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SOLUTION OF A NONLINEAR DELAY INTEGRAL EQUATION VIA A FASTER ITERATIVE METHOD

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Abstract. In this article, we study the Picard-Ishikawa iterative method for approximating the fixed point of generalized α-Reich-Suzuki nonexpanisive mappings. The weak and strong convergence theorems of the considered method are established with mild assumptions. Numerical example is provided to illustrate the computational efficiency of the studied method. We apply our results to the solution of a nonlinear delay integral equation. The results in this article are improvements of well-known results.

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1. Introduction

In this article, let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{R}$ the set of all real numbers and $\mathbb{C}$ the set of all complex numbers. Let $\mathcal{G}$ be a nonempty subset of a Banach space $\mathcal{M}$. Then, a self-mapping $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ is said to be nonexpansive if $\|\mathcal{L}u - \mathcal{L}v\| \leq \|u - v\|$ for all $u, v \in \mathcal{G}$ and it is said to be quasi-nonexpansive if $\|\mathcal{L}u - u^*\| \leq \|u - u^*\|$ for all $u \in \mathcal{G}$ and $u^* \in F(\mathcal{L}) = \{u^* \in \mathcal{G} : u^* = \mathcal{L}u^*\}$, the set of all fixed points of $\mathcal{L}$.

In the past few years, the fixed point theory for nonexpansive mappings has attracted several authors as a results of their vast applications in integral equation, differential equation, convex optimization and control theory, signal processing, game theory and many more. Several extensions and generalizations of the class of nonexpansive mappings have been studied in the past two decades (see [4, 8, 12, 13, 14, 15, 16, 17, 19, 21, 22, 27] and the references therein).

One of these important generalizations of the class of nonexpansive mappings known as Suzuki generalized nonexpansive mappings was given by Suzuki [27]. This class of mappings are also known as mappings satisfying the condition (C). In [4], Aoyama and Kohsaka introduced the class nonexpansive-type mappings called α-nonexpansive mappings.

In [22], Pant and Shukla considered another generalized nonexpansive-type mapping called generalized α-nonexpansive mappings. The authors showed that this class of mappings is more general than the class of mappings satisfying the condition (C). In [21], Pant and Pandey considered the Reich-Suzuki nonexpansive mappings. The authors proved that this class of mappings is more general than the class of mappings satisfying the condition (C). Furthermore, they proved some existence and fixed point results for such mappings.

Very recently, Pandy et al. [20] combined to the classes of generalized α-nonexpansive mappings and Reich-Suzuki nonexpansive mappings to defined a new class of mappings as follows:

Definition 1.1. Let $\mathcal{L}$ be a self mapping defined on a nonempty subset $\mathcal{G}$ of a Banach space $\mathcal{M}$. Then $\mathcal{L}$ is said to be a generalized α-Reich-Suzuki nonexpansive mapping if for all $u, v \in \mathcal{G}$, there exists $\alpha \in (0, 1]$ such that

$$\frac{1}{2}\|u - \mathcal{L}u\| \leq \|u - v\| \text{ implies } \|\mathcal{L}u - \mathcal{L}v\| \leq \max\{\Upsilon(u, v), \Omega(u, v)\},$$

where

$$\Upsilon(u, v) = \alpha\|u - \mathcal{L}v\| + \alpha\|v - \mathcal{L}u\| + (1 - 2\alpha)\|u - v\|$$

and

$$\Omega(u, v) = \alpha\|u - \mathcal{L}u\| + \alpha\|v - \mathcal{L}v\| + (1 - 2\alpha)\|u - v\|.$$

For Banach contraction principle, it is well known that the Picard iterative algorithm approximate the fixed points of contraction mappings, but fails to converge to the fixed point of nonexpansive mappings even when their fixed points exist. This posed a serious problem in the field of nonlinear analysis and the problem captured the interest of several authors.

For some years now, several iterative methods have been developed to approximate the fixed points of nonexpansive mappings and other general classes of mappings. Some of these iterative methods are: Abbas [1], Agarwal et al. [2], Ali et al. [3], Ishikawa [10], Noor [11], Thakur [28] and Ullah et al. ([29], [30]) iterative algorithms.

Very recently, in [18], Okeke introduced the Picard-Ishikawa iterative method as follows:

$$\begin{cases} u_0 \in \mathcal{G}, \\ w_m = (1-\beta_m)u_m + \beta_m \mathcal{L}u_m, \\ v_m = (1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m, \\ u_{m+1} = \mathcal{L}v_m, \end{cases} \qquad m \in \mathbb{N}, \tag{1.1}$$

where the sequences $\{\alpha_m\}, \{\beta_m\} \subset (0,1)$. The author showed that (1.1) has a better rate of convergence than most leading iterative algorithms in the literature.

Motivated by the above results, in this article, we prove the weak and strong convergence results of (1.1) for generalized α-Reich-Suzuki nonexpansive mappings. Numerical example is provided to illustrate the computational efficiency of the studied method. Furthermore, We apply our results to the solution of a nonlinear delay integral equations.

2. Preliminaries

Definition 2.1. A Banach space $\mathcal{M}$ is said to be satisfied the Opial's condition if for any sequence $\{u_m\}$ in $\mathcal{M}$ such that $u_m \rightharpoonup u \in \mathcal{M}$ implies

$$\limsup_{m\to\infty} \|u_m - u\| < \limsup_{m\to\infty} \|u_m - v\|, \quad \forall\, v \in \mathcal{M},\ v \neq u.$$

Definition 2.2. A Banach space $\mathcal{M}$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ with $u, v \in \mathcal{M}$ satisfying $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| > \epsilon$ such that

$$\left\| \frac{u+v}{2} \right\| < 1 - \delta.$$

Let $\mathcal{M}$ be a Banach space and $\mathcal{G}$ a nonempty closed convex subset of $\mathcal{M}$. Let $\{u_m\}$ be a bounded sequence in $\mathcal{M}$. For $u \in \mathcal{M}$, we set

$$r(u, \{u_m\}) := \limsup_{m\to\infty} \|u_m - u\|.$$

The asymptotic radius of $\{u_m\}$ relative to $\mathcal{G}$ is defined by

$$r(\mathcal{G}, \{u_m\}) := \inf\{r(u, \{u_m\}) : u \in \mathcal{G}\}.$$

The asymptotic center of $\{u_m\}$ relative to $\mathcal{G}$ is given as:

$$A(\mathcal{G}, \{u_m\}) := \{u \in \mathcal{G} : r(u, \{u_m\}) = r(\mathcal{G}, \{u_m\})\}.$$

It is well known that $A(\mathcal{G}, \{u_m\})$ consist of exactly one point in a uniformly convex Banach space.

Let $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be a nonlinear operator such that $F(\mathcal{L}) \neq \emptyset$. Then $I - \mathcal{L}$ is called demiclosed at zero if for any $u_m \in \mathcal{G}$, $u_m \rightharpoonup u$ and $(I - \mathcal{L})u_m \to 0$, then $u \in F(\mathcal{L})$.

Lemma 2.3. ([26]) *Let $\{\phi_m\}$ and $\{\sigma_m\}$ be sequences in $[0, \infty)$ such that*

$$\phi_{m+1} \leq (1 - \lambda_m)\phi_m + \lambda_m \sigma_m,$$

where $\lambda_m \in (0, 1)$ with $\sum_{m=0}^{\infty} \lambda_m = \infty$ and $\sigma_m \geq 0$ for all $m \in \mathbb{N}$. Then

$$0 \leq \limsup_{m\to\infty} \phi_m \leq \limsup_{m\to\infty} \sigma_m.$$

Definition 2.4. ([25]) A self-mapping $\mathcal{L}$ defined on $\mathcal{G}$ is said to be satisfied the condition (I), if a nondecreasing function $g : [0, \infty) \to [0, \infty)$ exists with $g(0) = 0$ and $g(s) > 0$ for all $s > 0$ such that

$$\|u - \mathcal{L}u\| \geq g(d(u, F(\mathcal{L}))))$$

for all $u \in \mathcal{G}$, where $d(f, F(\mathcal{L})) = \inf_{u^* \in F(\mathcal{L})} \|u - u^*\|$.

Lemma 2.5. ([24]) *Let $\{\mu_m\}$ be any sequence that satisfies $0 < u \leq \mu_m \leq v < 1$ for all $m \geq 1$ and $\{u_m\}$ and $\{v_m\}$ be any sequences in a uniformly convex Banach space $\mathcal{M}$ such that $\limsup_{m\to\infty} \|u_m\| \leq w$, $\limsup_{m\to\infty} \|v_m\| \leq w$ and*

$$\limsup_{m\to\infty} \|\mu_m u_m + (1 - \mu_m)v_m\| = w$$

hold for some $w \geq 0$. Then $\lim_{m\to\infty} \|u_m - v_m\| = 0$.

Proposition 2.6. ([22]) *Let $\mathcal{M}$ be a Banach space and $\mathcal{G}$ be a nonempty subset of $\mathcal{M}$. Then we have the followings:*

(i) *If $\mathcal{L}$ fulfills condition (C), then $\mathcal{L}$ is generalized α-Reich-Suzuki nonexpansive mapping.*

(ii) *If $\mathcal{L}$ is a generalized α-Reich-Suzuki nonexpansive mapping such that $F(\mathcal{L}) \neq \emptyset$, then $\mathcal{L}$ is quasi-nonexpansive.*

(iii) *If $\mathcal{L}$ is a generalized α-Reich-Suzuki nonexpansive mapping, then $F(\mathcal{L})$ is closed. Moreover, if $\mathcal{M}$ is strictly convex and $\mathcal{G}$ is convex, then $F(\mathcal{L})$ is also convex.*

(iv) *If $\mathcal{L}$ is a generalized α-Reich-Suzuki nonexpansive mapping, then we have*

$$\|u - \mathcal{L}v\| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \|u - \mathcal{L}u\| + \|u - v\|, \quad \forall\, u, v \in \mathcal{G}. \tag{2.1}$$

3. Weak and strong convergence theorems

In this part of the article, several weak and strong onvergence theorems will be stated and proved using the Picard-Ishikawa iterative method (1.1) for generalized α-Reich-Suzuki nonexpansive mappings. Further, we provide some novel numerical example. The provided example will be used to compare the computational efficiency of (1.1) with some well-known iterative methods in the literature.

Theorem 3.1. *Let $\mathcal{G}$ be a nonempty closed convex subset of a Banach space $\mathcal{M}$. Let $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be a generalized α-Reich-Suzuki nonexpansive mapping. If $\{u_m\}$ is the sequence defined by* (1.1), *then $\lim_{m\to\infty} \|u_m - u^*\|$ exists for all $u^* \in F(\mathcal{L})$.*

Proof. If $u^* \in F(\mathcal{L})$, then by Proposition 2.6(ii) and (1.1), we get

$$\begin{aligned} \|w_m - u^*\| &= \|(1-\beta_m)u_m + \beta_m \mathcal{L}u_m - u^*\| \\ &\leq (1-\beta_m)\|u_m - u^*\| + \beta_m\|\mathcal{L}u_m - u^*\| \\ &\leq (1-\gamma_m)\|u_m - u^*\| + \gamma_m\|u_m - u^*\| \\ &\leq \|u_m - u^*\|. \end{aligned} \tag{3.1}$$

Again, by (3.1), we have

$$\begin{aligned} \|v_m - u^*\| &= \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m - u^*\| \\ &\leq (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|\mathcal{L}w_m - u^*\| \\ &\leq (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|w_m - u^*\| \\ &\leq \|u_m - u^*\|. \end{aligned} \tag{3.2}$$

From (3.2), we have

$$\begin{aligned} \|u_{m+1} - u^*\| &= \|\mathcal{L}v_m - u^*\| \\ &\leq \|v_m - u^*\| \\ &\leq \|u_m - u^*\|, \end{aligned} \tag{3.3}$$

this means that the sequence $\{\|u_m - u^*\|\}$ is bounded and decreasing. Therefore, $\lim_{m\to\infty} \|u_m - \ell\|$ exists for all $u^* \in F(\mathcal{L})$. $\square$

Theorem 3.2. *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be defined as given in Theorem 3.1. Then, $F(\mathcal{L}) \neq \emptyset$ if and only if the sequence $\{u_m\}$ is bounded and*

$$\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0.$$

Proof. We have shown in Theorem 3.1 that $\{u_m\}$ is a bounded sequence and $\lim_{m\to\infty} \|u_m - u^*\|$ exists for any $u^* \in F(\mathcal{L})$. Set

$$\lim_{m\to\infty} \|u_m - u^*\| = \ell, \tag{3.4}$$

it follows from (3.1) and (3.4) that

$$\limsup_{m\to\infty} \|w_m - u^*\| \leq \limsup_{m\to\infty} \|u_m - u^*\| = \ell. \tag{3.5}$$

By Proposition 2.6(ii), we have

$$\limsup_{m\to\infty} \|\mathcal{L}u_m - u^*\| \leq \limsup_{m\to\infty} \|u_m - u^*\| = \ell. \tag{3.6}$$

Now, from (1.1), we get

$$\begin{aligned}
\|u_{m+1} - u^*\| &= \|\mathcal{L}u_m - u^*\| \\
&\leq \|v_m - u^*\| \\
&= \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m - u^*\| \\
&\leq (1-\alpha_m)\|u_m - u^*\| + \alpha_m \|\mathcal{L}w_m - u^*\| \\
&\leq (1-\alpha_m)\|u_m - u^*\| + \alpha_m \|w_m - u^*\| \\
&= \|u_m - u^*\| - \alpha_m \|u_m - u^*\| + \alpha_m \|w_m - u^*\|,
\end{aligned}$$

this implies that

$$\frac{\|u_{m+1} - u^*\| - \|u_m - u^*\|}{\alpha_m} \leq \|w_m - u^*\| - \|u_m - u^*\|.$$

Therefore,

$$\begin{aligned}
\|u_{m+1} - u^*\| - \|u_m - u^*\| &\leq \frac{\|u_{m+1} - u^*\| - \|u_m - u^*\|}{\alpha_m} \\
&\leq \|w_m - u^*\| - \|u_m - u^*\|,
\end{aligned}$$

this gives

$$\|u_{m+1} - u^*\| \leq \|w_m - u^*\|. \tag{3.7}$$

Therefore,

$$\ell \leq \liminf_{m\to\infty} \|w_m - u^*\|. \tag{3.8}$$

By (3.5) and (3.8), we obtain

$$\begin{aligned}
\ell &= \lim_{m\to\infty} \|w_m - u^*\| \\
&= \lim_{m\to\infty} \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}u_m - u^*\| \\
&= \lim_{m\to\infty} \|\alpha_m(\mathcal{L}u_m - u^*) + (1-\alpha_m)(u_m - u^*)\|. \qquad (3.9)
\end{aligned}$$

Combining Lemma 2.5, (3.4), (3.6) and (3.9), then we get

$$\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0.$$

Let $u^* \in A(\mathcal{G}, \{u_m\})$. Then

$$\begin{aligned}
r(\{u_m\}, \mathcal{L}u^*) &= \limsup_{m\to\infty} \|u_m - \mathcal{L}u^*\| \\
&\leq \limsup_{m\to\infty} \left\{ \left(\frac{3+\alpha}{1-\alpha}\right) \|u_m - \mathcal{L}u_m\| + \|\mathcal{L}u_m - \ell\| \right\} \\
&= \limsup_{m\to\infty} \left(\frac{3+\alpha}{1-\alpha}\right) \|u_m - \mathcal{L}u_m\| + \limsup_{m\to\infty} \|\mathcal{L}u_m - u^*\| \\
&\leq \limsup_{m\to\infty} \|u_m - u^*\| \\
&= r(\{u_m\}, u^*). \qquad (3.10)
\end{aligned}$$

It follows that $\mathcal{L}u^* \in A(\mathcal{G}, \{u_m\})$. By uniformly convexity of $\mathcal{M}$, it implies that $A(\mathcal{G}, \{u_m\})$ is a singleton set and thus, one get $\mathcal{L}u^* = u^*$. Therefore, $F(\mathcal{L}) \neq \emptyset$.

The converse part is trivial. □

Theorem 3.3. *If $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ are same as in Theorem* 3.1 *with $F(\mathcal{L}) \neq \emptyset$. If the Opial's property is satisfied by $\mathcal{M}$, then $\{u_m\}$ weakly converges to a point in $F(\mathcal{L})$.*

Proof. Since $F(\mathcal{L}) \neq \emptyset$, it follows from Theorem 3.2 and Theorem 3.1 that $\lim_{m\to\infty} \|u_m - u^*\|$ exists and $\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0$.

Now, we show that $\{u_m\}$ has just a one weakly sub-sequential limit in $F(\mathcal{L})$. Assume that k and h are two weak sub-sequential limits of $\{u_{m_j}\}$ and $\{u_{m_k}\}$, respectively. From Theorem 3.2 and the demiclosedness of $(I - \mathcal{L})$ at 0, we know that $(I - \mathcal{L})k = 0$. Thus, $\mathcal{L}k = k$ and from similar approach, we have $\mathcal{L}h = h$.

Next, we prove uniqueness. Assume $k \neq h$, then by Opial's condition, we get

$$\begin{aligned}
\lim_{m\to\infty}\|u_m-k\| &= \lim_{m_j\to\infty}\|u_{m_j}-k\| \\
&< \lim_{m_j\to\infty}\|u_{m_j}-h\| \\
&= \lim_{m\to\infty}\|u_m-h\| \\
&= \lim_{m_k\to\infty}\|u_{m_k}-h\| \\
&< \lim_{m_k\to\infty}\|u_{m_k}-k\| \\
&= \lim_{m\to\infty}\|u_m-k\|.
\end{aligned}$$

This is a contraction, therefore, $k = h$. Thus, $\{u_m\}$ weakly converges to an element in $F(\mathcal{L})$. □

Theorem 3.4. *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be the same as in Theorem 3.1 with $F(\mathcal{L}) \neq \emptyset$. Then, $\{u_m\}$ converges strongly to an element of $F(\mathcal{L})$ if and only if $\lim_{m\to\infty} d(u_m, F(\mathcal{L}) = 0$, where $d(u_m, F(\mathcal{L})) = \inf\{\|u_m - u^*\| : u^* \in F(\mathcal{L})\}$.*

Proof. The necessity is not hard to show, so we will omit it. Next, we show the converse case. Let u^* be any fixed point of $\mathcal{L}$. Then $\liminf d(u_m, F(\mathcal{L})) = 0$. From Theorem 3.1, we knows that $\lim_{m\to\infty}\|u_m - u^*\|$ exists for each $u^* \in F(\mathcal{L})$ and this implies that $\liminf_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$. Next, we claim that the sequence $\{u_m\}$ is Cauchy in $\mathcal{G}$. Due to $\liminf_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$ in as much as for any $\wp > 0$, there exists a constant $m_0 \in \mathbb{N}$ such that

$$d(u_m, F(\mathcal{L})) < \frac{\wp}{2}$$

and

$$\inf\{\|u_m - u^*\| : u^* \in F(\mathcal{L})\} < \frac{\wp}{2}$$

for all $m \geq m_0$. Therefore, $\inf\{\|u_{m_0} - u^*\| : u^* \in F(\mathcal{L})\} < \frac{\wp}{2}$. Hence, there exists $u^* \in F(\mathcal{L})$ such that

$$\|u_{m_0} - u^*\| < \frac{\wp}{2}.$$

For $n, m \geq m_0$, we have

$$\begin{aligned}
\|u_{m+n} - u_m\| &\leq \|u_{m+n} - u^*\| + \|u_m - u^*\| \\
&\leq \|u_{m_0} - u^*\| + \|u_{m_0} - u^*\| \\
&= 2\|u_{m_0} - u^*\| \\
&< \wp.
\end{aligned}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $\mathcal{G}$. From the completeness of $\mathcal{G}$, we get $\lim_{m\to\infty} u_m = p$ for some $p \in \mathcal{G}$. Further, $\lim_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$ shows that $p \in F(\mathcal{L})$. This completes the proof. □

Theorem 3.5. *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be the same as in Theorem* 3.1 *with $F(\mathcal{L}) \neq \emptyset$. Suppose $\mathcal{G}$ is compact. Then $\{u_m\}$ strongly converges to a fixed point of $\mathcal{L}$.*

Proof. Owing to the hypothesis that $F(\mathcal{L}) \neq \emptyset$, we know from Theorem 3.2 that $\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0$. Since $\mathcal{G}$ compact, one can have a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ with $\lim_{m\to\infty} u_{m_j} \to u^* \in \mathcal{G}$. From Proposition 2.6, we obtain

$$\|u_{m_j} - \mathcal{L}u^*\| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \|\mathcal{L}u_{m_j} - u_{m_j}\| + \|u_{m_j} - u^*\|.$$

On taking $j \to \infty$, $\mathcal{L}u^* = u^*$, that is, $u^* \in F(\mathcal{L})$. By Theorem 3.1, $\lim_{m\to\infty} \|u_m - u^*\|$ exists for any $u^* \in F(\mathcal{L})$ and so the sequence $\{u_m\}$ strongly converges to u^*. □

Theorem 3.6. *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be the same as in Theorem* 3.1 *with $F(\mathcal{L}) \neq \emptyset$. If $\mathcal{L}$ satisfies condition (I), then $\{u_m\}$ strongly converges to an element of $F(\mathcal{L})$.*

Proof. Due to Theorem 3.2, we obtain

$$\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0. \tag{3.11}$$

From (3.11) and condition (I) in Definition 2.4, we gets

$$\lim_{m\to\infty} g(d(u_m, F(\mathcal{L}))) \leq \lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0, \tag{3.12}$$

this implies that $\lim_{m\to\infty} g(d(u_m, F(\mathcal{L}))) = 0$. We know that g is a nondecreasing self-function defined on $[0,\infty)$ with $g(0) = 0$, $g(s) > 0$ for all $s \in (0,\infty)$, therefore, we get

$$\lim_{m\to\infty} d(u_m, F(\mathcal{L})) = 0. \tag{3.13}$$

By Theorem 3.4, $\{u_m\}$ strongly converges an element of $F(\mathcal{L})$. □

Next we present an example of a mapping that generalized α-Reich-Suzuki nonexpansive but does not satisfy condition (C).

Example 3.7. Let $\mathcal{M} = \mathbb{R}$ with the usual norm and $\mathcal{G} = [6, 10]$. Define $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{L}u = \begin{cases} \frac{u+42}{7}, & \text{if } \ u < 10, \\ 6, & \text{if } \ u = 10, \end{cases}$$

for all $u \in \mathcal{L}$.

(1) Let $u = 9$ and $v = 10$. Then we have

$$\begin{aligned} \frac{1}{2}\|u - \mathcal{L}u\| &= \frac{1}{2}|g - \mathcal{L}u| = \frac{6}{7} \\ &< 1 = |u - v| \\ &= \|u - v\|. \end{aligned}$$

But

$$\begin{aligned} \|\mathcal{L}u - \mathcal{L}v\| &= |\mathcal{L}u - \mathcal{L}v| = \frac{9}{7} \\ &> 1 = |u - v| \\ &= \|u - v\|. \end{aligned}$$

Therefore, $\mathcal{L}$ does not satisfy the condition (C).

(2) We show now that $\mathcal{L}$ is a generalized α-Reich-Suzuki nonexpansive mappings. For this, we take $\alpha = \frac{1}{3}$ and consider the following cases:

Case I: Let $u, v < 10$. Then

$$\begin{aligned} \Upsilon(g, h) &= \alpha\|u - \mathcal{L}u\| + \alpha\|v - \mathcal{L}v\| + (1 - 2\alpha)\|u - v\| \\ &= \frac{1}{3}\left|u - \left(\frac{u + 42}{7}\right)\right| + \frac{1}{3}\left|v - \left(\frac{v + 42}{7}\right)\right| + \frac{1}{3}|u - v| \\ &= \frac{1}{3}\left|\frac{6u - 42}{7}\right| + \frac{1}{3}\left|\frac{6v - 42}{7}\right| + \frac{1}{3}|u - v| \\ &\geq \frac{1}{7}|u - v| = \|\mathcal{L}u - \mathcal{L}v\|. \end{aligned}$$

Also,

$$\begin{aligned} \Omega(u, v) &= \alpha\|u - \mathcal{L}v\| + \alpha\|v - \mathcal{L}u\| + (1 - 2\alpha)\|u - v\| \\ &= \frac{1}{3}\left|u - \left(\frac{v + 42}{7}\right)\right| + \frac{1}{3}\left|v - \left(\frac{u + 42}{7}\right)\right| + \frac{1}{3}|u - v| \\ &\geq \frac{1}{7}|u - v| = \|\mathcal{L}u - \mathcal{L}v\|. \end{aligned}$$

Case II: Let $u < 10$ and $v = 10$. Then we have

$$\begin{aligned} \Upsilon(u, v) &= \alpha\|u - \mathcal{L}u\| + \alpha\|v - \mathcal{L}v\| + (1 - 2\alpha)\|u - v\| \\ &= \frac{1}{3}\left|u - \left(\frac{u + 42}{7}\right)\right| + \frac{1}{3}|10 - 6| + \frac{1}{3}|u - 10| \\ &= \frac{1}{3}\left|\frac{6u - 42}{7}\right| + \frac{4}{3} + \frac{1}{3}|u - 10| \\ &\geq \frac{1}{7}|u| = \|\mathcal{L}u - \mathcal{L}v\|. \end{aligned}$$

Also,

$$\begin{aligned}
\Omega(u,v) &= \alpha\|u-\mathcal{L}v\| + \alpha\|v-\mathcal{L}u\| + (1-2\alpha)\|u-v\| \\
&= \frac{1}{3}\,|u-6| + \frac{1}{3}\left|10-\left(\frac{u+42}{7}\right)\right| + \frac{1}{3}|u-10| \\
&= \frac{1}{3}\,|u-6| + \frac{1}{3}\left|\frac{28-u}{7}\right| + \frac{1}{3}|u-10| \\
&\geq \frac{4}{3} + \frac{1}{3}\left|\frac{28-g}{7}\right| \\
&\geq \frac{1}{7}\,|u| = \|\mathcal{L}u-\mathcal{L}v\|.
\end{aligned}$$

Case III: Let $u=10$ and $v<10$. Then we obtain

$$\begin{aligned}
\Upsilon(u,v) &= \alpha\|u-\mathcal{L}u\| + \alpha\|v-\mathcal{L}v\| + (1-2\alpha)\|u-v\| \\
&= \frac{1}{3}\,|10-6| + \frac{1}{3}\left|v-\left(\frac{v+42}{7}\right)\right| + \frac{1}{3}|10-v| \\
&= \frac{4}{3} + \left|\frac{6v-42}{7}\right| + \frac{1}{3}|10-v| \\
&\geq \frac{1}{7}\,|v| = \|\mathcal{L}u-\mathcal{L}v\|.
\end{aligned}$$

Also,

$$\begin{aligned}
\Omega(u,v) &= \alpha\|u-\mathcal{L}v\| + \alpha\|v-\mathcal{L}u\| + (1-2\alpha)\|u-v\| \\
&= \frac{1}{3}\left|10-\left(\frac{v+42}{7}\right)\right| + \frac{1}{3}\,|v-6| + \frac{1}{3}|10-v| \\
&= \frac{1}{3}\left|\frac{28-v}{7}\right| + \frac{1}{3}\,|v-6| + \frac{1}{3}|10-v| \\
&\geq \frac{1}{3}\left|\frac{28-v}{7}\right| + \frac{4}{3} \\
&\geq \frac{1}{7}\,|v| = \|\mathcal{L}u-\mathcal{L}v\|.
\end{aligned}$$

Case IV: Let $u=v=10$. Then we obtain

$$\begin{aligned}
\Upsilon(u,v) &= \alpha\|u-\mathcal{L}u\| + \alpha\|v-\mathcal{L}v\| + (1-2\alpha)\|u-v\| \\
&\geq 0 = \|\mathcal{L}u-\mathcal{L}v\|.
\end{aligned}$$

Also,

$$\begin{aligned}
\Omega(u,v) &= \alpha\|u-\mathcal{L}v\| + \alpha\|v-\mathcal{L}u\| + (1-2\alpha)\|u-v\| \\
&\geq 0 = \|\mathcal{L}u-\mathcal{L}v\|.
\end{aligned}$$

From all the cases above, $\|\mathcal{L}u - \mathcal{L}v\| \leq \max\{\Upsilon(u,v), \Omega(u,v)\}$ for $\alpha = \frac{1}{3}$. Hence, $\mathcal{L}$ is a generalized α-Reich-Suzuki nonexpansive mapping with fixed point $u^* = 7$.

For parameters $\alpha_m = \beta_m = \frac{m}{m+10}$, for all $m \in \mathbb{N}$ and starting point $u_1 = 8$, then with the aid of MATLAB R2015a, we obtain the following Tables (3)–(3). Obviously, from Tables 3–3, it is evident that (1.1) converges faster to 7 than all other considered iterative algorithms.

TABLE 1. Convergence behavior of various iterative algorithms.

g_m	S-Iteration	Abbas-Iteration	CR-Iteration	S*-Iteration	Picard-Ishikawa
g_1	8.0000000000	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_2	7.1418451678	7.0303964883	7.0188179168	7.0026882738	7.0003541140
g_3	7.0201200516	7.0009239465	7.0003541140	7.0000072268	7.0000001254
g_4	7.0028539321	7.0000280847	7.0000066637	7.0000000194	7.0000000000
g_5	7.0004048165	7.0000008537	7.0000001254	7.0000000001	7.0000000000
g_6	7.0000574213	7.0000000259	7.0000000024	7.0000000000	7.0000000000
g_7	7.0000081449	7.0000000008	7.0000000000	7.0000000000	7.0000000000
g_8	7.0000011553	7.0000000000	7.0000000000	7.0000000000	7.0000000000
g_9	7.0000001639	7.0000000000	7.0000000000	7.0000000000	7.0000000000
g_{10}	7.0000000232	7.0000000000	7.0000000000	7.0000000000	7.0000000000

TABLE 2. Convergence behavior of various iterative algorithms.

g_m	Mann	Ishakawa	Picard-Mann	Thakur	Picard-Ishikawa
g_1	8.0000000000	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_2	7.9210528045	7.1317254174	7.0202635954	7.0173515856	7.0003541140
g_3	7.8483382687	7.0173515856	7.0004106133	7.0003010775	7.0000001254
g_4	7.7813643415	7.0022856449	7.0000083205	7.0000052242	7.0000000000
g_5	7.7196778181	7.0003010775	7.0000001686	7.0000000906	7.0000000000
g_6	7.6628612727	7.0000396596	7.0000000034	7.0000000016	7.0000000000
g_7	7.6105302342	7.0000052242	7.0000000001	7.0000000000	7.0000000000
g_8	7.5623305845	7.0000006882	7.0000000000	7.0000000000	7.0000000000
g_9	7.5179361619	7.0000000906	7.0000000000	7.0000000000	7.0000000000
g_{10}	7.4770465545	7.0000000119	7.0000000000	7.0000000000	7.0000000000
g_{11}	7.4393850669	7.0000000016	7.0000000000	7.0000000000	7.0000000000
g_{12}	7.4046968481	7.0000000002	7.0000000000	7.0000000000	7.0000000000
g_{13}	7.3727471669	7.0000000000	7.0000000000	7.0000000000	7.0000000000

4. An application to delay nonlinear integral equations

In this section, we discuss an application to nonlinear Volterra integral equation with delay. Consider the integral equation:

$$u(t) = g(t) + \lambda \int_a^t f(t, s, u(s), u(s-\tau))ds, \;\; t \in I = [a, b] \tag{4.1}$$

with initial function:

$$u(t) = \phi(t), \;\; t \in [a-\tau, a], \tag{4.2}$$

where $\phi \in C[a-\tau, a], \mathbb{R})$, $a, b \in \mathbb{R}$ and $\tau > 0$.

Let $C[a, b]$ denote the set all of continuous functions defined on $[a, b]$ endowed with infinity norm $\|u - v\|_\infty = \max\limits_{a \le t \le b} \|u(t) - v(t)\|$. It is well known that $(C[a, b], \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

Theorem 4.1. *Let $\mathcal{A}$ be a nonempty closed compact subset of a Banach space $\mathcal{B} = (C([a, b], \mathbb{R}), \|\cdot\|_\infty)$. Let $\{u_m\}$ be the iterative method* (1.1) *with $\alpha_m, \beta_m \in (0, 1)$. Let $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ be the operator defined by*

$$\mathcal{L}u(t) = g(t) + \lambda \int_a^t f(t, s, u(s), u(s-\tau))ds, \;\; t \in I = [a, b], \;\; \lambda \ge 0,$$

$$\mathcal{L}u(t) = \phi(t), \;\; t \in [a-\tau, a].$$

Suppose the following assumptions hold:

(a) *$g : I \to \mathbb{R}$ is continuous;*
(b) *$f : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in the sense that there exists a constant $L_f > 0$ such that*

$$|f(t, s, u_1, u_2) - f(t, s, v_1, v_2)| \le L_f(|u_1 - v_1| + |u_2 - v_2|)$$

for all $t, s \in I$, $u_i, v_i \in \mathbb{R}$ $(i = 1, 2)$;
(c) *$2\lambda L_f(b - a) < 1$.*

Then, the problem (4.1)-(4.2) *has a unique solution, say $u^* \in C[a, b]$. Moreover, $\{u_m\}$ converges strongly to u^*.*

Proof. Now, using the contraction principle, we show that $\mathcal{L}$ has a unique fixed point. For $u, v \in \mathcal{A}$, we have

$$|\mathcal{L}u(t) - \mathcal{L}p(t)| = 0, \;\; u, v \in C([a-\tau, a], \mathbb{R}), \;\; t \in [a-\tau, b].$$

Next, for any $t \in I$, we have

$$
\begin{aligned}
&|\mathcal{L}u(t) - \mathcal{L}q(t)| \\
&= |g(t) + \lambda \int_a^t f(t, s, u(s), u(s-\tau))ds \\
&\quad - g(t) - \lambda \int_a^t f(t, s, v(s), v(s-\tau))ds| \\
&\leq \lambda \int_a^t L_f \left\{|u(s) - v(s)| + |u(s-\tau) - v(s-\tau)|\right\} ds \\
&\leq \lambda \int_a^t L_f \left\{\max_{a-\tau \leq s \leq b} |u(s) - v(s)| + \max_{a-\tau \leq s \leq b} |u(s-\tau) - v(s-\tau)|\right\} ds \\
&= \lambda \int_a^t L_f \left\{\|u - v\|_\infty + \|u - v\|_\infty\right\} ds \\
&\leq 2\lambda L_f (b-a)\|u - v\|_\infty.
\end{aligned}
$$

Therefore,

$$
\|\mathcal{L}u - \mathcal{L}v\|_\infty \leq 2\lambda L_f (b-a)\|u - v\|_\infty.
$$

From condition (c), the operator $\mathcal{L}$ is a contraction and using the contraction principle we deduce that the operator $\mathcal{L}$ has a unique fixed point, $F(\mathcal{L}) = \{u^*\}$, that is, the problem (4.1)-(4.2) has a unique solution $u^* \in C[a, b]$.

Next, we show that $\{m_n\}$ converges strongly to u^*. For $u, v \in \mathcal{A}$, we have

$$
\begin{aligned}
&|u(t) - \mathcal{L}v(t)| \\
&\leq |u(t) - \mathcal{L}u(t)| + |\mathcal{L}u(t) - \mathcal{L}v(t)| \\
&= |u(t) - \mathcal{L}u(t)| + |g(t) + \lambda \int_a^t f(t, s, u(s), u(s-\tau))ds \\
&\quad - g(t) - \lambda \int_a^t f(t, s, v(s), v(s-\tau))ds| \\
&\leq |u(t) - \mathcal{L}u(t)| + \lambda \int_a^t L_f \left\{|u(s) - v(s)| + |u(s-\tau) - v(s-\tau)|\right\} ds \\
&\leq \max_{a-\tau \leq t \leq b} |u(t) - \mathcal{L}u(t)| \\
&\quad + \lambda \int_a^t L_f \left\{\max_{a-\tau \leq s \leq b} |u(s) - v(s)| + \max_{a-\tau \leq s \leq b} |u(\alpha(s)) - v(\alpha(s))|\right\} ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \max_{a-\tau\le t\le b} |u(t) - \mathcal{L}u(t)| \\
&\quad + \lambda \int_a^t L_f \left\{ \max_{a-\tau\le d_1\le b} |u(d_1) - v(d_1)| + \max_{a-\tau\le r_1\le b} |u(r_1) - v(r_1)| \right\} ds \\
&= \|u - \mathcal{L}u\|_\infty + \lambda \int_a^t L_f \left\{ \|u - v\|_\infty + \|v - v\|_\infty \right\} ds \\
&\le \|u - \mathcal{L}u\|_\infty + 2\lambda L_f (b-a) \|u - v\|_\infty \\
&\le \|u - \mathcal{L}u\|_\infty + \|u - v\|_\infty .
\end{aligned}
$$

Therefore,

$$
\|u - \mathcal{L}v\|_\infty \le \|u - \mathcal{L}u\|_\infty + \|u - v\|_\infty . \tag{4.3}
$$

From (4.3), it is clear that $\mathcal{L}$ is a mapping satisfying (2.1) with $\left(\frac{3+\alpha}{1-\alpha}\right) \ge 1$ (hence, it is a generalized α-Reich-Suzuki nonexpansive mapping). Taking $\mathcal{A} = \mathcal{G}$ and $\mathcal{B} = \mathcal{M}$, then all the assumptions of Theorem 3.5 are satisfied. Therefore, the sequence $\{u_m\}$ defined by the iterative algorithm (1.1) convergence strongly to the unique solution of the problem (4.1)-(4.2). □

5. Conclusion

In this article, we have used the Picard-Ishikawa method (1.1) to approximate the fixed point of the more generalized α-Reich-Suzuki nonexpansive mappings. We have proved several weak and strong convergence theorems of the considered method under mild assumptions. We have shown numerically that the studied method has a better rate of convergence than some well-known methods for generalized α-nonexpansive mappings. We solve a problem involving nonlinear delay integral equation via Picard-Ishikawa iterative method.

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