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DIVISIBLE SUBSPACES OF LINEAR OPERATORS ON BANACH SPACES

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Abstract. In this paper, we investigate the properties related to algebraic spectral subspaces and divisible subspaces of linear operators on a Banach space. In addition, using the concept of topological divisior of zero of a Banach algebra, we prove that the only closed divisible subspace of a bounded linear operator on a Banach space is trivial. We also give an example of a bounded linear operator on a Banach space with non-trivial divisible subspaces.

1. Algebraic spectral subspaces of linear operators

Throughout this paper we shall use the standard notions and some basic results on the theory of operator theory and functional analysis. Let X be a Banach space over the complex plane $\mathbb{C}$ and let $L(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X. Given an operator $T \in L(X)$, $\text{Lat}(T)$ denotes the collection of all closed T-invariant linear subspaces of X, and for $Y \in \text{Lat}(T)$, $T|Y$ denotes the restriction of T on Y, and $\sigma(T)$, $\rho(T)$ denote the spectrum and the resolvent set of T, respectively.

Let T be a normal operator on a Hilbert space H. For a closed subset F of $\mathbb{C}$, consider the corresponding spectral projection $E(F)$. Since $\sigma(T|E(F)H) \subseteq F$, we have

$$(T - \lambda)E(F)H = E(F)H \quad \text{for all } \lambda \notin F.$$

One can show that this relation actually characterizes the spaces $E(F)H$. Such subspaces may be considered in a more general situation. It turns out that these subspaces are useful in the study of the local spectral theory.

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Definition 1.1. *Let $T : X \to X$ be a linear operator on a Banach space X. Let F be a subset of the complex plane $\mathbb{C}$. Consider the class of all linear subspaces Y of X which satisfy $(T-\lambda)Y = Y$ for all $\lambda \notin F$, the algebraic linear span $E_T(F)$ of all such subspaces Y of X is called an algebraic spectral subspace of T.*

By the definition of the algebraic spectral subspace, it is clear that

$$E_T(F_1) \subseteq E_T(F_2) \quad \text{for} \quad F_1 \subseteq F_2.$$

Let $A \in L(X)$ be with $AT = TA$. For a given subset F of $\mathbb{C}$ and $\lambda \notin F$, we obtain

$$(T-\lambda)AE_T(F) = A(T-\lambda)E_T(F) = AE_T(F).$$

By the maximality of $E_T(F)$, we have

$$AE_T(F) \subseteq E_T(F).$$

Hence the space $E_T(F)$ is a hyper-invariant subspace of T. That is, if $TS = ST$ then $E_T(F)$ is a invariant subspace of S. We also note that

$$E_T(F) = E_T(F \cap \sigma(T)),$$

where $\sigma(T)$ is the spectrum of T.

Lemma 1.2. *The space $E_T(F)$ is the union of all sets $M \subseteq X$ such that $M \subseteq (T-\lambda)M$ for all $\lambda \notin F$.*

Proof. Denote by Z the union of all sets M with $M \subseteq (T-\lambda)M$ for all $\lambda \notin F$. Then Z is a linear subspace of X with the property that

$$Z \subseteq (T-\lambda)Z \quad \text{for all} \quad \lambda \notin F.$$

On the other hand, applying the operator $T-\lambda$ to both sides of the above inclusion we get

$$(T-\lambda)Z \subseteq (T-\lambda)[(T-\lambda)Z] \quad \text{for all} \quad \lambda \notin F.$$

Hence, the set $(T-\lambda)Z$ has the given property, and we have

$$(T-\lambda)Z \subseteq Z \quad \text{for all} \quad \lambda \notin F$$

by the definition of Z. Thus we have shown that $(T-\lambda)Z = Z$ for all $\lambda \notin F$. Since $E_T(F)$ is the largest linear subspace of X with this property, we have

$$Z \subseteq E_T(F).$$

But the inclusion $E_T(F) \subseteq Z$ is obvious. Therefore, $E_T(F) = Z$. □

PROPOSITION 1.3. *Let F be a subset of $\mathbb{C}$. If $\lambda_0 \in F$, then*

$$(T-\lambda_0)^{-1}(E_T(F)) = E_T(F).$$

Proof. Let $M = (T-\lambda_0)^{-1}(E_T(F))$. To prove $M \subseteq E_T(F)$, it suffices to show that

$$M \subseteq (T-\lambda)M \quad \text{for all } \lambda \notin F.$$

To show this, take an arbitrary $x \in M$ and $\lambda \notin F$. Since $(T-\lambda_0)x \in E_T(F)$, it follows that

$$(T-\lambda_0)x = (T-\lambda)e \quad \text{for a suitable } e \in E_T(F).$$

From $(T-\lambda_0)\dfrac{x-e}{\lambda_0-\lambda} = e \in E_T(F)$ and the definition of M, we obtain

$$\frac{x-e}{\lambda_0-\lambda} \in M.$$

Also we obtain

$$(T-\lambda)e = (T-\lambda_0)x = [(\lambda-\lambda_0)+(T-\lambda)]x.$$

Thus we have the following equality:

$$x = (T-\lambda)\frac{x-e}{\lambda_0-\lambda}.$$

Hence $x \in (T-\lambda)M$. Therefore,

$$M \subseteq E_T(F).$$

Since the space $E_T(F)$ is invariant with respect to T, the reverse inclusion is clear. □

PROPOSITION 1.4. *Let F be a subset of $\mathbb{C}$. Then*

$$E_T(F) = \bigcap_{\lambda\notin F} E_T(\mathbb{C}\setminus\{\lambda\}).$$

Proof. Let

$$P = \bigcap_{\lambda\notin F} E_T(\mathbb{C}\setminus\{\lambda\}).$$

Since $E_T(F) \subseteq E_T(\mathbb{C}\setminus\{\lambda\})$ for all $\lambda \notin F$, we obtain

$$E_T(F) \subseteq P.$$

To prove the reverse inclusion it is enough to show that

$$P \subseteq (T-\lambda)P \quad \text{for all} \quad \lambda \notin F.$$

To see this, let $p \in P$ and $\mu \notin F$ be given. Since $p \in P \subseteq E_T(\mathbb{C} \setminus \{\mu\})$, we have

$$p = (T - \mu)p' \quad \text{for a suitable} \quad p' \in E_T(\mathbb{C} \setminus \{\mu\}).$$

Now take an arbitrary $\lambda \notin F$ with $\lambda \neq \mu$. Since $\mu \in \mathbb{C} \setminus \{\lambda\}$ and $(T - \mu)p' = p \in E_T(\mathbb{C} \setminus \{\lambda\})$, we have

$$p' \in E_T(\mathbb{C} \setminus \{\lambda\})$$

by Proposition 1.3. Thus $p' \in P$ and the proof is complete. □

Proposition 1.5. *Let $\{F_\alpha\}$ be a family of subsets of $\mathbb{C}$. Then*

$$E_T(\bigcap_\alpha F_\alpha) = \bigcap_\alpha E_T(F_\alpha).$$

Proof. By Proposition 1.4, we have

$$\begin{aligned} \bigcap_\alpha E_T(F_\alpha) &= \bigcap_\alpha \bigcap_{\lambda \notin F_\alpha} E_T(\mathbb{C} \setminus \{\lambda\}) \\ &= \bigcap_{\lambda \in \cup(\mathbb{C} \setminus F_\alpha)} E_T(\mathbb{C} \setminus \{\lambda\}) \\ &= \bigcap_{\lambda \notin \cap F_\alpha} E_T(\mathbb{C} \setminus \{\lambda\}) \\ &= E_T(\bigcap_\alpha F_\alpha). \end{aligned}$$

□

It is clear from the definition that

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T - \lambda)^n X.$$

For a linear operator T which has no eigenvalues, we will show that the above inclusion becomes in fact an equality. For example, since shift operators and Volterra operators have no eigenvalues, the algebraic spectral subspaces of these operators can be represented by the right hand side of the above inclusion.

Proposition 1.6. *If $T \in L(X)$ has no eigenvalues, then*

$$E_T(F) = \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T - \lambda)^n X$$

for any subset F of $\mathbb{C}$.

Proof. Suppose that $\lambda \notin F$. Let $x \in \bigcap_{n\in\mathbb{N}} (T-\lambda)^n X$. For each $n \in \mathbb{N}$, there is $x_n \in X$ such that

$$x = (T-\lambda)^n x_n.$$

Since $T-\lambda$ is one to one, we have

$$x_1 = (T-\lambda)x_2 = (T-\lambda)^2 x_3 = \dots .$$

Thus

$$x_1 \in \bigcap_{n\in\mathbb{N}} (T-\lambda)^n X.$$

But $x = (T-\lambda)x_1$ and we get

$$\bigcap_{n\in\mathbb{N}} (T-\lambda)^n X \subseteq (T-\lambda)[\bigcap_{n\in\mathbb{N}} (T-\lambda)^n X].$$

By Lemma 1.2,

$$\bigcap_{n\in\mathbb{N}} (T-\lambda)^n X \subseteq E_T(\mathbb{C} \setminus \{\lambda\}) \quad \text{for all} \quad \lambda \notin F.$$

Since $E_T(\cdot)$ preserves an arbitrary intersection, we have

$$\begin{aligned} \bigcap_{\lambda\notin F, n\in\mathbb{N}} (T-\lambda)^n X &\subseteq \bigcap_{\lambda\notin F} E_T(\mathbb{C} \setminus \{\lambda\}) \\ &= E_T(\bigcap_{\lambda\notin F} \mathbb{C} \setminus \{\lambda\}) \\ &= E_T(F). \end{aligned}$$

□

In general $E_T(F)$ need not be closed. But the closedness of $E_T(F)$ is closely related to automatic continuity theory of intertwining linear operators on Banach spaces.

2. Divisible subspaces of linear operators

Now we introduce the divisible subspace of operators on a Banach space. In the theory of automatic continuity, it is open crucial to exclude the existence of non-trivial divisible subspaces, because their presence tends to preclude the desired continuity conclusions.

Definition 2.1. *A linear subspace Z of X is called a T-divisible subspace if*

$$(T-\lambda)Z = Z \quad \textit{for all } \lambda \in \mathbb{C}.$$

It is clear that $E_T(\emptyset)$ is precisely the largest T-divisible subspace of X. Many important operators do not have non trivial divisible subspaces. For example, hyponormal operators on Hilbert spaces do not have non-trivial divisible subspaces.

Let $\mathbb{C}[\chi]$ denote the ring of polynomials with complex coefficients in an indeterminate χ. We regard X as $\mathbb{C}[\chi]$-module via the linear operator T, that is, if $p \in \mathbb{C}[\chi]$ we define $p \cdot x$ by $p \cdot x = p(T)x$. By the fundamental theorem of algebra, elements in $\mathbb{C}[\chi]$ may be factorized into linear factors. Thus, Z is T-divisible if and only if $p \cdot Z = Z$ for all non zero $p \in \mathbb{C}[\chi]$, and so Z is T-divisible if and only if Z is divisible as a $\mathbb{C}[\chi]$-module. Since the ring $\mathbb{C}[\chi]$ is a principal ideal domain, a $\mathbb{C}[\chi]$-submodule Z is divisible if and only if it is injective $\mathbb{C}[\chi]$-submodule.

Definition 2.2. *Let A be bounded linear operator on a Banach space X. Then A is sad to be a topological divisisor of zero if there is a sequence B_n with $\|B_n\| = 1$ for all $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \|B_n A\| = \lim_{n\to\infty} \|AB_n\| = 0.$$

Proposition 2.3. *Let T be a bounded linear operator on a Banach space X, and let λ be a boundary point of the spectrum of T. Then $T - \lambda$ is a topological divisior of zero.*

The proof of above proposition is found in [3]. We now present that there is no non-trivial closed T-divisible subspace of a bounded linear operator T on a Banach space.

Theorem 2.4. *Let $T \in L(X)$. If Z is a closed T-divisible subspace of a Banach space X, then $Z = \{0\}$.*

Proof. Let $\lambda \in \partial\sigma(T|Z)$, where $\partial\sigma(T|Z)$ denotes the boundary of the spectrum of T restricted to Z. By Proposition 2.3, there exists a sequence $\langle T_n \rangle$ of linear operators in Z with $\|T_n\| = 1$ for all $n \in \mathbb{N}$ and $T_n(T-\lambda) \to 0$. Since $(T-\lambda)Z = Z$, by the open mapping theorem,

$$kB \subseteq (T-\lambda)B \quad \text{for some} \quad k > 0$$

where B is the unit ball in Z. For sufficiently large natural number n with $\|T_n(T-\lambda)\| < \frac{k}{2}$, we have

$$kT_nB \subseteq T_n(T-\lambda)B \subseteq \frac{k}{2}B.$$

Then $T_nB \subseteq \frac{1}{2}B$, which contradicts the assumption that $\|T_n\| = 1$. $\square$

Proposition 2.5. *Let $T \in L(X)$ and let M be the maximal T-divisible subspace of X. Then M is characterized by M being the maximal subspace with respect to*

$$(T-\lambda)M = M \quad \text{for all } \lambda \in \sigma(T).$$

Proof. Let M be the maximal subspace for with respect to $(T-\lambda)M = M$ for all $\lambda \in \sigma(T)$. It is enough to show that $(T-\mu)M = M$ for all $\mu \in \rho(T)$, where $\rho(T)$ denotes the resolvent set of T. For each $\mu \in \rho(T)$

$$\begin{aligned}(T-\lambda)(T-\mu)^{-1}M &= (T-\mu)^{-1}(T-\lambda)M \\ &= (T-\mu)^{-1}M\end{aligned}$$

for all $\lambda \in \sigma(T)$, it follows that $(T-\mu)^{-1}M \subseteq M$ by the maximality of M. Hence we have

$$\begin{aligned}M &= (T-\mu)(T-\mu)^{-1}M \subseteq (T-\mu)M \\ &\subseteq (T-\lambda+(\lambda-\mu))M \\ &\subseteq M.\end{aligned}$$

□

We present a compact and quasi-nilpotent operator $T \in L(X)$ on a Banach space X such that T has a non-trivial divisible subspace.

Example 2.6. *We consider the Volterra operator T on $C[0,1]$ defined by*

$$(Tf)(s) = \int_0^s f(t)dt \quad \text{for all } f \in C[0,1] \text{ and } s \in [0,1].$$

Then T is both compact and quasi-nilpotent operator on $C[0,1]$, and T has the following non-trivial divisible subspace

$$Y = \{f \in C^\infty[0,1] : f^{(k)}(0) = 0 \quad \text{for all} \quad k = 0,1,2,\dots\}.$$

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