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MULTIPLE NONTRIVIAL SOLUTIONS FOR CRITICAL p-KIRCHHOFF TYPE PROBLEMS IN $\mathbb{R}^N$

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Abstract. In this paper, we study the existence and multiplicity of nontrivial solutions for a p-Kirchhoff equation involving critical Sobolev-Hardy exponent by using variational methods and we need to estimate the energy levels.

1. Introduction

This paper deals with the existence and multiplicity of nontrivial solutions to the following Kirchhoff problem

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\left(a\left\|u\right\|^p+b\right)\Delta_p u=\left|x\right|^{-s}u^{p^*(s)-1}+\lambda f(x)u^{q-1} & \text{in } \mathbb{R}^N \\ u\geq 0,\ u\in W^{1,p}\left(\mathbb{R}^N\right), & \end{cases}$$

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where a and b are two positive constants, $1 < p < N$, $1 < q < p$, λ is a positive parameter, $f \not\equiv 0$, Δ_p is the p-Laplacian operator, that is,

$$\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right),$$

$0 \leq s < p$, $p^*(s) = p(N-s)/(N-p)$ is the critical Sobolev-Hardy exponent and $\|.\|$ is the usual norm in $W^{1,p}\left(\mathbb{R}^N\right)$ given by

$$\|u\|^p = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Kirchhoff type problems are often referred to as being nonlocal because of the presence of the term $\int_{\mathbb{R}^N} |\nabla u|^p dx$ which implies that the equation in $(\mathcal{P}_\lambda)$ is no longer a pointwise identity. It is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u = g(x,u),$$

where $\Omega \subset \mathbb{R}^N$, u denotes the displacement, $g(x,u)$ is the external force and b is the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string.

In recent years, Kirchhoff type problems in bounded or unbounded domain have been studied in many papers by using variational methods. Some interesting studies can be found in [2, 4, 7, 8, 9, 11, 12, 14]. This problems in the whole space $\mathbb{R}^N$ considered in general without the critical exponent, when the difficulty is due to the lack of compactness embedding from $W^{1,p}(\mathbb{R}^N)$ into the space $L^r\left(\mathbb{R}^N\right)$ for $1 < r < p^*(0)$. In this subcritical case, many authors considering the following equation

$$(\mathcal{P}_V) \qquad -\left(a\|u\| + b\right)\Delta_p u + V(x)u = h(x,u) \quad \text{in } \mathbb{R}^N,$$

where $1 < p < N$, $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and $h \in C\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ is subcritical. In such problems, some conditions are imposed on the weight function $V(x)$ which are key points for recovering the compactness of Sobolev embedding. See for example [8] and [14].

On the other hand, the problem $(\mathcal{P}_\lambda)$ without the nonlocal term $a\|u\|$ is treated by Alves [1], he proves the existence of two nonnegative solutions for $(\mathcal{P}_\lambda)$ where $a = s = 0$, $b = 1$ and f is a nonnegative function.

A natural and interesting question is whether results concerning the solutions of problem $(\mathcal{P}_\lambda)$ with $a = s = 0$ remain valid for $a \neq 0$ and $s \neq 0$. Our answer is affirmative, but the adaptation to the procedure to our problem is not trivial at all, since the appearance of nonlocal term. In this context, we need more delicate estimates. We are concerned in finding conditions on N,

s, f and λ for which problem $(\mathcal{P}_\lambda)$ possesses multiple nontrivial solutions by mean of variational methods and concentration compactness. To the best of our knowledge, there is no result on the multiple nontrivial solutions to the critical problem $(\mathcal{P}_\lambda)$ in $\mathbb{R}^N$.

Before stating our results, recall that, the best Sobolev-Hardy constant

$$S_s = \inf_{u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}} \frac{\|u\|^p}{\|u\|^p_{p^*(s)}}$$

with

$$\|u\|_{p^*(s)} = \left(\int_{\mathbb{R}^N} |x|^{-s} |u|^{p^*(s)} \, dx \right)^{1/p^*(s)}$$

is attained in $\mathbb{R}^N$ by a function $U(x)$, see [13]. We introduce the following condition on f.

(H_f) $\quad$ $f \geq 0$ and $f \in L^{q_0}\left(\mathbb{R}^N\right)$ with $q_0 = pN/\left[(p-q)\,N + qp\right]$.

2. Main results

In this paper, we use the following notation: B_r is the ball centered at 0 and of radius r, $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence, $u^{\pm} = \max(\pm u,\ 0)$ and $\circ_n(1)$ denotes $\circ_n(1) \to 0$ as $n \to \infty$.

The starting point of the variational approach to our problem is the following Sobolev-Hardy inequality, which is essentially due to Caffarelli, Kohn and Nirenberg [6]. Assume that $1 < p < N$ and $0 \leq s < p$. Then

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(s)}}{|x|^s} dx \right)^{1/p^*(s)} \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p} \text{ for all } u \in C_0^\infty\left(\mathbb{R}^N\right), \quad (2.1)$$

for some positive constant C. Since our approach is variational, we define the functional I_λ by

$$I_\lambda(u) = \frac{a}{2p}\|u\|^{2p} + \frac{b}{p}\|u\|^p - \frac{1}{p^*(s)} \int_{\mathbb{R}^N} |x|^{-s} \left(u^+\right)^{p^*(s)} dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} f(x) \left(u^+\right)^q dx$$

for all $u \in W^{1,p}\left(\mathbb{R}^N\right)$.

Using (H_f), it is clear that I_λ is well defined in $W^{1,p}\left(\mathbb{R}^N\right)$ and belongs to $C^1\left(W^{1,p}\left(\mathbb{R}^N\right),\ \mathbb{R}\right)$.

Definition 2.1. Let $u \in W^{1,p}\left(\mathbb{R}^N\right) \setminus \{0\}$. Then u is said to be a weak solution of problem $(\mathcal{P}_\lambda)$ if it satisfies $u \geq 0$ and

$$\left(a\|u\|^p + b\right)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v\, dx - \int_{\mathbb{R}^N} |x|^{-s}\left(u^+\right)^{p^*(s)-1} v\, dx - \lambda \int_{\mathbb{R}^N} f(x)\left(u^+\right)^{q-1} v\, dx = 0$$

for all $v \in W^{1,p}\left(\mathbb{R}^N\right)$.

We need the following lemmas.

Lemma 2.2. *Assume that $a, b > 0$, $1 < p < N \leq 2p$, $1 < q < p$, $0 \leq s \leq 2p - N$ and f satisfies (H_f). Then there exist positive numbers Λ_1, r_1 and δ_1 such that for all $\lambda \in (0, \Lambda_1)$ we have*

(i) $I_\lambda(u) \geq \delta_1 > 0$, *for all $u \in W^{1,p}\left(\mathbb{R}^N\right)$ with $\|u\| = r_1$,*

(ii) $I_\lambda(u) \geq -\frac{p-q}{p}\left[\left(\frac{2q}{b}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS_0^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)}$ *for all $u \in B_{r_1}$.*

Proof. Let $u \in W^{1,p}\left(\mathbb{R}^N\right) \setminus \{0\}$. Then by Sobolev-Hardy and Hölder inequalities, we have

$$I_\lambda(u) \geq \frac{a}{2p}\|u\|^{2p} + \frac{b}{p}\|u\|^p - \frac{S_s^{-p^*(s)/p}}{p^*(s)}\|u\|^{p^*(s)} - \frac{\lambda}{q}S_0^{-q/p}\|f\|_{q_0}\|u\|^q.$$

Let $\eta > 0$, $r = \|u\|$ and

$$h(r) = \frac{a}{2p}r^{2p} + \frac{b}{p}r^p - \frac{S_s^{-p^*(s)/p}}{p^*(s)}r^{p^*(s)} - \frac{\lambda}{q}S_0^{-q/p}\|f\|_{q_0} r^q.$$

Then

$$\begin{aligned}\frac{\lambda}{q}S_0^{-q/p}\|f\|_{q_0} r^q &= \left[\left(\frac{\eta p}{q}\right)^{\frac{q}{p}} r^q\right]\left[\left(\frac{\eta p}{q}\right)^{-\frac{q}{p}} \frac{\lambda}{q}S_0^{-q/p}\|f\|_{q_0}\right] \\ &\leq \frac{\eta p}{q}r^p + \frac{p-q}{p}\left[\left(\frac{q}{p\eta}\right)^{\frac{q}{p}} \frac{S_0^{-q/p}}{q}\|f\|_{q_0}\right]^{p/(p-q)} \lambda^{p/(p-q)}.\end{aligned}$$

Therefore,

$$h(r) \geq \left(\frac{b}{p} - \eta\right) r^p - \frac{S_s^{-\frac{p^*(s)}{p}}}{p^*(s)} r^{p^*(s)} - \frac{p-q}{p}\left[\left(\frac{q}{p\eta}\right)^{\frac{q}{p}} \frac{S_0^{-\frac{q}{p}}}{q}\|f\|_{q_0}\right]^{\frac{p}{p-q}} \lambda^{\frac{p}{p-q}}.$$

Choosing $\eta = b/2p$, we get

$$h(r) \geq \frac{b}{2p} r^p - \frac{S_s^{-\frac{p^*(s)}{p}}}{p^*(s)} r^{p^*(s)} - \frac{p-q}{p} \left[\left(\frac{2q}{b} \right)^{\frac{q}{p}} \frac{S_0^{-\frac{q}{p}}}{q} \|f\|_{q_0} \right]^{\frac{p}{p-q}} \lambda^{\frac{p}{p-q}}.$$

Easy computations show that

$$\begin{aligned} \max_{r \geq 0} h(r) &= h\left(\left[\frac{b}{2} S_s^{\frac{p^*(s)}{p}} \right]^{1/(p^*(s)-p)} \right) \\ &= \frac{1}{N} \left(\frac{b}{2} S_s \right)^{\frac{N}{p}} - \frac{p-q}{p} \left[\left(\frac{2q}{b} \right)^{\frac{q}{p}} \frac{S_0^{-\frac{q}{p}}}{q} \|f\|_{q_0} \right]^{\frac{p}{p-q}} \lambda^{\frac{p}{p-q}}. \end{aligned}$$

Taking

$$r_1 = \left[\frac{b}{2} S_s^{\frac{p^*(s)}{p}} \right]^{1/(p^*(s)-p)}, \ \delta_1 = \frac{1}{2N} \left(\frac{b}{2} S_s \right)^{\frac{N}{p}}$$

and

$$\Lambda_1 = \frac{1}{q \|f\|_{q_0}} \left(\frac{b}{2q} S_0 \right)^{\frac{q}{p}} \left(\frac{p}{2N(p-q)} \left(\frac{b}{2} S_s \right)^{\frac{N}{p}} \right)^{\frac{p-q}{p}}.$$

Then the conclusion holds. □

Lemma 2.3. *Assume that $a, b > 0$, $1 < p < N \leq 2p$, $1 < q < p$, $0 \leq s \leq 2p - N$ and f satisfies (H_f). If $\{u_n\}$ is a $(PS)_c$ sequence of I_λ, then $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ for some u with $I'_\lambda(u) = 0$.*

Proof. We have

$$c + \circ_n(1) = I_\lambda(u_n) \quad \text{and} \quad \circ_n(1) = \left\langle I'_\lambda(u_n), u_n \right\rangle, \tag{2.2}$$

this implies that

$$\begin{aligned} c + \circ_n(1) &= I_\lambda(u_n) - \frac{1}{p^*(s)} \left\langle I'_\lambda(u_n), u_n \right\rangle \\ &\geq b \left(\frac{1}{p} - \frac{1}{p^*(s)} \right) \|u_n\|^p - \lambda \left(\frac{1}{q} - p^*(s) \right) S_0^{-q/p} \|f\|_{q_0} \|u_n\|^q. \end{aligned}$$

Then $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Up to a subsequence if necessary, we obtain

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W^{1,p}(\mathbb{R}^N), \ u_n \rightharpoonup u \text{ in } L^{p^*(s)}\left(\mathbb{R}^N, \ |x|^{-s}\right), \\ u_n &\to u \text{ a.e. and } \nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N, \end{aligned}$$

then $\langle I'_\lambda(u_n), v \rangle = 0$ for all $v \in C_0^\infty(\mathbb{R}^N)$, which means that $I'_\lambda(u) = 0$. □

Lemma 2.4. *Assume that* $0 < a < S_s^{-2}$, $b > 0$, $1 < p < N \leq 2p$, $s = 2p - N$, $1 < q < p$ *and* f *satisfies* (H_f). *Let* $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ *be a* $(PS)_c$ *sequence for* I_λ *for some* $c \in \mathbb{R}$ *such that* $u_n \rightharpoonup u$ *in* $W^{1,p}(\mathbb{R}^N)$. *Then*

$$\text{either } u_n \to u \quad \text{or} \quad c \geq I_\lambda(u) + \frac{b^2}{2p\left(S_s^{-2} - a\right)}.$$

Proof. As $s = 2p - N$, then $p^*(s) = 2p$. By the proof of Lemma 2.3 we have $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Then by (H_f) we get

$$\int_{\mathbb{R}^N} f(x)\left(u_n^+\right)^q dx \to \int_{\mathbb{R}^N} f(x)\left(u^+\right)^q dx. \tag{2.3}$$

Furthermore, if we write $v_n = u_n - u$; we derive $v_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, and by using Brézis-Lieb Lemma [5] we have

$$\|u_n\|^p = \|v_n\|^p + \|u\|^p + o_n(1) \tag{2.4}$$

and

$$\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s} dx = \int_{\mathbb{R}^N} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx + \int_{\mathbb{R}^N} \frac{(u^+)^{p^*(s)}}{|x|^s} dx + o_n(1). \tag{2.5}$$

Putting together (2.2) and (2.3), we get

$$\begin{aligned} c + o_n(1) &= I_\lambda(u) + \frac{a}{2p}\|v_n\|^{2p} + \frac{b}{p}\|v_n\|^p + \frac{a}{p}\|v_n\|^p\|u\|^p \\ &\quad - \frac{1}{p^*(s)}\int_{\mathbb{R}^N} |x|^{-s}\left(v_n^+\right)^{p^*(s)} dx \end{aligned}$$

and

$$o_n(1) = a\|v_n\|^{2p} + b\|v_n\|^p + 2a\|v_n\|^p\|u\|^p - \int_{\mathbb{R}^N} |x|^{-s}\left(v_n^+\right)^{p^*(s)} dx. \tag{2.6}$$

Then, as $s = 2p - N$ we have $p^*(s) = 2p$ and

$$c + o_n(1) \geq I_\lambda(u) + \frac{b}{2p}\|v_n\|^p. \tag{2.7}$$

Assume that $\|v_n\| \to l > 0$, then by (2.4) and the Sobolev-Hardy inequality we obtain

$$l^p \geq S_s\left(bl^p + al^{2p}\right)^{\frac{1}{2}},$$

this implies that

$$\left(S_s^{-2} - a\right)l^p - b \geq 0,$$

that is,

$$l^p \geq \frac{b}{S_s^{-2} - a}.$$

From the above inequality we conclude:

$$c \geq I_\lambda(u) + \frac{b^2}{2p\left(S_s^{-2} - a\right)}.$$

The last inequality completes the proof of Lemma 2.4. □

Remark 2.5. By Lemma 2.4, we ensure the local compactness of the $(\mathrm{PS})_c$ sequence for I_λ.

Theorem 2.6. *Assume that* $a, b > 0$, $1 < p < N \leq 2p$, $1 < q < p$, $0 \leq s \leq 2p - N$ *and* f *satisfies* (H_f). *Then there exists* $\Lambda_1 > 0$ *such that problem* $(\mathcal{P}_\lambda)$ *has at least one nontrivial solution for any* $\lambda \in (0,\ \Lambda_1)$.

Proof. By Lemma 2.2, we define

$$c_1 = \inf_{u \in \bar{B}_{r_1}} I_\lambda(u).$$

As $U > 0$ and $f \geq 0$ $f \not\equiv 0$, we have $\int_{\mathbb{R}^N} f(x) U^q dx > 0$. Then there exists $t_0 > 0$ small enough such that $\|t_0 U\| < r_1$ and

$$\begin{aligned} I_\lambda(t_0 v) &= \frac{a}{2p} t_0^{2p} \|U\|^{2p} + \frac{b}{p} t_0^p \|U\|^p \\ &\quad - \frac{t_0^{p^*(s)}}{p^*(s)} \int_{\mathbb{R}^N} |x|^{-s} U^{p^*(s)} dx - \lambda \frac{t_0^q}{q} \int_{\mathbb{R}^N} f(x) U^q dx \\ &< 0, \end{aligned}$$

which implies that $c_1 < 0$. Using the Ekeland's variational principle [10], for the complete metric space $\bar{B}_{r_1}$ with respect to the norm of $W^{1,p}\left(\mathbb{R}^N\right)$, we obtain by Lemma 2.3, the result that there exists a $(\mathrm{PS})_{c_1}$ sequence $\{u_n\} \subset \bar{B}_{r_1}$ such that $u_n \rightharpoonup u_1$ in $W^{1,p}\left(\mathbb{R}^N\right)$ for some u_1 with $\|u_1\| \leq r_1$. After a direct calculation, we derive $\left\|u_1^-\right\| = \left\langle I_\lambda'(u_1), u_1^-\right\rangle = 0$, which implies $u_1 \geq 0$. As $I_\lambda(0) = 0 > c_1$, then $u_1 \neq 0$. Assume $u_n \not\rightharpoonup u_1$ in $W^{1,p}\left(\mathbb{R}^N\right)$. Then $\left\|u_1^-\right\| < \liminf_{n \to +\infty} \|u_n\|$, which implies that

$$\begin{aligned} c_- &\leq I_\lambda(u_1) \\ &= I_\lambda(u_1) - \frac{1}{p^*(s)} \left\langle I_\lambda'(u_1), u_1\right\rangle \end{aligned}$$

$$\begin{aligned}
&= a\left(\frac{1}{2p}-\frac{1}{p^*(s)}\right)\|u_1\|^{2p}+b\left(\frac{1}{p}-\frac{1}{p^*(s)}\right)\|u_1\|^{p} \\
&\quad -\lambda\left(\frac{1}{q}-\frac{1}{p^*(s)}\right)\int_{\mathbb{R}^N} f(x)(u_1)^q\,dx \\
&< \liminf_{n\to+\infty}\left[a\left(\frac{1}{2p}-\frac{1}{p^*(s)}\right)\|u_n\|^{2p}+b\left(\frac{1}{p}-\frac{1}{p^*(s)}\right)\|u_n\|^{p}\right. \\
&\quad \left.-\lambda\left(\frac{1}{q}-\frac{1}{p^*(s)}\right)\int_{\mathbb{R}^N} f(x)(u_1)^q\,dx\right] \\
&= \liminf_{n\to+\infty}\left[I_\lambda(u_n)-\frac{1}{p^*(s)}\langle I'_\lambda(u_n),u_n\rangle\right] \\
&= c_-,
\end{aligned}$$

which is a contradiction. We conclude that $u_n \to u_1$ strongly in $W^{1,p}(\mathbb{R}^N)$. Therefore, $I'_\lambda(u_1)=0$ and $I_\lambda(u_1)=c_-<0=I_\lambda(0)$. Hence u_1 is a positive solution of $(\mathcal{P}_\lambda)$ with negative energy. □

Theorem 2.7. *In addition to the assumptions of Theorem* 2.6, *we assume that* $a<S_s^{-2}$, $N\le 2p$ *and* $s=2p-N$. *Then there exists* $\Lambda_*>0$ *such that problem* $(\mathcal{P}_\lambda)$ *has at least two nontrivial solutions for any* $\lambda\in(0,\ \Lambda_*)$.

Proof. Note that $I_\lambda(0)=0$ and $I_\lambda(t_3U)<0$ for t_3 large enough, also from Lemma 2.2, we know that

$$I_\lambda(u)|_{\partial B_{r_1}}\ge\delta_1>0 \quad \text{for all} \quad \lambda\in(0,\Lambda_1).$$

Then, by the Mountain Pass theorem [3], there exists a $(\text{PS})_{c_2}$ sequence, where

$$c_2=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t))$$

with

$$\Gamma=\left\{\gamma\in C([0,\ 1],\ W^{1,p}(\mathbb{R}^N)),\ \gamma(0)=0 \text{ and } \gamma(1)=t_3U\right\}.$$

Using Lemma 2.3, we have $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u_2$ in $W^{1,p}(\mathbb{R}^N)$, for some u_2. As $\|u_2^-\|=\langle I'_\lambda(u_2),u_2^-\rangle=0$, we conclude that $u_2\ge 0$. We consider the functions

$$\Phi_1(t)=a\frac{t^{2p}}{2p}\|U\|^{2p}+b\frac{t^p}{p}\|U\|^p-\frac{t^{p^*(s)}}{p^*(s)}\int_{\mathbb{R}^N}|x|^{-s}U^{p^*(s)}dx$$

and

$$\Phi_2(t)=\Phi_1(t)-\lambda\frac{t^q}{q}\int_{\mathbb{R}^N} f(x)U^q dx.$$

So, for all $\lambda \in (0, \Lambda_2)$ we have

$$\Phi_2(0) = 0 < -\frac{p-q}{p}\left[\left(\frac{2q}{b}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS_0^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)} + \frac{b^2}{2p\left(S_s^{-2} - a\right)}.$$

Hence, by the continuity of $\Phi_2(t)$, there exists $t_1 > 0$ small enough such that

$$\Phi_2(t) < -\frac{p-q}{p}\left[\left(\frac{2q}{b}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS_0^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)} + \frac{b^2}{2p\left(S_s^{-2} - a\right)}$$

for all $t \in (0, t_1)$.

Moreover, U satisfies

$$\|U\|^p = \int_{\mathbb{R}^N} \frac{U^{p^*(s)}}{|x|^s} dx = S_s^{\frac{N}{p}}. \tag{2.8}$$

Then, the function $\Phi_1(t)$ attains its maximum at

$$t_{\max} = \left(\frac{b}{1 - aS_s^2}\right)^{\frac{1}{p}}.$$

Therefore

$$\Phi_1(t_{\max}) = \frac{b^2}{2p\left(S_s^{-2} - a\right)}.$$

Thus we deduce that

$$\Phi_1(t) \leq \frac{b^2}{2p\left(S_s^{-2} - a\right)} \quad \text{for} \quad t > 0.$$

On the other hand, using Lemma 2.2, we see that

$$c_1 \geq -\frac{p-q}{p}\left[\left(\frac{2q}{b}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS_0^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)} \quad \text{for all} \quad \lambda \in (0, \Lambda_1),$$

furthermore, if

$$\lambda < \left[\left(\frac{2q}{b}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS_0^{q/p}}\right]^{-p/q} \left[\frac{p}{q\,(p-q)} \int_{\mathbb{R}^N} f(x) U^q dx\right]^{(p-q)/q},$$

we get

$$c_1 > -\lambda \frac{t_1}{q} \int_{\mathbb{R}^N} f(x) U^q \, dx.$$

Taking

$$\Lambda_* = \min\left\{\Lambda_1,\ \Lambda_2,\ \left[\left(\frac{2q}{b}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS_0^{q/p}}\right]^{-\frac{p}{q}} \left[\frac{p}{q\,(p-q)} \int_{\mathbb{R}^N} f(x) U^q \, dx\right]^{\frac{p-q}{q}}\right\},$$

then we deduce that

$$\sup_{t\geq 0} I_\lambda(tU) < c_1 + \frac{b^2}{2p\left(S_s^{-2} - a\right)} \quad \text{for all} \quad \lambda \in (0, \Lambda_*).$$

Then from Lemma 2.4 we deduce that $u_n \to u_2$ in $W^{1,p}(\mathbb{R}^N)$. Thus we obtain a critical point u_2 of I_λ satisfying $I_\lambda(u_2) > 0$, and we conclude that u_2 is a nontrivial solution of $(\mathcal{P}_\lambda)$ with positive energy. □

Corollary 2.8. *Let $s = 0$, $N = 2p$ and $a < S_0^{-2}$. Then there exists $\widetilde{\Lambda}_* > 0$ such that problem $(\mathcal{P}_\lambda)$ has at least two nontrivial solutions for any $\lambda \in \left(0, \ \widetilde{\Lambda}_*\right)$.*

Proof. Using the Sobolev inequality, we give the proof similarly to that of Theorem 2.7. □

Remark 2.9. In Theorem 2.7, if $s = 0$ we have $N = 2p$, then our results imply that suitable real s can realize the restriction on the spatial dimension N.

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References

[1] C.O. Alves, *Multiple positive solutions for equations involving critical Sobolev exponent,* Electron. J. Diff. Equ., **13** (1997), 1-10.

[2] C.O. Alves, F.J.S.A. Corrêa and T.F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type,* Comput. Math. Appl., **49** (2005), 85-93.

[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349-381.

[4] A. Benaissa and A. Matallah, *Nonhomogeneous elliptic Kirchhoff equations of the p-Laplacian Type*, Ukr. Math. Math. J., **72** (2020), 203-210.

[5] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals,* Proc. Amer. Math. Soc., **88** (1983), 486-490.

[6] L. Caffarelli, R. Kohn and L. Nirenberg, *First order interpolation inequality with weights,* Compos. Math., **53**(3) (1984) 259-275.

[7] C.S. Chen, J.C. Huang and L.H. Liu, *Multiple solutions to the nonhomogeneous p-Kirchhoff elliptic equation with concave-convex nonlinearities,* Appl. Math. Lett., **26** (2013), 754-759.

[8] S.J. Chen and L. Li, *Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbb{R}^N$,* Nonlinear Anal. Real World Appl., **14** (2013), 1477-1486

[9] F.J.S.A. Corrêa and G.M. Figueiredo, *On a elliptic equation of p-Kirchhoff type via variational methods,* Bull. Aust. Math. Soc., **74** (2006), 263-277.

[10] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl., **47** (1974), 324-353.

[11] G.M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument,* J. Math. Anal. Appl., **401** (2013), 706-713.

[12] A. Fiscella and E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator,* Nonlinear Anal., **94** (2014), 156-170.

[13] N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents,* Trans. Amer. Math. Soc., **352** (2000), 5703-5743.

[14] X. Wu, *Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in* $\mathbb{R}^N$*,* Nonlinear Anal. Real World Appl., **12** (2011), 1278-1287.