a}_n, \daleth\breve{a}_{n-1}) = \rho(A, B) \end{cases} \tag{2.2}$$

for all $n \in \mathbb{N}$. Now, from (2.1) with $\partial = \breve{a}_n$, $\eth = \breve{a}_{n+1}$, $\breve{a} = \breve{a}_{n-1}$ and $\tilde{a} = \breve{a}_n$, we get,

$$\pounds\Big(\rho(\breve{a}_n, \breve{a}_{n+1}), \rho(\breve{a}_{n-1}, \breve{a}_n), \rho(\breve{a}_{n-1}, \breve{a}_n), \rho(\breve{a}_n, \breve{a}_{n+1}), \rho(\breve{a}_{n-1}, \breve{a}_{n+1}), \rho(\breve{a}_n, \breve{a}_n)\Big) \leq 0.$$

Now, since $\pounds$ is decreasing in τ_5 then,

$$\pounds\Big(\rho(\breve{a}_n,\breve{a}_{n+1}),\rho(\breve{a}_{n-1},\breve{a}_n),\rho(\breve{a}_{n-1},\breve{a}_n),\rho(\breve{a}_n,\breve{a}_{n+1}),\\ \rho(\breve{a}_n,\breve{a}_{n+1})+\rho(\breve{a}_{n-1},\breve{a}_n),0\Big)\leq 0$$

and so from $(F1)$ we get,

$$\rho(\breve{a}_n,\breve{a}_{n+1})\leq\theta(\rho(\breve{a}_{n-1},\breve{a}_n)).$$

By induction, we have

$$\rho(\breve{a}_n,\breve{a}_{n+1})\leq\theta^n(\rho(\breve{a}_0,\breve{a}_1)).$$

Fix $\epsilon>0$, there exists $N\in\mathbb{N}$ such that

$$\sum_{n\geq N}\theta^n(\rho(\breve{a}_0,\breve{a}_1))<\epsilon,\quad\forall n\in\mathbb{N}.$$

Let $m,n\in\mathbb{N}$ with $m>n\geq N$. Then by triangular inequality we get

$$\rho(\breve{a}_n,\breve{a}_m)\leq\sum_{k=n}^{m-1}\rho(\breve{a}_k,\breve{a}_{k+1})\leq\sum_{n\geq N}\theta^n(\rho(\breve{a}_0,\breve{a}_1))<\epsilon.$$

Consequently $\lim_{m,n,\to+\infty}\rho(\breve{a}_n,\breve{a}_m)=0$. Hence $\{\breve{a}_n\}$ is a Cauchy sequence. Since A is complete, so there is $\hat{a}\in A$ such that $\breve{a}_n\to\hat{a}$. Since $\urcorner$ is continuous, so $\urcorner\breve{a}_n\to\urcorner\hat{a}$ as $n\to\infty$. Hence,

$$\rho(A,B)=\lim_{n\to\infty}\rho(\breve{a}_{n+1},\urcorner\breve{a}_n)=\rho(\hat{a},\urcorner\hat{a}).$$

Thus $\hat{a}$ is desired best proximity point of $\urcorner$.

Let $\breve{a},\tilde{a}\in A$ be two best proximity point of $\urcorner$ such that $\breve{a}\neq\tilde{a}$. That is, $\rho(\breve{a},\urcorner\breve{a})=\rho(A,B)=\rho(\tilde{a},\urcorner\tilde{a})$. From (iii) we get, $\sqsupset(\breve{a},\tilde{a})\geq 1$. So, by (2.1) we derive,

$$\pounds\big(\rho(\breve{a},\tilde{a}),\rho(\breve{a},\tilde{a}),\rho(\breve{a},\breve{a}),\rho(\tilde{a},\tilde{a}),\rho(\tilde{a},\breve{a}),\rho(\breve{a},\tilde{a})\big)\leq 0,$$

which implies

$$\pounds\big(\rho(\breve{a},\tilde{a}),\rho(\breve{a},\tilde{a}),0,0,\rho(\breve{a},\tilde{a}),\rho(\breve{a},\tilde{a})\big)\leq 0,$$

which is a contradiction to (F4). Hence, $\urcorner$ has a unique best proximity point. □

Example 2.6. Let $\mathfrak{X} = \{0, 1, 2, 3, 4, 5\}$ be endowed with the function ρ given as:

ρ	0	1	2	3	4	5
0	0	1.2	1.1	1	1.3	1.4
1	1.2	0	2	1.5	1	1.6
2	1.1	2	0	1.7	1.8	1
3	1	1.5	1.7	0	1.9	2
4	1.3	1	1.8	1.9	0	1.5
5	1.4	1.6	1	2	1.5	0

It is easy to see that $(\mathfrak{X}, \rho)$ is metric space. Suppose $A = \{0, 1, 2\}$ and $B = \{3, 4, 5\}$. After a simple calculation, $\rho(A, B) = 1$ and $A_0 = A$ and $B_0 = B$. Now, define a mapping $\daleth : A \to B$ by the following

$$\daleth\breve{a} = \begin{cases} 3, & \text{if } \breve{a} = \{0, 2\}, \\ 5, & \text{if } \breve{a} = 1. \end{cases}$$

Clearly, $\daleth(A_0) \subseteq B_0$. Now, we have to show that $\daleth$ satisfies an implicit relation type $\beth$-proximal contraction

$$\pounds(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\breve{a}, \eth), \rho(\tilde{a}, \partial)) \leq 0$$

for all $\breve{a}, \tilde{a}, \partial, \eth \in A$ and for the function $\beth : A \times A \to [0, +\infty)$ defined by

$$\beth(\breve{a}, \tilde{a}) = \rho(\breve{a}, \tilde{a}) + 1.$$

Since

$$\rho(0, \daleth 2) = \rho(0, 3) = \rho(A, B),$$
$$\rho(2, \daleth 1) = \rho(2, 5) = \rho(A, B).$$

After routine calculations, $\beth(\breve{a}, \tilde{a}) = \rho(2, 1) + 1 = 3$.
Suppose

$$\pounds(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - a\tau_2 - \frac{b[1 + \tau_3]\tau_4}{1 + \tau_2} - c[\tau_3 + \tau_4] - e[\tau_5 + \tau_6],$$

where $a + b + 2c + 2e < 1$. Then,

$$\pounds(1.1, 2, 1.1, 2, 0, 1.2) = 1.1 - 2a - 1.4b - 3.1c - 1.2e.$$

After choosing $a = \frac{1}{2}$, $b = \frac{1}{7}$, $c = \frac{1}{11}$ and $e = \frac{1}{13}$, where $a + b + 2c + 2e = 0.9785 < 1$ and

$$\pounds(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = -0.4741 \leq 0.$$

Hence, 0 is the unique best proximity point of the mapping $\daleth$.

Theorem 2.7. *Let two nonempty subsets A and B of a metric space $(\mathfrak{X}, \rho)$ such that A is complete, B is approximatively compact with respect to A and A_0 is nonempty. Assume that an implicit relation type $\beth$-proximal contraction define as $\daleth : A \to B$ such that the following conditions hold:*

(i) $\urcorner$ *is an* $\sqsupset$*-proximal admissible mapping and* $\urcorner(A_0) \subseteq B_0$,

(ii) *there exist* $\breve{a}_0, \breve{a}_1 \in A_0$ *such that*

$$\rho(\breve{a}_1, \urcorner\breve{a}_0) = \rho(A, B) \quad \textit{and} \quad \sqsupset(\breve{a}_0, \breve{a}_1) \geq 1,$$

(iii) *if* $\{\breve{a}_n\}$ *is a sequence in* $\mathfrak{X}$ *such that* $\sqsupset(\breve{a}_n, \breve{a}_{n+1}) \geq 1$ *with* $\breve{a}_n \to \breve{a}$ *as* $n \to \infty$*, then* $\sqsupset(\breve{a}_n, \breve{a}) \geq 1$.

Then $\urcorner$ *has a best proximity point. Further, the best proximity point is unique if,*

(iv) *for every* $\breve{a}, \tilde{a} \in A$ *where* $\rho(\breve{a}, \urcorner\breve{a}) = \rho(A, B) = \rho(\tilde{a}, \urcorner\tilde{a})$*, we have* $\sqsupset(\breve{a}, \tilde{a}) \geq 1$.

Proof. Following the proof of Theorem 2.5, there exist a Cauchy sequence $\{\breve{a}_n\} \subseteq A$ and $\hat{a} \in A$ such that (2.1) holds and $\breve{a}_n \to \hat{a}$ as $n \to +\infty$. On the other hand, for all $n \in \mathbb{N}$, we can write

$$\begin{aligned} \rho(\hat{a}, B) &\leq \rho(\hat{a}, \urcorner\breve{a}_n) \\ &\leq \rho(\hat{a}, \breve{a}_{n+1}) + \rho(\breve{a}_{n+1}, \urcorner\breve{a}_n) \\ &= \rho(\hat{a}, \breve{a}_{n+1}) + \rho(A, B). \end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality, we get

$$\lim_{n \to +\infty} \rho(\hat{a}, \urcorner\breve{a}_n) = \rho(\hat{a}, B) = \rho(A, B). \tag{2.3}$$

Since, B is approximatively compact with respect to A, the sequence, $\{\urcorner\breve{a}_n\}$ has a subsequence $\{\urcorner\breve{a}_{n_k}\}$ that converges to some $\tilde{a}^* \in B$. Hence,

$$\rho(\hat{a}, \tilde{a}^*) = \lim_{n \to \infty} \rho(\breve{a}_{n_k+1}, \urcorner\breve{a}_{n_k}) = \rho(A, B)$$

and so $\hat{a} \in A_0$. Now, since $\urcorner(A_0) \subseteq B_0$ so, $\rho(w, \urcorner\hat{a}) = \rho(A, B)$ for some $w \in A$. Now, by (iii) and (2.2), we have $\sqsupset(\breve{a}_n, \hat{a}) \geq 1$ and $\rho(\breve{a}_{n+1}, \urcorner\breve{a}_n) = \rho(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$. Also, since $\urcorner$ is an implicit relation type $\sqsupset$-proximal contraction, we get,

$$\pounds\big(\rho(\breve{a}_{n+1}, w), \rho(\breve{a}_n, \hat{a}), \rho(\breve{a}_n, \breve{a}_{n+1}), \rho(\hat{a}, w), \rho(\breve{a}_n, w), \rho(\hat{a}, \breve{a}_{n+1})\big) \leq 0.$$

Taking the limit as $n \to +\infty$ in the above inequality and applying continuity of $\pounds$, we have

$$\pounds\big(\rho(\hat{a}, w), 0, 0, \rho(\hat{a}, w), \rho(\hat{a}, w), 0\big) \leq 0.$$

Now if we take $\partial = \rho(\hat{a}, w)$ and $\eth = 0$, then we have

$$\pounds\big(\partial, \eth, 0, \partial + \eth, \partial, \eth\big) \leq 0$$

and so, from (F3) we get, $\partial \leq \theta(\eth)$. That is $\rho(\hat{a}, w) \leq \theta(0) = 0$. Thus, $\hat{a} = \eth$. Hence $\hat{a}$ is a best proximity point of $\urcorner$. Uniqueness follows similarly as in the proof of Theorem 2.5. □

By using Example 2.2 and Theorem 2.7 we obtain following main result of Hussain et al. [8] for $\wp(\breve{a}, \tilde{a}) = 1$.

Corollary 2.8. ([8, Theorem 11]) *Let A and B be nonempty closed subsets of a complete metric space $(\mathfrak{X}, \rho)$ such that B is approximatively compact with respect to A. Assume $\beth : A \times A \to [0, \infty)$, A_0 and B_0 are nonempty and $\daleth : A \to B$ is an $\beth$-rational proximal contraction of the first kind satisfying the following assertions:*

(i) $\daleth(A_0) \subseteq B_0$;
(ii) $\daleth$ *is* $\beth$*-proximal admissible;*
(iii) *There exist elements $\breve{a}_0$ and $\breve{a}_1$ in A_0 such that,*

$$\rho(\breve{a}_1, \daleth\breve{a}_0) = \rho(A, B) \quad \text{and} \quad \beth(\breve{a}_0, \breve{a}_1) \geq 1;$$

(iv) *If $\{\breve{a}_n\}$ is a sequence in A such that $\beth(\breve{a}_n, \breve{a}_{n+1}) \geq 1$ and $\breve{a}_n \to \breve{a} \in A$ as $n \to \infty$, then $\beth(\breve{a}_n, \breve{a}) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists $\hat{a} \in A_0$ such that

$$\rho(\hat{a}, \daleth\hat{a}) = \rho(A, B).$$

Moreover, if $\beth(\breve{a}, \tilde{a}) \geq 1$ for all $\breve{a}, \tilde{a} \in B_{est}(\daleth)$, then $\hat{a}$ is the unique best proximity point of $\daleth$.

By taking $\beth(\breve{a}, \tilde{a}) = 1$ in Theorem 2.7 we deduce the following corollary.

Corollary 2.9. *Let A and B be two nonempty subsets of a metric space $(\mathfrak{X}, \rho)$ such that A is complete, B is approximatively compact with respect to A and A_0 is nonempty. Assume that $\daleth : A \to B$ is a nonself-mapping such that $\daleth A_0 \subseteq B_0$ and for all $\breve{a}, \tilde{a}, \partial, \eth \in A$,*

$$\begin{cases} \rho(\partial, \daleth\breve{a}) = \rho(A, B), \\ \rho(\eth, \daleth\tilde{a}) = \rho(A, B) \end{cases}$$

then

$$\pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\tilde{a}, \partial), \rho(\breve{a}, \eth)\big) \leq 0,$$

where $\pounds \in \mathcal{F}$. Then $\daleth$ has a unique best proximity point.

By using Example 2.1 and Corollary 2.9 we deduce the following result.

Corollary 2.10. *Let A and B be two nonempty subsets of a metric space $(\mathfrak{X}, \rho)$ such that A is complete, B is approximatively compact with respect to A and A_0 is nonempty. Assume that $\daleth : A \to B$ is a nonself-mapping such that $\daleth A_0 \subseteq B_0$ and for all $\breve{a}, \tilde{a}, \partial, \eth \in A$,*

$$\begin{cases} \rho(\partial, \daleth\breve{a}) = \rho(A, B), \\ \rho(\eth, \daleth\tilde{a}) = \rho(A, B) \end{cases}$$

then

$$\rho(\partial,\eth)\le\theta\bigg(\max\bigg\{\rho(\breve{a},\tilde{a}),\rho(\breve{a},\partial),\rho(\tilde{a},\eth),\frac{\rho(\tilde{a},\partial)+\rho(\breve{a},\eth)}{2}\bigg\}\bigg),$$

where $\theta\in\Theta$. Then $\daleth$ has a unique best proximity point.

By using Example 2.2 and Corollary 2.9 we obtain the following result.

Corollary 2.11. (Main result of [13]) *Let A and B be two nonempty subsets of a metric space $(\mathfrak{X},\rho)$ such that A is complete, B is approximatively compact with respect to A and A_0 is nonempty. Assume that $\daleth: A\to B$ is a nonself-mapping such that $\daleth A_0\subseteq B_0$ and there exists nonnegative real numbers a,b,c,d with $a+b+2c+2d<1$ such that for all $\breve{a},\tilde{a},\partial,\eth\in A$,*

$$\begin{cases}\rho(\partial,\daleth\breve{a})=\rho(A,B)\\ \rho(\eth,\daleth\tilde{a})=\rho(A,B)\end{cases}\implies\quad\begin{aligned}\rho(\partial,\eth)\ &\le a\rho(\breve{a},\tilde{a})+b\frac{[1+\rho(\breve{a},\partial)]\rho(\tilde{a},\eth)}{1+\rho(\breve{a},\tilde{a})}\\ &\quad+c[\rho(\breve{a},\partial)+\rho(\tilde{a},\eth)]\\ &\quad+\rho[\rho(\breve{a},\eth)+\rho(\tilde{a},\partial)].\end{aligned}$$

Then $\daleth$ has a unique best proximity point.

3. Some results in metric spaces endowed with a graph

Following Jachymski [10], let $(\mathfrak{X},\rho)$ be a metric space and let Δ denote the diagonal of the Cartesian product $\mathfrak{X}\times\mathfrak{X}$. Let G be a directed graph such that the set of its vertices $V(G)$ coincides with $\mathfrak{X}$, and the set of its edges $\bigwedge(\mathcal{G})$ contains all loops, that is, $\bigwedge(\mathcal{G})\supseteq\Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G),\bigwedge(\mathcal{G}))$. Moreover, we may treat G as a weighted graph (see [10]) by assigning to each edge the distance between its vertices. If $\breve{a}$ and $\tilde{a}$ are vertices in a graph G, then a path in G from $\breve{a}$ to $\tilde{a}$ of length N ($N\in\mathbb{N}$) is a sequence $\{\breve{a}_i\}_{i=0}^N$ of $N+1$ vertices such that $\breve{a}_0=\breve{a}$, $\breve{a}_N=\tilde{a}$ and $(\breve{a}_{n-1},\breve{a}_n)\in\bigwedge(\mathcal{G})$ for $i=1,\dots,N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected (see for details [5, 9, 10, 22, 25]).

In 2006, Espinola and Kirk [6] established an important combination of fixed point theory and graph theory.

Definition 3.1. Let A,B be two nonempty closed subsets of a metric space $(\mathfrak{X},\rho)$ equipped with a graph G. An implicit relation type G-proximal contraction $\daleth: A\to B$ is defined as follows: for all $\breve{a},\tilde{a},\partial,\eth\in A$,

$$\begin{cases}(\breve{a},\tilde{a})\in\bigwedge(\mathcal{G})\\ \rho(\partial,\daleth\breve{a})=\rho(A,B)\\ \rho(\eth,\daleth\breve{a})=\rho(A,B)\end{cases}\Rightarrow\quad(\partial,\eth)\in\bigwedge(\mathcal{G})$$

and
$$\begin{cases} (\breve{a}, y) \in \bigwedge(\mathcal{G}) \\ \rho(\partial, \urcorner\breve{a}) = \rho(A, B) \\ \rho(\eth, \urcorner\tilde{a}) = \rho(A, B) \end{cases}$$
then
$$\pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\tilde{a}, \partial), \rho(\breve{a}, \eth)\big) \leq 0,$$
where $\pounds \in \mathcal{F}$.

Theorem 3.2. *Let A, B be two nonempty closed subsets of a metric space $(\mathfrak{X}, \rho)$ endowed with a graph G. Let A be complete, A_0 be nonempty and $\urcorner : A \to B$ be a continuous implicit relation type G-proximal contraction Satisfying the following conditions:*

(i) $\urcorner(A_0) \subseteq B_0$,

(ii) *there exist elements $\breve{a}_0, \breve{a}_1 \in A_0$ such that*
$$\rho(\breve{a}_1, \urcorner\breve{a}_0) = \rho(A, B) \quad \textit{and} \quad (\breve{a}_0, \breve{a}_1) \in \bigwedge(\mathcal{G}).$$

Then $\urcorner$ has a best proximity point. Further, the best proximity point is unique, if for every $\breve{a}, y \in A$ such that $\rho(\breve{a}, \urcorner\breve{a}) = \rho(A, B) = \rho(\tilde{a}, \urcorner\tilde{a})$, we have $(\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G})$.

Proof. Define $\sqsupset : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ by
$$\sqsupset(\breve{a}, \tilde{a}) = \begin{cases} 1, & \text{if } (\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G}), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$
Firstly, we prove that $\urcorner$ is a triangular $\sqsupset$-proximal admissible mapping. To this aim, assume
$$\begin{cases} \sqsupset(\breve{a}, \tilde{a}) \geq 1, \\ \rho(\partial, \urcorner\breve{a}) = \rho(A, B), \\ \rho(\eth, \urcorner\tilde{a}) = \rho(A, B). \end{cases}$$
Therefore, we have
$$\begin{cases} (\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G}), \\ \rho(\partial, \urcorner\breve{a}) = \rho(A, B), \\ \rho(\eth, \urcorner\tilde{a}) = \rho(A, B). \end{cases}$$
Since $\urcorner$ is an implicit relation type G-proximal contraction, we get $(\partial, \eth) \in \bigwedge(\mathcal{G})$, that is $\sqsupset(\partial, v) \geq 1$ and
$$\pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\tilde{a}, \partial), \rho(\breve{a}, \eth)\big) \leq 0.$$
Thus, $\urcorner$ is an $\sqsupset$-proximal admissible mapping with $\urcorner(A_0) \subseteq B_0$ and continuous implicit relation type G-proximal contraction. From (ii) there exist $\breve{a}_0, \breve{a}_1 \in A_0$ such that $\rho(\breve{a}_1, \urcorner\breve{a}_0) = \rho(A, B)$ and $(\breve{a}_0, \breve{a}_1) \in \bigwedge(\mathcal{G})$, that is $\rho(\breve{a}_1, \urcorner\breve{a}_0) = \rho(A, B)$ and $\sqsupset(\breve{a}_0, \breve{a}_1) \geq 1$. Hence, all the conditions of Theorem 2.5 are satisfied and $\urcorner$ has a best fixed point. □

Similarly, by using Theorem 2.7, we can prove the following theorem.

Theorem 3.3. *Let A and B be two nonempty closed subsets of a metric space $(\mathfrak{X}, \rho)$ endowed with a graph G. Assume that A is complete, B is approximatively compact with respect to A and A_0 is nonempty. Also suppose that $\urcorner : A \to B$ is an implicit relation type G-proximal contraction mapping such that the following conditions hold:*

(i) $\urcorner(A_0) \subseteq B_0$,

(ii) *there exist elements $\breve{a}_0, \breve{a}_1 \in A_0$ such that*

$$\rho(\breve{a}_1, \urcorner\breve{a}_0) = \rho(A, B) \quad \textit{and} \quad (\breve{a}_0, \breve{a}_1) \in \bigwedge(\mathcal{G}),$$

(iii) *if $\{\breve{a}_n\}$ is a sequence in $\mathfrak{X}$ such that $(\breve{a}_n, \breve{a}_{n+1}) \in \bigwedge(\mathcal{G})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\breve{a}_n \to \breve{a}$ as $n \to +\infty$, then $(\breve{a}_n, \breve{a}) \in \bigwedge(\mathcal{G})$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then $\urcorner$ has a best proximity point. Further, the best proximity point is unique, if for every $\breve{a}, \tilde{a} \in A$ such that $\rho(\breve{a}, \urcorner\breve{a}) = \rho(A, B) = \rho(\tilde{a}, \urcorner\tilde{a})$, we have $(\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G})$.

4. Some results in metric spaces endowed with a partial order

Theorem 4.1. *Let two nonempty closed subsets A and B of a partially ordered complete metric space $(\mathfrak{X}, \rho, \preceq)$ such that A_0 is nonempty. Assume that $\urcorner : A \to B$ satisfy the following conditions:*

(i) *$\urcorner$ is continuous and proximally ordered-preserving such that $\urcorner(A_0) \subseteq B_0$,*

(ii) *there exist elements $\breve{a}_0, \breve{a}_1 \in A_0$ such that*

$$\rho(\breve{a}_1, \urcorner\breve{a}_0) = \rho(A, B) \quad \textit{and} \quad \breve{a}_0 \preceq \breve{a}_1,$$

(iii) *for all $\breve{a}, \tilde{a}, \partial, \eth \in A$,*

$$\begin{cases} \breve{a} \preceq \tilde{a} \\ \rho(\partial, \urcorner\breve{a}) = \rho(A, B) \\ \rho(\tilde{a}, \urcorner\tilde{a}) = \rho(A, B) \end{cases} \Rightarrow \pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\tilde{a}, \partial), \rho(\breve{a}, \eth)\big) \leq 0.$$

Then $\urcorner$ has a best proximity point.

Proof. Define $\sqsupset : A \times A \to [0, +\infty)$ by

$$\sqsupset(\breve{a}, \tilde{a}) = \begin{cases} 1, & \text{if } \breve{a} \preceq \tilde{a}, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

Firstly, we prove that $\urcorner$ is an $\sqsupset$-proximal admissible mapping. To this aim, assume

$$\begin{cases} \sqsupset(\breve{a}, \tilde{a}) \geq 1, \\ \rho(\partial, \urcorner\breve{a}) = \rho(A, B), \\ \rho(\eth, \urcorner\tilde{a}) = \rho(A, B). \end{cases}$$

Then, we have

$$\begin{cases} \breve{a} \preceq \tilde{a}, \\ \rho(\partial, \daleth\breve{a}) = \rho(A, B), \\ \rho(\eth, \daleth\tilde{a}) = \rho(A, B). \end{cases}$$

Now, since $\daleth$ is proximally ordered-preserving, $\partial \preceq \eth$, that is $\sqsupset(\partial, \eth) \geq 1$. Further, by (ii) we have

$$\rho(\breve{a}_1, \daleth\breve{a}_0) = \rho(A, B) \quad \text{and} \quad \sqsupset(\breve{a}_0, \breve{a}_1) \geq 1.$$

Moreover, from (iii) we get

$$\begin{cases} \sqsupset(\breve{a}, \tilde{a}) \geq 1 \\ \rho(\partial, \daleth\breve{a}) = \rho(A, B) \Rightarrow \pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\tilde{a}, \partial), \rho(\breve{a}, \eth)\big) \leq 0. \\ \rho(\tilde{a}, \daleth\tilde{a}) = \rho(A, B) \end{cases}$$

Thus, all the conditions of Theorem 2.5 hold and $\daleth$ has a best proximity point. □

Theorem 4.2. *Let two nonempty closed subsets A and B of a partially ordered complete metric space $(\mathfrak{X}, \rho, \preceq)$ such that A_0 is nonempty and B is approximatively compact with respect to A. Assume that $\daleth : A \to B$ satisfy the following conditions:*

(i) *$\daleth$ is proximally ordered-preserving such that $\daleth(A_0) \subseteq B_0$,*

(ii) *there exist elements $\breve{a}_0, \breve{a}_1 \in A_0$ such that*

$$\rho(\breve{a}_1, \daleth\breve{a}_0) = \rho(A, B) \quad \textit{and} \quad \breve{a}_0 \preceq \breve{a}_1,$$

(iii) *for all $\breve{a}, \tilde{a}, \partial, \eth \in A$,*

$$\begin{cases} \breve{a} \preceq \tilde{a} \\ \rho(\partial, \daleth\breve{a}) = \rho(A, B) \Rightarrow \pounds\big(\rho(\partial, \eth), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \partial), \rho(\tilde{a}, \eth), \rho(\tilde{a}, \partial), \rho(\breve{a}, \eth)\big) \leq 0. \\ \rho(\tilde{a}, \daleth\tilde{a}) = \rho(A, B) \end{cases}$$

(iv) *if $\{\breve{a}_n\}$ is an increasing sequence in A converging to $\breve{a} \in A$, then $\breve{a}_n \preceq \breve{a}$ for all $n \in \mathbb{N}$.*

Then $\daleth$ has a best proximity point.

5. Application to fixed point theory

5.1. Implicit relation type modified $\sqsupset$-contraction.

Definition 5.1. Let $(\mathfrak{X}, \rho)$ be a metric space and a function defined as $\sqsupset : A \times A \to [0, \infty)$. Then $\daleth : \mathfrak{X} \to \mathfrak{X}$ is called an implicit relation type $\sqsupset$-contraction, if for all $\breve{a}, \tilde{a} \in \mathfrak{X}$ with $\sqsupset(\breve{a}, \tilde{a}) \geq 1$, we have

$$\pounds\big(\rho(\daleth\breve{a}, \daleth\tilde{a}), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \daleth\breve{a}), \rho(\tilde{a}, \daleth\tilde{a}), \rho(\tilde{a}, \daleth\breve{a}), \rho(\breve{a}, \daleth\tilde{a})\big) \leq 0, \qquad (5.1)$$

where $\pounds \in \mathcal{F}$.

Theorem 5.2. *Let $(\mathfrak{X}, \rho)$ be a complete metric space. Assume that continuous self-mapping is $\daleth : \mathfrak{X} \to \mathfrak{X}$ satisfying the following conditions:*

(i) *$\daleth$ is $\beth$-admissible,*
(ii) *there exists $\breve{a}_0$ in $\mathfrak{X}$ such that $\beth(\breve{a}_0, \daleth\breve{a}_0) \geq 1$,*
(iii) *$\daleth$ is an implicit relation type modified $\beth$-contraction.*

Then $\daleth$ has a fixed point.

Theorem 5.3. *Let $(\mathfrak{X}, \rho)$ be a complete metric space. Assume that self-mapping $\daleth : \mathfrak{X} \to \mathfrak{X}$ is satisfied the following conditions:*

(i) *$\daleth$ is $\beth$-admissible,*
(ii) *there exists $\breve{a}_0$ in $\mathfrak{X}$ such that $\beth(\breve{a}_0, \daleth\breve{a}_0) \geq 1$,*
(iii) *$\daleth$ is an implicit relation type modified $\beth$-contraction,*
(iv) *if $\{\breve{a}_n\}$ is a sequence in $\mathfrak{X}$ such that $\beth(\breve{a}_n, \breve{a}_{n+1}) \geq 1$ and $\breve{a}_n \to \breve{a}$ as $n \to +\infty$, then $\beth(\breve{a}_n, \breve{a}) \geq 1$ for all $n \in \mathbb{N}$.*

Then $\daleth$ has a fixed point.

5.2. **Implicit relation type G-contraction.** Recently, many fixed point results have appeared for different kinds of contractive mappings defined on metric space $(\mathfrak{X}, \rho)$ endowed with a graph (see [10]).

Definition 5.4. ([10]) We say that a mapping $\daleth : \mathfrak{X} \to \mathfrak{X}$ is a Banach G-contraction or simply G-contraction, if $\daleth$ preserves edges of G, that is

$$\forall \breve{a}, \tilde{a} \in \mathfrak{X}((\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G}),\ (\daleth(\breve{a}), \daleth(\tilde{a})) \in \bigwedge(\mathcal{G}))$$

and $\daleth$ decreases weights of edges of G in the following way:

$$\exists\, \beth \in (0,1), \forall \breve{a}, \tilde{a} \in \mathfrak{X}((\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G}),\ \rho(\daleth(\breve{a}), \daleth(\tilde{a})) \leq \beth\rho(\breve{a}, \tilde{a})).$$

Definition 5.5. ([10]) A mapping $\daleth : \mathfrak{X} \to \mathfrak{X}$ is called G-continuous, if given $\breve{a} \in \mathfrak{X}$ and sequence $\{\breve{a}_n\}$

$$\breve{a}_n \to \breve{a} \text{ as } n \to \infty \text{ and } (\breve{a}_n, \breve{a}_{n+1}) \in \bigwedge(\mathcal{G}) \text{ for all } n \in \mathbb{N} \text{ imply } \daleth\breve{a}_n \to \daleth\breve{a}.$$

Definition 5.6. Let $(\mathfrak{X}, \rho)$ be a metric space endowed with a graph G. Then $\daleth : \mathfrak{X} \to \mathfrak{X}$ is said to be an implicit relation type G-contraction, if for all $\breve{a}, \tilde{a} \in \mathfrak{X}$,

$$(\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G}) \quad \Rightarrow \quad (\daleth\breve{a}, \daleth\tilde{a}) \in \bigwedge(\mathcal{G})$$

and

$$(\breve{a}, \tilde{a}) \in \bigwedge(\mathcal{G}) \Rightarrow \pounds\big(\rho(\daleth\breve{a}, \daleth\tilde{a}), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \daleth\breve{a}), \rho(\tilde{a}, \daleth\tilde{a}), \rho(\tilde{a}, \daleth\breve{a}), \rho(\breve{a}, \daleth\tilde{a})\big) \leq 0,$$

where $\pounds \in \mathcal{F}$.

Theorem 5.7. *Let $(\mathfrak{X}, \rho)$ be a complete metric space endowed with a graph G. Assume that continuous self-mapping $\urcorner : \mathfrak{X} \to \mathfrak{X}$ satisfies the following conditions:*

(i) *there exists $\breve{a}_0$ in $\mathfrak{X}$ such that $(\breve{a}_0, \urcorner\breve{a}_0) \in \bigwedge(\mathcal{G})$,*

(ii) *$\urcorner$ is an implicit relation type G-contraction.*

Then $\urcorner$ has a fixed point.

Theorem 5.8. *Let $(\mathfrak{X}, \rho)$ be a complete metric space endowed with a graph G. Assume that self-mapping $\urcorner : \mathfrak{X} \to \mathfrak{X}$ satisfies the following conditions:*

(ii) *there exists $\breve{a}_0$ in $\mathfrak{X}$ such that $(\breve{a}_0, \urcorner\breve{a}_0) \in \bigwedge(\mathcal{G})$,*

(iii) *$\urcorner$ is an implicit relation type G-contraction,*

(iv) *if $\{\breve{a}_n\}$ is a sequence in $\mathfrak{X}$ such that $(\breve{a}_n, \breve{a}_{n+1}) \in \bigwedge(\mathcal{G})$ and $\breve{a}_n \to x$ as $n \to +\infty$, then $(\breve{a}_n, \breve{a}) \in \bigwedge(\mathcal{G})$ for all $n \in \mathbb{N}$.*

Then $\urcorner$ has a fixed point.

Theorem 5.9. ([3, Theorem 3.2]) *Let $(\mathfrak{X}, \rho, \preceq)$ be a partially ordered complete metric space. Assume that self-mapping $\urcorner : \mathfrak{X} \to \mathfrak{X}$ satisfying the following conditions:*

(i) *there exists $\breve{a}_0$ in $\mathfrak{X}$ such that $\breve{a}_0 \preceq \urcorner\breve{a}_0$,*

(ii) *for all $\breve{a}, \tilde{a} \in \mathfrak{X}$ with $\breve{a} \preceq \tilde{a}$, we have*

$$\pounds\big(\rho(\urcorner\breve{a}, \urcorner\tilde{a}), \rho(\breve{a}, \tilde{a}), \rho(\breve{a}, \urcorner\breve{a}), \rho(\tilde{a}, \urcorner\tilde{a}), \rho(\tilde{a}, \urcorner\breve{a}), \rho(\breve{a}, \urcorner\tilde{a})\big) \leq 0,$$

where $\pounds \in \mathcal{F}$,

(iii) *either $\urcorner$ is continuous or if $\{\breve{a}_n\}$ is an increasing sequence in $\mathfrak{X}$ such that $\breve{a}_n \to \breve{a}$ as $n \to +\infty$, then $\breve{a}_n \preceq \breve{a}$ for all $n \in \mathbb{N}$.*

Then $\urcorner$ has a fixed point.

6. Conclusion

The best proximity point theorem states that there is a solution $\breve{a}$ which approximates a given point $\urcorner\breve{a}$ in the optimal sense that the error $\rho(\breve{a}, \urcorner\breve{a})$ is minimized globally at $\rho(A, B)$. In this study, we introduced the concept of $\sqsupset-$proximal contraction of implicit relation type for nonself-mappings, to establish new and innovative theorems on best proximity points. Within the framework of metric spaces, we developed some significant results on the uniqueness of proximal point outcomes. Using the approach of implicit relations, we derived a plethora of results that enhance, extend, unify, and generalize various well-known results in the current state-of-the-art. Subsequently, we applied our findings to examine modified results based on modified contractions and graph contractions.

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