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ULAM STABILITIES FOR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In the present paper, we establish Ulam-Hyres and Ulam-Hyers-Rassias stabilities for nonlinear impulsive integro-differential equations with non-local condition in Banach space. The generalization of Grownwall type inequality is used to obtain our results.

1. Introduction

A question raised by Ulam [20] in 1940, is answered by Hyers [7] in case of Banach space. Furthermore, Rassias [19] generalised the concept of Ulam-Hyers stability in 1978. Rassias introduced new function variables. Therefore the new concept of stability named with Ulam-Hyers-Rassias stability. The

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Ulam type stability problems attracted many researchers ([12,14,15,18–23]). Kucche and Shikare [12] studied Ulam type stabilities of the following problem:

$$\begin{cases} x'(t)=Ax(t)+f(t,x_t,\int_0^t g_1(t,s,x_s)ds,\int_0^b g_2(t,s,x_s)ds), t\in(0,b], 0<b<\infty, \\ x(t)=\phi(t), t\in[-r,0]. \end{cases}$$

In [18], Parthasarathy studied the Ulam problem for impulsive differential equation of the type:

$$\begin{cases} x'(t)=f(t,x(t)), \quad t\in J':=J\{t_1,t_2,...,t_m\}, \quad J=[0,T], \\ x(t_k^+)=x(t_k^-)+I_k(x(t_k^-)), \quad k=1,2,...,m. \end{cases}$$

As per best of our knowledge, Ulam type stabilities for impulsive integro-differential equation with nonlocal condition is not investigated yet. In the present paper we consider impulsive integro-differential equation of first order of the type:

$$\begin{cases} u'(t)=Au(t)+f(t,u_t,\int_0^t k(t,s)h(s,u_s)ds), \\ \qquad t\in(0,T], \ \ t\neq\tau_k, \ \ k=1,2,..,m, \\ u(t)+(g(u_{t_1},...,u_{t_p}))(t)=\phi(t), \ \ -r\leq t\leq 0, \\ \Delta u(\tau_k)=I_k u(\tau_k), \quad k=1,2,...,m, \end{cases} \tag{1.1}$$

where $0<t_1<t_2<...<t_p\leq T$, $p\in\mathbb{N}$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ and $I_k(k=1,2,...,m)$ are the linear operators acting in a Banach space X.

Let $k:[0,T]\times[0,T]\to\mathbb{R}$ be a continuous function and the functions f,h, ϕ and g are given functions satisfying some assumptions. The impulsive moments τ_k are such that $0\leq\tau_0<\tau_1<\tau_2<...<\tau_m<\tau_{m+1}\leq T$, $m\in\mathbb{N}$, $\Delta u(\tau_k)=u(\tau_k+0)-u(\tau_k-0)$, where $u(\tau_k+0)$ and $u(\tau_k-0)$ are the right and the left limits of u at τ_k, respectively.

Many authors studied existence, uniqueness and other qualitative properties of equations (1.1) and their special forms, see ([1,3,6,8–11]) and the references therein. For more details on impulsive differential equations, see ([3,13,16]). The aim of the present paper is to investigate Ulam-Hyres and Ulam-Hyers-Rassias stabilities of mild solution of the problem (1.1). We use generalization of Grownwall type inequality to derive the result.

The paper is organized as follows: In Section 2, we present the preliminaries, hypotheses. In Section 3, we give proof of Ulam-Hyers stability results and Section 4, contains Ulam-Hyers-Rassias stability results.

2. Preliminaries and hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r,0],X)$, $0 < r < \infty$ be the Banach space of all continuous functions $\psi : [-r,0] \to X$ endowed with supremum norm $\|\psi\|_C = \sup\{\|\psi(t)\| : -r \le t \le 0\}$. $PC([-r,T],X) = \{u : [-r,T] \to X | u(t)$ is piecewise continuous at $t \neq \tau_k$, left continuous at $t = \tau_k$, and the right limit $u(\tau_k+0)$ exists for $k = 1,2,...,m\}$. Then $PC([-r,T],X)$ is a Banach space with the supremum norm

$$\|u\|_{PC} = \sup\{\|u(t)\| : t \in [-r,T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}\}.$$

For any $u \in PC([-r,T],X)$ and $t \in [0,T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}$, we denote u_t the element of C given by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r,0]$ and ϕ is a given element of C.

Definition 2.1. A function $u \in PC([-r,T],X)$ satisfied the equations:

$$\begin{aligned} u(t) &= T(t)\phi(0) - T(t)(g(u_{t_1}, ..., u_{t_p}))(0) \\ &\quad + \int_0^t T(t-s) f(s, u_s, \int_0^s k(s,\tau)h(\tau,u_\tau)d\tau)ds \\ &\quad + \sum_{0<\tau_k<t} T(t-\tau_k) I_k u(\tau_k), \in (0,T], \\ u(t) &+ (g(u_{t_1}, ..., u_{t_p}))(t) = \phi(t),\ -r \le t \le 0 \end{aligned}$$

is said to be the mild solution of the initial value problem (1.1).

Theorem 2.2. ([17]) *Let* $\{T(t)\}_{t\ge 0}$ *be a* C_0 *semigroup. Then there exist constants* $\omega \ge 0$ *and* $M \ge 1$ *such that* $\|T(t)\| \le Me^{\omega t}$, $0 < t < \infty$.

Lemma 2.3. ([2]) *Let for* $t \ge t_0$*, the following inequality hold:*

$$u(t) \le a(t) + \int_{t_0}^{t} b(t,s)u(s)ds + \int_{t_0}^{t} (\int_{t_0}^{s} k(t,s,\tau)u(\tau)d\tau)ds + \sum_{t_0<\tau_k<t} \beta_k(t)u(t_k),$$

where, $u, a \in PC([t_0,\infty),\mathbb{R}_+)$*,* a *is nondecreasing,* $b(t,s)$ *and* $k(t,s,\tau)$ *are continuous and non-negative functions for* $t, s, \tau \ge t_0$ *and are nondecreasing with respect to* t*,* $\beta_k(t)(k \in \mathbb{N})$ *are nondecreasing for* $t \ge t_0$*. Then for* $t \ge t_0$ *the following inequality hold :*

$$u(t) \le a(t) \prod_{t_0<\tau_k<t} (1+\beta_k(t)) exp(\int_{t_0}^{t} b(t,s)ds) + \int_{t_0}^{t} \int_{t_0}^{s} k(t,s,\tau)d\tau)ds.$$

Definition 2.4. We say that equation (1.1) has the Ulam-Hyers stability, if there exists a non-negative constant c_1 such that for each $\epsilon > 0$ and u in $PC([-r, T], X)$ satisfies:

$$\begin{cases} \|u'(t) - Au(t) - f(t, u_t \int_0^t k(t,s)h(s,u_s)ds)\| \leq \epsilon, \\ \qquad t \in (0, T], \quad t \neq \tau_k, \quad k = 1, 2, .., m, \\ \|u(t) + (g(u_{t_1}, ..., u_{t_p}))(t) - \phi(t)\| \leq \epsilon, \quad -r \leq t \leq 0, \\ \|\Delta u(\tau_k) - I_k u(\tau_k), \| \leq \epsilon, \quad k = 1, 2, ..., m, \end{cases} \tag{2.1}$$

then there exists a solution v of the equation (1.1) with

$$\|u - v\|_{PC} \leq c_1 \epsilon.$$

Definition 2.5. We say that equation (1.1) has the generalized Ulam-Hyers stability, if there exists a piecewise continuous function ξ(depend upon f) with $\xi(0) = 0$ such that for each solution v in $PC([-r, T], X)$ of the equation (1.1) with

$$\|u - v\|_{PC} \leq \xi(\epsilon).$$

Definition 2.6. We say that equation (1.1) has the Ulam-Hyers-Rassias stability, if there exists a positive piecewise continuous function $\bar{\psi}(t) : [-r, T] \to R$ such that for each $\epsilon > 0$ and u in $PC([-r, T], X)$, there exists $c_2 \geq 0$ (depending upon f and $\bar{\psi}(t)$) such that for every $\epsilon > 0$, $\bar{\psi} \geq 0$, if $u \in PC([-r, T], X)$ satisfies:

$$\begin{cases} \|u'(t) - Au(t) - f(t, u_t, \int_0^t k(t,s)h(s,u_s)ds)\| \leq \epsilon\bar{\psi}(t), \\ \qquad t \in (0, T], \quad t \neq \tau_k, \quad k = 1, 2, ..., m, \\ \|u(t) + (g(u_{t_1}, ..., u_{t_p}))(t) - \phi(t)\| \leq \epsilon\bar{\psi}, \quad -r \leq t \leq 0, \\ \|\Delta u(\tau_k) - I_k u(\tau_k), \| \leq \epsilon\bar{\psi}, \quad k = 1, 2, ..., m, \end{cases} \tag{2.2}$$

then there exists a solution $v : [-r, T] \to X$ of the equation (1.1) with

$$\|u - v\|_{PC} \leq \epsilon c_2(\bar{\psi}(t) + \bar{\psi}(k+1)).$$

Definition 2.7. We say that equation (1.1) has the generalized Ulam-Hyers-Rassias stability, if there exists a positive piecewise continuous function $\bar{\psi}(t) : [-r, T] \to R$ such that for each $\epsilon > 0$ and u in $PC([-r, T], X)$, there exists $c_2 \geq 0$ (depending upon f and $\bar{\psi}(t)$) such that for every $\epsilon > 0$, $\bar{\psi} \geq 0$ if $u \in PC([-r, T], X)$ satisfies :

$$\begin{cases} \|u'(t) - Au(t) - f(t, u_t, \int_0^t k(t,s)h(s,u_s)ds)\| \leq \bar{\psi}(t), \\ \qquad t \in (0, T], \quad t \neq \tau_k, \quad k = 1, 2, .., m, \\ \|u(t) + (g(u_{t_1}, ..., u_{t_p}))(t) - \phi(t)\| \leq \bar{\psi}, \quad -r \leq t \leq 0, \\ \|\Delta u(\tau_k) - I_k u(\tau_k), \| \leq \bar{\psi}, \quad k = 1, 2, ..., m, \end{cases} \tag{2.3}$$

then there exists a solution $v : [-r, T] \to X$ of the equation (1.1) with

$$\|u - v\|_{PC} \leq c_2(\bar{\psi}(t) + \bar{\psi}(k+1))$$

Remark 2.8. A function $u \in PC([-r, T], X)$ is solution of the inequality (2.1). If there exists $b_u \in PC([-r, T], X)$ and a sequence b_k, $k = 1, 2, ..., m$ (which depend on u) such that

(1) $\|b_u(t)\| \leq \epsilon, \in [-r, T]$ and $\|b_k(\tau_k)\| \leq \epsilon, \quad k = 1, 2, ..., m,$

(2) $u'(t) = Au(t) + f(t, u_t, \int_0^t k(t, s)h(s, u_s)ds) + b_u(t), t \in (0, T], \quad t \neq \tau_k,$

(3) $u(t) + (g(u_{t_1}, ..., u_{t_p}))(t) + b_u(t) = \phi(t), \quad -r \leq t \leq 0,$

(4) $\Delta u(\tau_k) = I_k u(\tau_k) + b_k(\tau_k), \quad k = 1, 2, ..., m.$

Proposition 2.9. *If $u \in PC([-r, T], X)$ satisfies the set of inequalities* (2.1), *then u is the solution of following integro-differential equations:*

$$\begin{aligned}
&\|u(t) - T(t)\phi(0) + T(t)(g(u_{t_1}, ..., u_{t_p}))(0) \\
&\quad - \int_0^t T(t-s)f(s, u_s, \int_0^s k(s, \tau)h(\tau, u_\tau)d\tau)ds - \sum_{0<\tau_k<t} T(t-\tau_k)I_k u(\tau_k)\| \\
&\leq \epsilon\Big(\int_0^t \|T(t-s)\|ds + \|T(t)\| + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\Big).
\end{aligned}$$

Proof. With Remark 2.8, we have

$$\begin{cases} u'(t) = Au(t) + f(t, u_t, \int_0^t k(t, s)h(s, u_s)ds) + b_u(t), \\ \qquad t \in (0, T], \quad t \neq \tau_k, \\ u(t) + (g(u_{t_1}, ..., u_{t_p}))(t) + b_u(t) = \phi(t), \quad -r \leq t \leq 0, \\ \Delta u(\tau_k) = I_k u(\tau_k) + b_k(\tau_k), \quad k = 1, 2, ..., m. \end{cases} \tag{2.4}$$

Clearly, the solution of system of equations (2.4) is given by

$$\begin{aligned}
u(t) &= T(t)\phi(0) - T(t)(g(u_{t_1}, ..., u_{t_p}))(0) - T(t)b_u(t) \\
&\quad + \int_0^t T(t-s)f(s, u_s, \int_0^s k(s, \tau)h(\tau, u_\tau)d\tau)ds + \int_0^t T(t-s)b_u(t)ds \\
&\quad + \sum_{0<\tau_k<t} T(t-\tau_k)I_k u(\tau_k) + \sum_{0<\tau_k<t} T(t-\tau_k)b_k(u(\tau_k)).
\end{aligned}$$

It follows that,

$$\begin{aligned}
&\|u(t) - T(t)\phi(0) + T(t)(g(u_{t_1}, ..., u_{t_p}))(0) \\
&\quad - \int_0^t T(t-s)f(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau)ds - \sum_{0<\tau_k<t} T(t-\tau_k)I_k u(\tau_k)\| \\
&\leq \int_0^t \|T(t-s)\|\|b_u(t)\|ds + \|T(t)\|\|b_u(t)\| + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\|b_k(u(\tau_k))\| \\
&\leq \epsilon\big(\int_0^t \|T(t-s)\|ds + \|T(t)\| + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\big).
\end{aligned}$$

□

Let us introduce the following hypotheses which are assumed thereafter for our convenience.

(H_1) Let $f : [0,T] \times C \times X \to X$ and $h : [0,T] \times C \to X$ be continuous functions such that there exists a continuous nondecreasing function $p : [0,T] \to \mathbb{R}_+ = [0,\infty)$ and $q : [0,T] \to \mathbb{R}_+$ such that

$$\|f(t,\psi,u) - f(t,\phi,v)\| \leq p(t)(\|\psi - \phi\|_C + \|u - v\|),$$
$$\|h(t,\psi) - h(t,\phi)\| \leq q(t)\|\psi - \phi\|_C$$

for every $t \in [0,T]$, $\psi \in C$ and $u, v \in X$.

(H_2) Let $g : C^p \to C$ such that there exists a constant $G \geq 0$ such that

$$\|g(u_{t_1}, u_{t_2}, ..., u_{t_p}) - g(u_{t_1}, v_{t_2}, ..., v_{t_p})\| \leq G\|u - v\|.$$

(H_3) Let $I_k : X \to X$ are functions such that there exists constants L_k satisfying

$$\|I_k(u) - I_k(v)\| \leq L_k\|u - v\|, u, v \in X, k = 1, 2, ..., m.$$

3. Ulam-Hyres stability

Theorem 3.1. *Suppose that the hypotheses (H_1)-(H_3) hold. Then the impulsive initial-value problem* (1.1) *is Ulam-Hyres stable on* $[-r,T]$, *whenever* $1 - MGe^{\omega T} > 0$.

Proof. Let $u \in PC([-r,T],X)$ satisfies inequalities (2.2), $v \in PC([-r,T],X)$ be the mild solution of equations (1.1).

$$\begin{aligned}
v(t) &= T(t)\phi(0) - T(t)(g(v_{t_1}, ..., v_{t_p}))(0) \\
&\quad + \int_0^t T(t-s)f(s, v_s, \int_0^s k(s,\tau)h(\tau, v_\tau)d\tau)ds + \sum_{0<\tau_k<t} T(t-\tau_k)I_k v(\tau_k).
\end{aligned}$$

Using Proposition 2.9 and Theorem 2.2, we obtain,

$$\begin{aligned}
&\|u(t) - T(t)\phi(0) + T(t)(g(u_{t_1}, ..., u_{t_p}))(0) \\
&\quad - \int_0^t T(t-s) f(s, u_s, \int_0^s k(s,\tau) h(\tau, u_\tau) d\tau) ds - \sum_{0<\tau_k<t} T(t-\tau_k) I_k u(\tau_k)\| \\
&\leq \epsilon\Big(\int_0^t \|T(t-s)\| ds + \|T(t)\| + \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \Big) \\
&\leq \epsilon\Big(\int_0^t M e^{\omega(t-s)} ds + M e^{\omega t} + \sum_{0<\tau_k<t} M e^{\omega(t-\tau_k)} \Big) \\
&\leq \epsilon\Big(\frac{M}{\omega} e^{\omega t - 1} + M e^{\omega t} + k M e^{\omega(t-\tau_k)} \Big) \\
&\leq \epsilon\Big(\frac{M}{\omega} e^{\omega T - 1} + M e^{\omega T} + k M e^{\omega T} \Big) \\
&\leq \epsilon\Big(\frac{M}{\omega} e^{\omega T - 1} + M e^{\omega T}(k+1) \Big).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|u(t) - v(t)\| &= \|u(t) - T(t)\phi(0) + T(t)(g(v_{t_1}, ..., v_{t_p}))(0) \\
&\quad - \int_0^t T(t-s) f(s, v_s, \int_0^s k(s,\tau) h(\tau, v_\tau) d\tau) ds \\
&\quad - \sum_{0<\tau_k<t} T(t-\tau_k) I_k v(\tau_k)\| \\
&\leq \|u(t) - T(t)\phi(0) + T(t)(g(u_{t_1}, ..., u_{t_p}))(0) \\
&\quad - \int_0^t T(t-s) f(s, u_s, \int_0^s k(s,\tau) h(\tau, u_\tau) d\tau) ds \\
&\quad - \sum_{0<\tau_k<t} T(t-\tau_k) I_k u(\tau_k)\| \\
&\quad + \|T(t)\| \|(g(v_{t_1}, ..., v_{t_p}))(0) - (g(u_{t_1}, ..., u_{t_p}))(0)\| \\
&\quad + \int_0^t \|T(t-s)\| \|f(s, u_s, \int_0^s k(s,\tau) h(\tau, u_\tau) d\tau) \\
&\quad - f(s, v_s, \int_0^s k(s,\tau) h(\tau, v_\tau) d\tau)\| ds \\
&\quad + \sum_{0<\tau_k<t} \|T(t-\tau_k)\| I_k u(\tau_k) - I_k v(\tau_k)\|
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon\big(\frac{M}{\omega}e^{\omega T-1}ds + Me^{\omega T}(k+1)\big) + MGe^{\omega t}\|v-u\| \\
&\quad + \int_0^t Me^{\omega(t-s)}p(s)[\|u_s - v_s\|_C + \int_0^s Lq(\tau)(\|u_\tau - v_\tau\|_C)\|d\tau]ds \\
&\quad + \sum_{0<\tau_k<t} Me^{\omega(t-\tau_k)}L_k\|u(\tau_k) - v(\tau_k)\|.
\end{aligned}$$

Let $R(t) = \sup\{p(t), Lq(t)\}$ and $R^* = \sup\{R(t) : t \in [-r, T]\}$. Define the function $z : [-r, T] \to \mathbb{R}$ by $z(t) = \sup\{\|u(s) - v(s)\| : -r \leq s \leq t\}, t \in [0, T]$. Let $t^* \in [-r, t]$ be such that $z(t) = \|u(t^*) - v(t^*)\|$. If $t^* \in [0, t]$, then

$$\begin{aligned}
z(t) &\leq \epsilon\big(\frac{M}{\omega}e^{\omega T-1}ds + Me^{\omega T}(k+1)\big) + Me^{\omega t}Gz(t) \\
&\quad + \int_0^{t*} Me^{\omega(t-s)}p(s)[z(s) + \int_0^s Lq(\tau)(z(\tau)d\tau]ds \\
&\quad + \sum_{0<\tau_k<t} Me^{\omega(t-\tau_k)}L_k z(\tau_k),
\end{aligned}$$

$$\begin{aligned}
(1 - MGe^{\omega t})z(t) &\leq \epsilon\big(\frac{M}{\omega}e^{\omega T-1} + Me^{\omega T}(k+1)\big) + \int_0^t Me^{\omega T}p(s)z(s)ds \\
&\quad + \int_0^t Me^{\omega T}\int_0^s Lq(\tau)(z(\tau)d\tau ds + \sum_{0<\tau_k<t} Me^{\omega T}L_k z(\tau_k) \\
&\leq \frac{\epsilon}{(1 - MGe^{\omega T})}\big(\frac{M}{\omega}e^{\omega T-1} + Me^{\omega T}(k+1)\big) \\
&\quad + \int_0^t \frac{Me^{\omega T}}{(1 - MGe^{\omega T})}p(s)z(s)ds \\
&\quad + \int_0^t \frac{MLe^{\omega T}}{(1 - MGe^{\omega T})}\int_0^s q(\tau)(z(\tau)d\tau ds \\
&\quad + \sum_{0<\tau_k<t} \frac{L_k Me^{\omega T}}{(1 - MGe^{\omega T})}z(\tau_k). \qquad (3.1)
\end{aligned}$$

If $t^* \in [-r, 0]$, then

$$\begin{aligned}
z(t) &\leq \|(g(v_{t_1}, ..., v_{t_p}))(0) - (g(u_{t_1}, ..., u_{t_p}))(0)\| + \|b_v(t) - b_u(t)\| \\
&\leq G\|u - v\| + 2\epsilon. \qquad (3.2)
\end{aligned}$$

In the view of inequality (3.1) and (3.2), the inequality (3.1) holds good for $t \in [-r, T]$. Now applying impulsive Lemma 2.3 to (3.1), we get

$$z(t) \leq \frac{\epsilon}{(1 - MGe^{\omega T})}\big(\frac{M}{\omega}e^{\omega T-1} + Me^{\omega T}(k+1)\big) \prod_{0<\tau_k<t}(1 + \frac{L_k M e^{\omega T}}{(1 - MGe^{\omega T})})$$
$$\times \exp\{\int_0^t \frac{Me^{\omega T}}{(1-MGe^{\omega T})}R(s)ds + \int_0^t \int_0^s [\frac{MLe^{\omega T}}{(1-MGe^{\omega T})}R(s)R(\tau)d\tau]ds\}$$
$$\leq \frac{\epsilon}{(1 - MGe^{\omega T})}\big(\frac{M}{\omega}e^{\omega T-1} + Me^{\omega T}(k+1)\big)$$
$$\times \prod_{0<\tau_k<t}(1 + \frac{L_k M e^{\omega T}}{(1 - MGe^{\omega T})}) \exp\{\frac{Me^{\omega T}}{(1 - MGe^{\omega T})}.R^* T(1 + LR * \frac{T}{2})\}.$$

Therefore, $\|u(t) - v(t)\|_{PC} \leq \epsilon c_1$, where $\hat{c}$ is depend upon f only.

$$c_1 = \frac{\epsilon}{(1 - MGe^{\omega T})}\big(\frac{M}{\omega}e^{\omega T-1} + Me^{\omega T}(k+1)\big)$$
$$\times \prod_{0<\tau_k<t}(1 + \frac{L_k M e^{\omega T}}{(1 - MGe^{\omega T})}) exp\{\frac{Me^{\omega T}}{(1 - MGe^{\omega T})}R^* T(1 + LR * \frac{T}{2})\}$$

for $1 - MGe^{\omega T} > 0$. □

Corollary 3.2. *Assume that the hypotheses* (H_1)-(H_3) *hold. Then the impulsive initial-value problem* (1.1) *is generalized Ulam-Hyres stable on* $[-r, T]$ *for* $1 - MGe^{\omega T} > 0$.

Proof. Define,

$$\xi(\epsilon) \leq \frac{\epsilon}{(1 - MGe^{\omega T})}\big(\frac{M}{\omega}e^{\omega T-1}ds + Me^{\omega T}(k+1)\big)$$
$$\times \prod_{0<\tau_k<t}(1 + \frac{L_k M e^{\omega T}}{(1 - MGe^{\omega T})}) exp\{\frac{Me^{\omega T}}{(1 - MGe^{\omega T})}R^* T(1 + LR * \frac{T}{2})\}.$$

Then $\xi(\epsilon)$ is piecewise continuous and $\xi(0) = 0$. Therefore,

$$\|u(t) - v(t)\|_{PC} \leq \xi(\epsilon).$$

□

4. Ulam-Hyres-Rassias stability

Theorem 4.1. *Assume that the hypotheses* (H_1)-(H_3) *hold. Consider* $\bar{\psi} : [-r, T] \to \mathbb{R}_+$ *is positive nondecreasing continuous function and there exists* $\lambda > 0$ *such that*

$$\int_0^t \bar{\psi}(s)ds \leq \lambda\bar{\psi}(t), \quad t \in [-r, T].$$

Then the impulsive initial-value problem (1.1) *is Ulam-Hyers-Rassias stable with respect to* $\bar{\psi}(t)$, $\bar{\psi}$ *provided that* $1 - MGe^{\omega T} > 0$.

Proof. Let $u \in PC([-r,T],X)$ satisfies the inequalities (2.3), $v \in PC([-r,T],X)$ be the mild solution of equations (1.1).

$$v(t) = T(t)\phi(0) - T(t)(g(v_{t_1},...,v_{t_p}))(0) + \int_0^t T(t-s)f(s,v_s,\int_0^s k(s,\tau)h(\tau,v_\tau)d\tau)ds + \sum_{0<\tau_k<t} T(t-\tau_k)I_k v(\tau_k).$$

Using inequalities (2.3) and Theorem 2.2, we obtain,

$$\begin{aligned}
&\|u(t) - T(t)\phi(0) + T(t)(g(u_{t_1},...,u_{t_p}))(0) \\
&\quad - \int_0^t T(t-s)f(s,u_s,\int_0^s k(s,\tau)h(\tau,u_\tau)d\tau)ds \\
&\quad - \sum_{0<\tau_k<t} T(t-\tau_k)I_k u(\tau_k)\| \\
&\leq \big(\int_0^t \|T(t-s)\|ds + \|T(t)\| + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\big) \\
&\leq \big(\int_0^t Me^{\omega(t-s)}\epsilon\bar{\psi}(s)ds + Me^{\omega t}\epsilon\bar{\psi} + \sum_{0<\tau_k<t} Me^{\omega(t-\tau_k)}\epsilon\bar{\psi}\big) \\
&\leq \epsilon\lambda\bar{\psi}(t)Me^{\omega T} + Me^{\omega T}\epsilon\bar{\psi} + \sum_{0<\tau_k<t} Me^{\omega T}\epsilon\bar{\psi}\big) \\
&\leq \epsilon Me^{\omega T}(\lambda\bar{\psi}(t) + \bar{\psi}(k+1)).
\end{aligned}$$

Proceeding in same way as in Theorem 3.1 we obtain,

$$\begin{aligned}
z(t) &\leq \frac{\epsilon}{(1-MGe^{\omega T})}(\lambda\bar{\psi}(t) + \bar{\psi}(k+1)) \prod_{0<\tau_k<t}(1 + \frac{L_k Me^{\omega T}}{(1-MGe^{\omega T})}) \\
&\quad \times \exp\{\int_0^t \frac{Me^{\omega T}}{(1-MGe^{\omega T})}R(s)ds + \int_0^t\int_0^s [\frac{MLe^{\omega T}}{(1-MGe^{\omega T})}R(s)R(\tau)d\tau]ds\} \\
&\leq \frac{\epsilon}{(1-MGe^{\omega T})}(\lambda\bar{\psi}(t) + \bar{\psi}(k+1)) \\
&\quad \times \prod_{0<\tau_k<t}(1 + \frac{L_k Me^{\omega T}}{(1-MGe^{\omega T})}) \exp\{\frac{Me^{\omega T}}{(1-MGe^{\omega T})}R^*T(1 + LR * \frac{T}{2})\},
\end{aligned}$$

$$\|u(t) - v(t)\|_{PC} \leq \epsilon(\lambda\bar{\psi}(t) + \bar{\psi}(k+1))c_2,$$

where, c_2 is depend upon f and $\bar{\psi}(t)$ and

$$c_2 = \frac{1}{(1 - MGe^{\omega T})} \prod_{0<\tau_k<t} (1 + \frac{L_k M e^{\omega T}}{(1 - MGe^{\omega T})}) \times \exp\left\{ \frac{Me^{\omega T}}{(1 - MGe^{\omega T})} R^* T (1 + LR * \frac{T}{2}) \right\}$$

for $1 - MGe^{\omega T} > 0$. □

Corollary 4.2. *Assume that the hypotheses* (H_1)-(H_3) *hold. Then the impulsive initial-value problem* (1.1) *is generalized Ulam-Hyres-Rassias stable with respect to* $\bar{\psi}(t)$, $\bar{\psi}$ *on* $[-r, T]$ *for* $1 - MGe^{\omega T} > 0$.

References

[1] M Asadi, *Exact Solutions of Singular IVPs Lane-Emden Equation*, Comm. Nonlinear Anal., **6**(1) (2019), 60–63.

[2] D. Bainov and Snezhana G. Haristova , *Integral Inequalities of Gronwall type for piecewise continuous functions*, J. Appl. Math. Stoc. Anal., **10**(1) (1997), 89–94.

[3] D. Bainov and Pavel Simeonov, *Impusive Differential Equations*, Periodic Solutions and Applications, 1993.

[4] M. Benchora, J. Henderson and S.K. Ntouyas, *An existence Result for first order impulsive functional differential equations in Banach spaces*, Comput. Math. Appl., **42** (2001), 1303–1310.

[5] M. Gabeleh, M. Asadi and P.R. Patle, *Simulation functions and MeirKeeler condensing operators with application to integral equations*, Asian-Europ. J. Math., **15**(9) (2022), Artical No. 2250171.

[6] S. Ghezellou, M. Azhini and M. Asadi, *New theorems by minimax inequalities on H-space*, Bull. Math. Anal. Appl., **12**(2) (2020), 27–34.

[7] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., **27** (1941), 222–224.

[8] Rupali S. Jain and M.B. Dhakne, *On impulsive nonlocal integro-differential equations with finite delay*, Int. J. of Mathematics Research, **5**(4) (2013), 361–373.

[9] Rupali S. Jain and M.B. Dhakne, *On Global Existence of solutions for abstract nonlinear functional Integro-Differentai Equations With Nonlocal Condition*, Contemporary Mathematics and Statistics, **2023**(1) (2013), 44–53.

[10] Rupali S. Jain and M.B. Dhakne, *On mild solutions of nonlocal semilinear functional integro-differential equations*, Malaya Journal of Matematik, **3**(1) (2013), 27-33.

[11] S.D. Kadam, R.S. Jain and B. Surendranath Reddy, *On the global existence of mild solutions of impulsive integro-differential equations with nonlocal condition*, Open J. of Applied and theoretical mathematics, **2**(4) (2016), 1027–1040.

[12] K.D. Kucche and P.U. Shikhare, *Ulam stabilities via pachpatte's inequality for Volterra-Fredholm delay integrodifferential equations in Banach spaces*, Note di Matematica, Note Mat., **38**(1) (2018), 67–82.

[13] G.E. Ladas and V. Lakshmikantham, *Differential equations in abstract space*, Acsdemic press, 1972.

[14] Xuezhu Li and Jinrong Wang, *Ulam-Hyers-Rassias stability of semilinear differential equations with impulses*, Electronic Journal of Differential Equations, **2013**(172) (2013), 1–8.

[15] Zeng Lin, WeiWei and JinRong Wang, *Existence and stability results for impulsive integro-differential equations*, FACTA UNIVERSITATIS ($NI\check{S}$) Ser. Math. Inform., **29**(2) (2014), 119-130.

[16] V. Lkashmikanthamm, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*, World scientific Singapore, 1889.

[17] B.G. Pachpatte, *Integral and finite difference inequalities and applications*, Northholland Mathematics Studies, 205, Elsevier Science, B.V., Amsterdam, 2006.

[18] C. Parthasarathy, *Existence and Hyers-Ulam Stability of nonlinear impulsive differential equations with nonlocal conditions*, Electronic Journal of Mathematical Analysis and Applications, **4**(1) (2016), 106–115.

[19] Th.M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.

[20] S.M. Ulam, *A collection of mathematical problems*, Interscience Publishers, New York, 1968.

[21] JinRong Wang, Michal Feckan and Yong Zhou, *On the stability of first order impulsive evolution equations*, Opuscula Math., **34**(3) (2014), 639-657.

[22] JinRong Wanga, Michal Fečkan and Yong Zhou, *Ulams type stability of impulsive ordinary differential equations*, J. Math. Anal. Appl., **395** (2012), 258-264.

[23] Akbar Zada, Shah Faisal and Yongjin Li, *On the Hyers-Ulam stability of first-order impulsive delay differential equations*, Hindawi Publishing Corporation Journal of Function Spaces, Volume 2016, 6 pages, http://dx.doi.org/10.1155/2016/8164978 .