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FIXED POINT THEOREMS FOR A PAIR OF (α, η, ψ)-GERAGHTY CONTRACTION TYPE MAPS IN COMPLETE METRIC SPACES

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Abstract. In this paper, we prove the existence of common fixed point for a pair of $\alpha-\eta-\psi$-Geraghty contraction type maps in complete metric spaces using new type of α-admissible. These results extend and generalize some of the previously known results.

1. Introduction and preliminaries

Fixed point theory is one of the out standing subfields of nonlinear functional analysis. It has been used in research area of mathematics and nonlinear sciences. In 1992, Banach [3] proved a fixed point theorem for contraction

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mappings is one of the pivotal results in analysis. This theorem that has been extended and generalized by several authors.

In 1973, Geraghty [6] studied a generalization of Banach contraction mapping principle in a complete metric space. In 2012, Samet Vetro and Vetro [16] introduced a new concept namely (α, ψ)-contractive type mappings and established various fixed point theorems for such class of mappings defined on complete metric spaces. Afterwards, Abdeljawad [1] introduced a pair of α-admissible mappings and obtained fixed point and common fixed point theorems. For more works on α-admissible, we refer [12, 15].

In 2013, Cho, Bae and Karapinar [5] defined the concept of α-Geraghty contraction type maps in a metric space and proved the existence and uniqueness of a fixed point for the mappings satisfying this conditions. Recently, karapinar [11] defined the concept of (α, ψ)-Geraghty contraction type mappings. For more details we refer [2, 4, 10, 13]. Hussain and Adheel [8] and Hussain *et al.* [7] introduced the new contractive-type mapping called θ-contraction and generalized the Banach contraction principle. Balajee *et al.* [14] established a new class category of nonexpansive mappings in a metric space which is wider than the class category of mappings satisfying contractive condition. Jagannadha Rao *et al.* [9] discussed the existence of best proximity points of certain mappings via simulation functions in the frame of complete metric-like spaces.

In this paper, we prove the existence of common fixed point theorem for a pair of (α, η, ψ)-Geraghty contraction type maps in complete metric spaces using new type of α-admissible.

In this section, we give the definitions which we use in the later development.

Definition 1.1. ([13]) Let X be a nonempty set. A function $\alpha : X \times X \to \mathbb{R}^+$ is said to be triangular function if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$ for $x, y, z \in X$.

Definition 1.2. ([13]) Let X be a nonempty set. Let $f : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$. We say that f is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$.

Definition 1.3. ([16]) Let X be a nonempty set. Let $f, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$. We say that f and g are triangular α-admissible if

(i) α is triangular,
(ii) $\alpha(x, y) \geq 1$ implies $\alpha(fx, gy) \geq 1$ and $\alpha(gy, fx) \geq 1$.

Definition 1.4. ([16]) Let X be a nonempty set. Let $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. We say that α is η triangular if $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ implies $\alpha(x, z) \geq \eta(x, z)$ for $x, y, z \in X$.

Definition 1.5. ([16]) Let X be a nonempty set. Let $f : X \to X$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. We say that f is α-admissible with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha(fx, fy) \geq \eta(fx, fy)$.

Definition 1.6. ([16]) Let X be a nonempty set. Let $f, g : X \to X$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. We say that f and g are α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha(fx, gy) \geq \eta(fx, gy)$ and $\alpha(gy, fx) \geq \eta(gy, fx)$.

Lemma 1.7. *Let $f, g : X \to X$ be triangular α-admissible maps. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$. Define sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n = 0, 1, 2, \cdots$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$, that is, $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_1, x_0) \geq 1$. By the definition, we have $\alpha(fx_0, gx_1) \geq 1$ and $\alpha(gx_1, fx_0) \geq 1$, that is, $\alpha(x_1, x_2) \geq 1$ and $\alpha(x_2, x_1) \geq 1$. Now $\alpha(x_2, x_1) \geq 1$ implies $\alpha(fx_2, gx_1) \geq 1$ and $\alpha(gx_1, fx_2) \geq 1$, that is, $\alpha(x_3, x_2) \geq 1$ and $\alpha(x_2, x_3) \geq 1$.

By induction it can be proved that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n.

Now $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_{n+2}) \geq 1$ implies $\alpha(x_n, x_{n+2}) \geq 1$. By induction it can be shown that $\alpha(x_n, x_m) \geq 1$ for $n < m$ and similarly, we can shown that $\alpha(x_m, x_n) \geq 1$ for $m > n$. This completes the proof. □

Lemma 1.8. *If $\{P_n\}$ is a sequence in $\mathbb{R}^+$ such that $\psi(P_{n+1}) \leq \beta(\psi(P_n))\psi(P_n)$. Then $\psi(P_n) \to 0$ as $n \to \infty$.*

Proof. Assume that $\{\psi(P_n)\}$ converges to s and $\{P_n\}$ converges to r (say).

Case (i): $\beta(\psi(P_n)) \leq \lambda < 1$ (bounded by a quantity less than 1).

Now $\psi(P_{n+1}) < \lambda\psi(P_n)$ for large $n \geq N$ implies that

$$\psi(P_{n+2}) < \lambda\psi(P_{n+1}) < \lambda^2\psi(P_n).$$

By induction, we get $\psi(P_{n+k}) < \lambda^k\psi(P_n)$. Now allowing $k \to \infty$, we have $\psi(P_{n+k}) \to 0$. Therefore, $s = 0$ and $\psi(r) \leq s = 0$, implies that $\psi(r) = 0$, so that $r = 0$.

Case (ii): Suppose $\overline{\lim}\beta(\psi(P_n)) = 1$. Then there exists n_k such that

$$\lim \beta(\psi(P_{n_k})) = 1,$$

which implies that $\psi(P_{n_k})) \to 0$. Hence $s = 0$. But $0 \leq \psi(r) \leq s = 0$, so that $\psi(r) = 0$. Hence $r = 0$. □

We write $\Gamma = \{\beta : [0,\infty) \to [0,1) |\ \beta(t_n) \to 1$ implies $t_n \to 0\}$ and $\Psi = \{\psi : [0,\infty) \to [0,\infty) | \psi$ is continuous, monotonically increasing and $\psi(0) = 0\}$.

Theorem 1.9. ([6]) *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. If f satisfies the following inequality:*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

for any $x, y \in X$, where $\beta \in \Gamma$, then f has a unique fixed point.

Definition 1.10. ([10]) Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Two mappings $f, g : X \to X$ are called generalized (α, ψ)-Geraghty contraction type mappings if there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that for all $x, y \in X$,

$$\alpha(x, y)\psi(d(fx, gy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, fx), d(y, gy), \frac{d(y,fx)+d(x,gy)}{2}\}$.

Definition 1.11. ([10]) Let (X, d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. Two mappings $f, g : X \to X$ are called generalized (α, η, ψ)-Geraghty contraction type mappings if there exist $\beta \in \Gamma$ and $\psi \in \Psi$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow \psi(d(fx, gy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(y, fx) + d(x, gy)}{2}\right\}.$$

2. Main results

In this section, we prove the existence of common fixed point involving (α, β, ψ) and η functions in complete metric spaces.

Theorem 2.1. *Let (X, d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Let $f, g : X \to X$ be two mappings. Suppose that the following conditions are satisfied:*

(i) *f and g is a generalized $\alpha - \psi$-Geraghty type mappings,*
(ii) *f and g is triangular α-admissible,*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$,*
(iv) *either f or g is continuous.*

Then f and g have common fixed point.

Proof. Let $x_1 \in X$ such that $x_1 = fx_0$ and $x_2 = gx_1$. By induction, we define a sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n = 0, 1, 2, \cdots$. By assumption $\alpha(x_0, x_1) \geq 1$ and f and g are triangular α-admissible and by Lemma 1.7, we have $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Suppose $x_{2n} = x_{2n+1}$ for some n, that is, $x_{2n} = fx_{2n}$. Therefore, x_{2n} is a fixed point of f.

Now, we show that $x_{2n} = x$ (say) is a fixed point of g. Consider

$$\psi(d(x_{2n+1}, x_{2n+2})) = \psi(d(fx_{2n}, gfx_{2n})) = \psi(d(fx, gfx)),$$

then

$$\psi(d(fx, gfx)) \leq \alpha(x, fx)\psi(d(fx, gfx)) \leq \beta(\psi(M(d(x, fx)))).\psi(M(d(x, fx))),$$

where

$$\begin{aligned} M(x, fx) &= \max\{d(x, fx), d(x, fx), d(fx, gfx), \frac{d(fx, fx) + d(x, gfx)}{2}\} \\ &= \max\{0, 0, d(x, gx), \frac{d(x, gx)}{2}\} = d(x, gx). \end{aligned}$$

Therefore, $\psi(d(x, gx)) \leq \beta(\psi(d(x, gx)))\psi(d(x, gx)) < \psi(d(x, gx))$, which is a contradiction. Therefore, $x = gx$. Hence, x is a common fixed point of f and g.

Assume that $x_n \neq x_{n+1}$ for all n. Now

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \alpha(x_{2n}, x_{2n+1})\psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \beta(\psi(M(x_{2n}, x_{2n+1})))\psi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \\ &\qquad \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Therefore, from (2.1), we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &\leq \beta(\psi(M(x_{2n}, x_{2n+1})))\psi(M(x_{2n}, x_{2n+1})) \\ &\leq \beta(\psi(d(x_{2n}, x_{2n+1})))\psi(d(x_{2n}, x_{2n+1})) \\ &< \psi(d(x_{2n}, x_{2n+1})) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\psi(d(x_{2n+2}, x_{2n+3})) &\leq \psi(d(gx_{2n+1}, fx_{2n+2}))\\ &= \psi(d(fx_{2n+2}, gx_{2n+1}))\\ &\leq \alpha(x_{2n+2}, x_{2n+1})\psi(d(fx_{2n+2}, gx_{2n+1}))\\ &\leq \beta(\psi(M(x_{2n+2}, x_{2n+1})))\psi(M(x_{2n+2}, x_{2n+1})),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}M(x_{2n+2}, x_{2n+1}) &= \max\{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, fx_{2n+2}), d(x_{2n+1}, gx_{2n+1}),\\ &\qquad \frac{d(x_{2n+2}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n+2})}{2}\}\\ &= \max\{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}),\\ &\qquad \frac{d(x_{2n+2}, x_{2n+2}) + d(x_{2n+1}, x_{2n+3})}{2}\}\\ &= \max\{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3})\}\end{aligned}$$

and

$$\begin{aligned}\psi(d(x_{2n+2}, x_{2n+3})) &\leq \beta(\psi(d(x_{2n+2}, x_{2n+1})))\psi(d(x_{2n+2}, x_{2n+1}))\\ &\leq \beta(\psi(d(x_{2n}, x_{2n+1})))\psi(d(x_{2n}, x_{2n+1}))\\ &< \psi(d(x_{2n+2}, x_{2n+1})).\end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we have $\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore,

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence $\{\psi(d(x_{n+1}, x_{n+2}))\}$ is decreasing sequence, and it converges to say $s(\geq 0)$. And so, $\{d(x_{n+1}, x_{n+2})\}$ is decreasing sequence, and it converges to say $r(\geq 0)$. Now,

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1}))\\ &\leq \beta(\psi(d(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})).\end{aligned}$$

By Lemma 1.8, we have $s = 0$ and hence $r = 0$.

Now, we show that the sequence $\{x_n\}$ is Cauchy. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that for all $k > 0$, we have $m_k > n_k > k$, $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k-1}, x_{n_k}) < \epsilon$.

Suppose n_k is even and m_k is odd for infinitely many n. Now

$$\begin{aligned}\psi(d(x_{n_k+1}, x_{m_k+2})) &= \psi(d(fx_{n_k}, gx_{m_k+1}))\\ &\leq \alpha(x_{n_k}, x_{m_k+1})\psi(d(fx_{n_k}, gx_{m_k+1}))\\ &\leq \beta(\psi(M(x_{n_k}, x_{m_k+1})))\psi(M(x_{n_k}, x_{m_k+1})).\end{aligned}$$

On letting $k \to \infty$, we have

$$\begin{aligned}\psi(\epsilon) &\le \underline{\lim}\beta(\psi(M(x_{n_k}, x_{m_k+1})))\psi(\epsilon)\\ &\le \overline{\lim}\beta(\psi(M(x_{n_k}, x_{m_k+1})))\psi(\epsilon)\\ &\le \psi(\epsilon).\end{aligned}$$

Therefore, the limit exists and equal to 1. Hence $\psi(M(x_{n_k}, x_{m_k+1})) \to 0$, implies that $\psi(\epsilon) = 0$. Hence we have $\epsilon = 0$ which is a contradiction.

Similarly, we can proceed in the above manner for other cases. Therefore, sequence $\{x_n\}$ is Cauchy.

Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ implies that $x_{2n+1} \to x^*$ and $x_{2n+2} \to x^*$. Since f and g are continuous, we get $x_{2n+1} = fx_{2n} \to fx^*$ and $x_{2n+2} = gx_{2n+1} \to gx^*$. Hence by uniqueness of limit, we have $fx^* = x^*$ and $gx^* = x^*$. Therefore, $fx^* = gx^* = x^*$. Hence f and g have a common fixed point x^* in X. □

Theorem 2.2. *Suppose hypothesis of Theorem* 2.1 *except* (*iv*) *holds. Further assume that* $\{z_n\}$ *is a sequence in* X *such that* $\alpha(z_n, z_{n+1}) \ge 1$ *and* $\alpha(z_{n+1}, z_n) \ge 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $z_n \to z^*$ *as* $n \to \infty$, *then there exists a subsequence* $\{z_{n_k}\}$ *of* $\{z_n\}$ *such that* $\alpha(z_{n_k}, z^*) \ge 1$ *and* $\alpha(z^*, z_{n_k}) \ge 1$ *for all* k. *Then* f *and* g *have common fixed point.*

Proof. Following the proof of Theorem 2.1, we get the sequence $\{x_n\}$ is Cauchy and hence convergent to x^* (upto this stage we did not use the continuity of either f or g). Also we have shown that $\alpha(x_n, x_{n+1}) \ge 1$ and $\alpha(x_{n+1}, x_n) \ge 1$ for $n = 0, 1, 2, \cdots$.

Now from our assumption there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \ge 1$ and $\alpha(z^*, z_{n_k}) \ge 1$ for all k. Then, there exists a subsequence $\{n_{k_l}\}$ of $\{n_k\}$ such that either n_{k_l} even for all l or n_{k_l} odd for all l. Without loss of generality, we may suppose that n_{k_l} is even for all l. Hence n_{k_l} can be written as $n_{k_l} = 2m_l$. Since $\alpha(x_{2m_l}, x^*) \ge 1$, suppose $gx^* \ne x^*$. Now

$$\begin{aligned}\psi(d(x_{2m_l+1}, gx^*)) &= \psi(d(fx_{2m_l}, gx^*))\\ &\le \alpha(x_{2m_l}, x^*)\psi(d(fx_{2m_l}, gx^*))\\ &\le \beta(\psi(M(x_{2m_l}, x^*)))\psi(M(x_{2m_l}, x^*)),\end{aligned}$$

where

$$\begin{aligned}M(x_{2m_l}, x^*) = \max\Big\{&d(x_{2m_l}, x^*), d(x_{2m_l}, fx_{2m_l}), d(x^*, gx^*),\\ &\frac{1}{2}\Big[d(x_{2m_l}, gx^*) + d(x^*, fx_{2m_l})\Big]\Big\}.\end{aligned}$$

Therefore, $M(x_{2m_l}, x^*) = d(x^*, gx^*)$ for large l. Hence, for large l,

$$\psi(d(x_{2m_l+1}, gx^*)) \le \beta(\psi(d(x^*, gx^*)))\psi(d(x^*, gx^*)).$$

On letting $l \to \infty$, we have

$$\begin{aligned}\psi(d(x^*, gx^*)) &\leq \beta(\psi(d(x^*, gx^*)))\psi(d(x^*, gx^*)) \\ &< \psi(d(x^*, gx^*)),\end{aligned}$$

which is a contradiction. Therefore, $x^* = gx^*$. Similarly, we can show that $x^* = fx^*$. Which shows that x^* is the common fixed point of f and g. □

Theorem 2.3. *Let (X, d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Let $f, g : X \to X$ be two mappings. Suppose that the following conditions are satisfied:*

(i) *f and g are generalized (α, η, ψ)-Geraghty type mappings,*
(ii) *f and g are triangular α-admissible with respect to η,*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$ and $\alpha(fx_0, x_0) \geq \eta(fx_0, x_0)$,*
(iv) *either f or g is continuous.*

Then f and g have common fixed point.

Proof. Let $x_1 \in X$ such that $x_1 = fx_0$ and $x_2 = gx_1$. Then, by induction, we define a sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n = 0, 1, 2, \cdots$. By assumption $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$, that is, $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and f and g are triangular α-admissible with respect to η, we have $\alpha(fx_0, gx_1) \geq \eta(fx_0, gx_1)$ and $\alpha(gx_1, fx_0) \geq \eta(gx_1, fx_0)$, that is, $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $\alpha(x_2, x_1) \geq \eta(x_2, x_1)$. By induction, we get

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

and

$$\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)$$

for every n. Assume that $x_n \neq x_{n+1}$ for all n. Now

$$\begin{aligned}\psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \beta(\psi(M(x_{2n}, x_{2n+1})))\psi(M(x_{2n}, x_{2n+1})),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \\ &\qquad \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.\end{aligned}$$

Therefore, from (2.5), we have

$$\begin{aligned}\psi(d(x_{2n+1},x_{2n+2})) &\leq \beta(\psi(d(x_{2n},x_{2n+1})))\psi(d(x_{2n},x_{2n+1}))\\ &< \psi(d(x_{2n},x_{2n+1}))\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\psi(d(x_{2n+2},x_{2n+3})) &= \psi(d(gx_{2n+1},fx_{2n+2}))\\ &= \psi(d(fx_{2n+2},gx_{2n+1}))\\ &\leq \beta(\psi(M(x_{2n+2},x_{2n+1})))\psi(M(x_{2n+2},x_{2n+1})),\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}M(x_{2n+2},x_{2n+1}) =& \max\{d(x_{2n+2},x_{2n+1}),d(x_{2n+2},fx_{2n+2}),d(x_{2n+1},gx_{2n+1}),\\ &\frac{d(x_{2n+1},fx_{2n+2})+d(x_{2n+2},gx_{2n+1})}{2}\}\\ =& \max\{d(x_{2n+2},x_{2n+1}),d(x_{2n+2},x_{2n+3}),d(x_{2n+1},x_{2n+2}),\\ &\frac{d(x_{2n+1},x_{2n+3})+d(x_{2n+2},x_{2n+2})}{2}\}\\ =& \max\{d(x_{2n+2},x_{2n+1}),d(x_{2n+2},x_{2n+3})\}.\end{aligned}$$

Therefore, from (2.7), we have

$$\begin{aligned}\psi(d(x_{2n+2},x_{2n+3})) &\leq \beta(\psi(d(x_{2n+1},x_{2n+2})))\psi(d(x_{2n+1},x_{2n+2}))\\ &< \psi(d(x_{2n+1},x_{2n+2})).\end{aligned} \tag{2.8}$$

From (2.6) and (2.8), we have, for every n,

$$\psi(d(x_{2n+1},x_{2n+2})) < \psi(d(x_{2n},x_{2n+1})). \tag{2.9}$$

Hence, $\{\psi(d(x_{n+1},x_{n+2}))\}$ is a decreasing sequence, so converges to say $s(\geq 0)$. Hence, $\{d(x_{n+1},x_{n+2})\}$ is a decreasing sequence, so converges to say $r(\geq 0)$. Now,

$$\begin{aligned}\psi(d(x_{n+1},x_{n+2})) &\leq \beta(\psi(M(x_n,x_{n+1})))\psi(M(x_n,x_{n+1}))\\ &\leq \beta(\psi(d(x_n,x_{n+1})))\psi(d(x_n,x_{n+1})).\end{aligned}$$

By Lemma 1.8, we have $s=0$ and hence $r=0$.

Now we show that the sequence $\{x_n\}$ is Cauchy. Suppose $\{x_n\}$ is not Cauchy. Then, there exists $\epsilon>0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that for all $k>0$, we have $m_k>n_k>k$, $d(x_{m_k},x_{n_k})\geq\epsilon$ and $d(x_{m_k-1},x_{n_k})<\epsilon$. Suppose n_k is even and m_k is odd for infinitely many n. Now

$$\begin{aligned}\psi(d(x_{n_k+1},x_{m_k+2})) &= \psi(d(fx_{n_k},gx_{m_k+1}))\\ &\leq \beta(\psi(M(x_{n_k},x_{m_k+1})))\psi(M(x_{n_k},x_{m_k+1})).\end{aligned}$$

On letting $k \to \infty$, we have

$$\begin{aligned}\psi(\epsilon) &\leq \underline{\lim}\beta(\psi(M(x_{n_k}, x_{m_k+1})))\psi(\epsilon)\\ &\leq \overline{\lim}\beta(\psi(M(x_{n_k}, x_{m_k+1})))\psi(\epsilon)\\ &\leq \psi(\epsilon).\end{aligned}$$

Therefore, the limit exists and equal to 1. Hence $\psi(M(x_{n_k}, x_{m_k+1})) \to 0$, implies that $\psi(\epsilon) = 0$. Hence $\epsilon = 0$, which is a contradiction.

Similarly, we can proceed in the above manner for other cases. Therefore sequence $\{x_n\}$ is Cauchy. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ implies that $x_{2n+1} \to x^*$ and $x_{2n+2} \to x^*$. Since f and g are continuous, we get $x_{2n+1} = fx_{2n} \to fx^*$ and $x_{2n+2} = gx_{2n+1} \to gx^*$. Hence by uniqueness of limit, we have $fx^* = x^*$ and $gx^* = x^*$. Therefore $fx^* = gx^* = x^*$. Hence f and g have a common fixed point x^* in X. □

Theorem 2.4. *Suppose hypothesis of Theorem* 2.3 *except* (*iv*) *holds. Further assume that* $\{z_n\}$ *is a sequence in* X *such that* $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ *and* $\alpha(z_{n+1}, z_n) \geq \eta(z_{n+1}, z_n)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $z_n \to z^*$ *as* $n \to \infty$, *then there exists a subsequence* $\{z_{n_k}\}$ *of* $\{z_n\}$ *such that* $\alpha(z_{n_k}, z^*) \geq \eta(z_{n_k}, z^*)$ *and* $\alpha(z^*, z_{n_k}) \geq \eta(z^*, z_{n_k})$ *for all* k. *Then* f *and* g *have common fixed point.*

Proof. Following the proof of Theorem 2.3 we get the sequence $\{x_n\}$ is Cauchy and hence convergent to x^*. Also we have shown that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for $n = 0, 1, 2, \cdots$.

Now from our assumption there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all k. There exists subsequence $\{n_{k_l}\}$ of $\{n_k\}$ such that either n_{k_l} is even for all l or n_{k_l} is odd for all l. Without loss of generality, we may suppose that n_{k_l} is even for all l. Then n_{k_l} can be written as $n_{k_l} = 2m_l$. Since $\alpha(x_{2m_l}, x^*) \geq \eta(x_{2m_l}, x^*)$, suppose $gx^* \neq x^*$.

Now

$$\begin{aligned}\psi(d(x_{2m_l+1}, gx^*)) &= \psi(d(fx_{2m_l}, gx^*))\\ &\leq \beta(\psi(M(x_{2m_l}, x^*)))\psi(M(x_{2m_l}, x^*)),\end{aligned}$$

where

$$\begin{aligned}M(x_{2m_l}, x^*) = \max\Big\{&d(x_{2m_l}, x^*), d(x_{2m_l}, fx_{2m_l}), d(x^*, gx^*),\\ &\frac{1}{2}[d(x_{2m_l}, gx^*) + d(x^*, fx_{2m_l})]\Big\}.\end{aligned}$$

Hence $M(x_{2m_l}, x^*) = d(x^*, gx^*)$ for large l. Therefore, for large l,

$$\psi(d(x_{2m_l+1}, gx^*)) \leq \beta(\psi(d(x^*, gx^*)))\psi(d(x^*, gx^*)).$$

On letting $l \to \infty$, we have

$$\begin{aligned}\psi(d(x^*, gx^*)) &\leq \beta(\psi(d(x^*, gx^*)))\psi(d(x^*, gx^*))\\ &< \psi(d(x^*, gx^*)),\end{aligned}$$

which is a contradiction. $x^* = gx^*$. Similarly, we can show that $x^* = fx^*$. Thus $x^* = fx^* = gx^*$. Which shows that x^* is the common fixed point of f and g. This completes the proof. □

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