SVN-Ostrowski Type Inequalities for $(\alpha, \beta, \gamma, \delta)$ −Convex Functions

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Summary

In this paper, we present the very first time the generalized notion of $(\alpha, \beta, \gamma, \delta)-$ convex (concave) function in mixed kind, which is the generalization of $(\alpha, \beta)-$ convex (concave) functions in 1^{st} and 2^{nd} kind, $(s, r)-$ convex (concave) functions in mixed kind, $s-$convex (concave) functions in 1^{st} and 2^{nd} kind, $P-$ convex (concave) functions, quasi convex(concave) functions and the class of convex (concave) functions. We would like to state the well-known Ostrowski inequality via SVN-Riemann Integrals for $(\alpha, \beta, \gamma, \delta)-$ convex (concave) function in mixed kind. Moreover we establish some SVN-Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $(\alpha, \beta, \gamma, \delta)$-convex (concave) functions in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases with respect to convexity of function.

Keywords:

Ostrowski inequality, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ −convex functions, Single valued Neutrosophic sets.

1. Introduction

Ostrowski inequality is most celebratted inequality in literature, there are many variants and generalizations are given in literature of Ostrowski inequality, in this article our main focus on the basis of the generalization of Ostrowski type inequalties via generalized convex functions. From literature, we recall and introduce some definitions for various convex (concave) functions.

Definition 1.1 *[3] A function $\phi: I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex (concave) function, if*

$$\phi(tx + (1-t)y) \leq (\geq) t\phi(x) + (1-t)\phi(y),$$

$\forall x, y \in I, t \in [0,1]$.

We recall here definition of $P-$convex(concave) function from [18].

Definition 1.2 *We say that $\phi: I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex(concave) function, if ϕ is a non-negative and $\forall x, y \in I$ and $t \in [0,1]$, we have*

$$\phi(tx + (1-t)y) \leq (\geq) \phi(x) + \phi(y).$$

Here we also have definition of quasi−convex(concave) function (for detailed discussion see [21].

Definition 1.3 *A function $\phi: I \subset \mathbb{R} \to \mathbb{R}$ is known as quasi−convex(concave), if*

$$\phi(tx + (1-t)y) \leq (\geq) \max\{\phi(x), \phi(y)\}$$

$\forall x, y \in I, t \in [0,1]$.

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Now we present definition of $s-$convex functions in the first kind as follows which are extracted from [32]:

Definition 1.4 *[5] Let $s \in [0,1]$. A function $\phi: I \subset [0, \infty) \to [0, \infty)$ is said to be $s-$convex (concave) function in the 1^{st} kind, if*

$$\phi(tx + (1-t)y) \leq (\geq) t^s\phi(x) + (1-t^s)\phi(y),$$

$\forall x, y \in I, t \in [0,1]$.

Remark 1.5 *Note that in this definition we also included $s = 0$. Further if we put $s = 0$, we get quasi−convexity (see Definition 0.3).*

For second kind convexity we recall definition from [32].

Definition 1.6 *Let $s \in [0,1]$. A function $\phi: I \subset [0, \infty) \to [0, \infty)$ is said to be $s-$convex (concave) function in the 2^{nd} kind, if*

$$\phi(tx + (1-t)y) \leq (\geq) t^s\phi(x) + (1-t)^s\phi(y),$$

$\forall x, y \in I, t \in [0,1]$.

Remark 1.7 *In the similar manner, we have slightly improved definition of second kind convexity by including $s = 0$. Further if we put $s = 0$, we easily get $P-$convexity (see Definition 0.2).*

Definition 1.8 *[2] Let $(s, r) \in [0,1]^2$. A function $\phi: I \subset [0, \infty) \to [0, \infty)$ is said to be $(s, r)-$convex (concave) function in mixed kind, if*

$$\phi(tx + (1-t)y) \leq (\geq) t^{rs}\phi(x) + (1-t^r)^s\phi(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.9 *[19] Let $(\alpha, \beta) \in [0,1]^2$. A function $\phi: I \subset [0, \infty) \to [0, \infty)$ is said to be $(\alpha, \beta)-$convex(concave) in the 1^{st} kind, if*

$$\phi(tx + (1-t)y) \leq (\geq) t^\alpha\phi(x) + (1-t^\beta)\phi(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.10 *[19] Let $(\alpha, \beta) \in [0,1]^2$. A function $\phi: I \subset [0, \infty) \to [0, \infty)$ is said to be $(\alpha, \beta)-$convex(concave) function in the 2^{nd} kind, if*

$$\phi(tx + (1-t)y) \leq (\geq) t^\alpha\phi(x) + (1-t)^\beta\phi(y),$$

$\forall x, y \in I, t \in [0,1]$.

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality.

Theorem 1.11 *[33] Let $\varphi:[\rho_a,\rho_b]\to\mathbb{R}$ be differentiable function on (ρ_a,ρ_b) with the property that $|\varphi'(t)|\le M$ for all $t\in(\rho_a,\rho_b)$. Then*

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\le M(\rho_b-\rho_a)\left[\frac{1}{4}+\left(\frac{x-\frac{\rho_a+\rho_b}{2}}{\rho_b-\rho_a}\right)^2\right],$$

$\forall x\in(\rho_a,\rho_b)$. The constant $\frac{1}{4}$ is the best possible in the kind that it cannot be replaced by a smaller quantity.

Now we present the extension of definitions of fuzzy numbers and their results as from the [7], [8], [28] and [20].

Definition 1.12 *[4] A SVN-Number is $\phi:\mathbb{R}\to[0,1]$ can be defined as*
1. $[\phi]^0=\text{Closure}(\{r\in\mathbb{R}:T\phi(r)>0,I\phi(r)>0,F\phi(r)>0\})$ is compact.
2. ϕ is Normal.(i.e, $\exists r_0\in\mathbb{R}$ such that $T\phi(r_0)=1,I\phi(r_0)=0$ and $F\phi(r_0)=0$).
3. ϕ is SVN-convex, i.e, $\forall r_1,r_2\in\mathbb{R},\eta\in[0,1]$
$$T\phi(\eta r_1+(1-\eta)r_2)\ge\min\{T\phi(r_1),T\phi(r_2)\},$$
$$I\phi(\eta r_1+(1-\eta)r_2)\le\max\{I\phi(r_1),I\phi(r_2)\},$$
$$F\phi(\eta r_1+(1-\eta)r_2)\le\max\{F\phi(r_1),F\phi(r_2)\}.$$
4. $\forall r_0\in R$ and $\epsilon>0,\exists$ Neighborhood $V(r_0)$, such that $\forall r\in\mathbb{R},T\phi(r)\le T\phi(r_0)+\epsilon,I\phi(r)\ge I\phi(r_0)-\epsilon$, and $F\phi(r)\ge\phi(r_0)-\epsilon$.

Definition 1.13 *[4, 22] For any $(\zeta_1,\zeta_2,\zeta_3)\in[0,1]^3$, and ϕ be any SVN-number, then $(\zeta_1,\zeta_2,\zeta_3)-$ level set $[\phi]^{(\zeta_1,\zeta_2,\zeta_3)}=\{r\in\mathbb{R}:T\phi(r)\ge\zeta_1,I\phi(r)\le\zeta_2,F\phi(r)\le\zeta_3\}$.*
Moreover $[\phi]^\zeta=\left[\phi_-^{(\zeta_1,\zeta_2,\zeta_3)},\phi_+^{(\zeta_1,\zeta_2,\zeta_3)}\right],\forall(\zeta_1,\zeta_2,\zeta_3)\in[0,1]^3$

Proposition 1.14 *[22, 30] Let $\phi,\varphi\in SVN_{\mathbb{R}}$ (Set of all SVN-Numbers) and $\eta\in\mathbb{R}$, then the following properties holds:*
1. $[\phi+\varphi]^{(\zeta_1,\zeta_2,\zeta_3)}=[\phi]^{(\zeta_1,\zeta_2,\zeta_3)}+[\varphi]^{(\zeta_1,\zeta_2,\zeta_3)}$.
2. $[\eta\odot\phi]^{(\zeta_1,\zeta_2,\zeta_3)}=\eta[\phi]^{(\zeta_1,\zeta_2,\zeta_3)}$.
3. $\phi\oplus\varphi=\varphi\oplus\phi$.
4. $\eta\odot\phi=\phi\odot\eta$.
5. $1\odot\phi=\phi$.

$\forall\zeta\in[0,1]$.

Definition 1.15 *[31] Let $D:SVN_{\mathbb{R}}\times SVN_{\mathbb{R}}\to\mathbb{R}_+\cup\{0\}$, defined as*
$$D(\phi,\varphi)=\sup_{\zeta\in[0,1]}\max\left\{\left|T\phi_-^{(\zeta)}-T\phi_+^{(\zeta)}\right|,\left|T\varphi_-^{(\zeta)}-T\varphi_+^{(\zeta)}\right|\right\}$$
$$+\inf_{\zeta\in[0,1]}\min\left\{\left|I\phi_-^{(\zeta)}-I\phi_+^{(\zeta)}\right|,\left|I\varphi_-^{(\zeta)}-I\varphi_+^{(\zeta)}\right|\right\}$$
$$+\inf_{\zeta\in[0,1]}\min\left\{\left|F\phi_-^{(\zeta)}-F\phi_+^{(\zeta)}\right|,\left|F\varphi_-^{(\zeta)}-F\varphi_+^{(\zeta)}\right|\right\}.$$

$\forall\phi,\varphi\in SVN_{\mathbb{R}}$. Then D is metric on $SVN_{\mathbb{R}}$.

Proposition 1.16 *[31] Let $\phi_1,\phi_2,\phi_3,\phi_4\in SVN_{\mathbb{R}}$ and $\eta\in SVN_{\mathbb{R}}$, we have*
1. $D(\phi_1\oplus\phi_3,\phi_2\oplus\phi_3)=D(\phi_1,\phi_2)$.
2. $D(\eta\odot\phi_1,\eta\odot\phi_2)=|\eta|D(\phi_1,\phi_2)$.
3. $D(\phi_1\oplus\phi_2,\phi_3\oplus\phi_4)\le D(\phi_1,\phi_3)+D(\phi_2,\phi_4)$.
4. $D(\phi_1\oplus\phi_2,\tilde{0})\le D(\phi_1,\tilde{0})+D(\phi_2,\tilde{0})$.
5. $D(\phi_1\oplus\phi_2,\phi_3)\le D(\phi_1,\phi_3)+D(\phi_2,\tilde{0})$,

where $\tilde{0}\in SVN_{\mathbb{R}}$, defined by $\forall r\in\mathbb{R},\tilde{0}(r)=(0,0,1)$.

Definition 1.17 *[30] Let $\phi,\varphi\in SVN_{\mathbb{R}}$, if $\exists\theta\in SVN_{\mathbb{R}}$, such that $\phi=\varphi\oplus\theta$, then θ is $H-$difference of ϕ and φ, denoted by $\theta=\phi\ominus\varphi$.*

Definition 1.18 *[31] A function $\phi:[r_0,r_0+\epsilon]\to SVN_{\mathbb{R}}$ is (1) $H-$differentiable at r, if $\exists\phi'(r)\in SVN_{\mathbb{R}}$, i.e both limits*
$$\lim_{h\to0^+}\frac{\phi(r+h)\ominus\phi(r)}{h},\ \lim_{h\to0^+}\frac{\phi(r)\ominus\phi(r-h)}{h}$$
exists and are equal to $\phi'(r)$.

Definition 1.19 *[30, 31] Let $\phi:[\rho_a,\rho_b]\to SVN_{\mathbb{R}}$, if $\forall\zeta>0,\exists\eta>0$, for any partition $P=\{[u,v]:\delta\}$ of $[\rho_a,\rho_b]$ with norm $\Delta(P)<\eta$, we have*
$$D\left(\sum_P^{*}(v-u)\phi(\delta),\varphi\right)<\zeta,$$
then we say that ϕ is SVN-Riemann integrable to $\varphi\in SVN_{\mathbb{R}}$, we write it as
$$\varphi=(SVNR)\int_{\rho_a}^{\rho_b}\phi(x)dx.$$
The main aim of our study is to generalize the ostrowski inequality $(\mathbf{1})$ *for* $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta})$ *−convex function, which is given in Section 2. Moreover we present* $\boldsymbol{SVN}-$ *Ostrowski type inequalities for which at the certain powers of absolute derivatives are* $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta})$ *−convex functions by using different techniques including Hölder's inequality [37] and power mean inequality [36]. Also we give the special cases of our results of midpoint inequalities.*

2. SVN-Ostrowski type Inequalities via $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta})$ −convex functions in mixed kind

In this section first we introduce $(\alpha,\beta,\gamma,\delta)$ −convex(concave) in mixed kind.

Definition 2.1 *Let $(\alpha,\beta,\gamma,\delta)\in[0,1]^4$. A function $\phi:I\subset[0,\infty)\to[0,\infty)$ is said to be $(\alpha,\beta,\gamma,\delta)$ −convex(concave) function in mixed kind, if*
$$\phi(tx+(1-t)y)\le(\ge)t^{\alpha\gamma}\phi(x)+(1-t^\beta)^\delta\phi(y),\qquad(2)$$
$\forall x,y\in I,t\in[0,1]$.

Remark 2.2 *In Definition* 0.20, *we have the following cases.*
1. If we choose $\gamma=\delta=1$ in (2), we get $(\alpha,\beta)-$ convex (concave) in 1^{st} kind function.
2. If we choose $\beta=\gamma=1$ in (2), we get $(\alpha,\beta)-$ convex (concave) in 2^{nd} kind function.
3. If we choose $\alpha=\delta=s,\beta=\gamma=r$, where $s,r\in[0,1]$ in (2), we get (s,r) −convex (concave) in mixed kind function.
4. If we choose $\alpha=\beta=s$ and $\gamma=\delta=1$ where $s\in[0,1]$ in (2), we get s −convex (concave) in 1^{st} kind function.
5. If we choose $\alpha=\beta=0$, and $\gamma=\delta=1$, in (2), we get quasi−convex (concave) function.
6. If we choose $\alpha=\delta=s,\beta=\gamma=1$ where $s\in[0,1]$ in (2), we get s −convex (concave) in 2^{nd} kind function.
7. If we choose $\alpha=\delta=0$, and $\beta=\gamma=1$, in (2), we get P −convex (concave) function.
8. If we choose $\alpha=\beta=\gamma=\delta=1$ in (2), gives us ordinary convex (concave) function.

In order to prove our main results, we need the following lemma that has been obtained in [34].

Lemma 2.3 *Let $\varphi:[\rho_a,\rho_b]\to F_{\mathbb{R}}$ be an absolutely continuous mapping on (ρ_a,ρ_b) with $\rho_a<\rho_b$. If $\varphi'\in C_F[\rho_a,\rho_b]\cap L_F[\rho_a,\rho_b]$, then for $x\in(\rho_a,\rho_b)$ the following identity holds:*

$$\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\oplus\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\odot(SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_a)dt$$

$$=\varphi(x)\oplus\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\odot(SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_b)dt \quad (3)$$

We make use of the beta function of Euler type, which is for $x,y>0$ defined as

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x)=\int_0^\infty e^{-u}u^{x-1}du.$

Theorem 2.4 *Suppose all the assumptions of Lemma 2.3 hold. Additionally, assume that $D(\varphi',\tilde{0})$ is $(\alpha,\beta,\gamma,\delta)$ −convex function on $[\rho_a,\rho_b]$ and $D(\varphi'(x),\tilde{0})\le M$. Then $\forall x\in(\rho_a,\rho_b)$ the following inequality holds:*

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\le M\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)\kappa(x), \quad (4)$$

where $\kappa(x)=\frac{(x-\rho_a)^2+(\rho_b-x)^2}{\rho_b-\rho_a}$.

Proof. From the Lemma 2.3

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\le D\left(\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\odot(SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_a)dt,\ \frac{(\rho_b-x)^2}{\rho_b-\rho_a}\odot(SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_b)dt\right),$$

$$\le D\left(\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\odot(SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_a)dt,\tilde{0}\right)+D\left(\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\odot(SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_b)dt,\tilde{0}\right),$$

$$=\frac{(x-\rho_a)^2}{\rho_b-\rho_a}D\left((SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_a)dt,\tilde{0}\right)+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}D\left((SVNR)\int_0^1 t\odot\varphi'(tx+(1-t)\rho_b)dt,\tilde{0}\right),$$

$$\le\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\int_0^1 tD\big(\varphi'(tx+(1-t)\rho_a),\tilde{0}\big)dt+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\int_0^1 tD\big(\varphi'(tx+(1-t)\rho_b),\tilde{0}\big)dt, \quad (5)$$

Since $D(\varphi',\tilde{0})$ be $(\alpha,\beta,\gamma,\delta)$ −convex function in mixed kind and $D(\varphi'(x),\tilde{0})\le M$, we have

$$D\big(\varphi'(tx+(1-t)\rho_a),\tilde{0}\big)\le t^{\alpha\gamma}D\big(\varphi'(x),\tilde{0}\big)+\big(1-t^\beta\big)^\delta D\big(\varphi'(\rho_a),\tilde{0}\big)\le M\left[t^{\alpha\gamma}+\big(1-t^\beta\big)^\delta\right] \quad (6)$$

$$D\big(\varphi'(tx+(1-t)\rho_b),\tilde{0}\big)\le t^{\alpha\gamma}D\big(\varphi'(x),\tilde{0}\big)+\big(1-t^\beta\big)^\delta D\big(\varphi'(\rho_b),\tilde{0}\big)\le M\left[t^{\alpha\gamma}+\big(1-t^\beta\big)^\delta\right] \quad (7)$$

Now using (6) and (7) in (5) we get (4).

Corollary 2.5 *In Theorem 2.4, one can see the following.*

1. If one takes $\gamma=\delta=1,\alpha\in[0,1]$ and $\beta\in(0,1]$, in (4), one has the SVN −Ostrowski inequality for (α,β) −convex functions in 1^{st} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le M\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)\kappa(x).$$

2. If one takes $\beta=\gamma=1,\alpha\in[0,1]$ and $\delta\in[0,1]$, in (4), then one has the SVN − Ostrowski inequality for (α,δ) − convex functions in 2^{nd} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le M\left(\frac{1}{\alpha+2}+\frac{1}{(\delta+1)(\delta+2)}\right)\kappa(x).$$

3. If one takes $\alpha=\delta=s,\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (4), then one has the SVN −Ostrowski inequality for (s,r) −convex functions in mixed kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le M\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)\kappa(x). \quad (4)$$

4. If one takes $\alpha=\beta=r$ and $\gamma=\delta=1$, where $r\in(0,1]$ in (4), then one has the SVN − Ostrowski inequality for r − convex functions in 1^{st} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le M\left(\frac{1}{r+2}+\frac{B\left(\frac{2}{r},2\right)}{r}\right)\kappa(x).$$

5. If one takes $\beta=\gamma=1,\alpha=\delta=s$ where $s\in[0,1]$, in (4), then one has the SVN − Ostrowski inequality for s − convex functions in 2^{nd} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le M\left(\frac{1}{s+1}\right)\kappa(x).$$

6. If one takes $\alpha=\delta=0$ and $\beta=\gamma=1$ in (4), then one has the SVN −Ostrowski inequality for P −convex functions:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le M\kappa(x).$$

7. If one takes $\alpha=\beta=\gamma=\delta=1$, in (4), then one has the SVN −Ostrowski inequality for convex functions:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{2}\kappa(x).$$

Theorem 2.6 *Suppose all the assumptions of Lemma* **2.3** *hold. Additionally, assume that* $[D(\varphi',\tilde{0})]^q$ *is* $(\alpha,\beta,\gamma,\delta)$ *−convex function on* $[\rho_a,\rho_b], q\geq 1$ *and* $D(\varphi'(x),\tilde{0})\leq M$*. Then for each* $x\in(\rho_a,\rho_b)$ *the following inequality holds:* $D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$

$$\leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\kappa(x). \quad (8)$$

Proof. From the Inequality (5) and power mean inequality [36]

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq \frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \quad (9)$$

Since $[D(\varphi',\tilde{0})]^q$ be $(\alpha,\beta,\gamma,\delta)-$ convex function in mixed kind and $D(\varphi'(x),\tilde{0})\leq M$, we have

$$\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q \leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q + \left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_a),\tilde{0}\right)\right]^q \leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right], \quad (10)$$

$$\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q \leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q + \left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_b),\tilde{0}\right)\right]^q \leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]. \quad (11)$$

Now using (10) and (11) in (9) we get (8).

Corollary 2.7*In Theorem* 2.6, *one can see the following.*

1. If one takes $q=1$, one has the Theorem 0.23.
2. If one takes $\gamma=\delta=1,\alpha\in[0,1]$ and $\beta\in(0,1]$, in (8), one has the SVN −Ostrowski inequality for (α,β) −convex functions in 1^{st} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}\kappa(x).$$

3. If one takes $\beta=\gamma=1,\alpha\in[0,1]$ and $\delta\in[0,1]$, in (8), then one has the $SVN-$Ostrowski inequality for $(\alpha,\delta)-$ convex functions in 2^{nd} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{(\alpha+2)}+\frac{1}{(\delta+1)(\delta+2)}\right)^{\frac{1}{q}}\kappa(x).$$

4. If one takes $\alpha=\delta=s,\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (8), then one has the SVN −Ostrowski inequality for (s,r) −convex functions in mixed kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\kappa(x).$$

5. If one takes $\alpha=\beta=r$ and $\gamma=\delta=1$, where $r\in(0,1]$ in (8), then one has the $SVN-$Ostrowski inequality for $r-$convex functions in 1^{st} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{r+2}+\frac{B\left(\frac{2}{r},2\right)}{r}\right)^{\frac{1}{q}}\kappa(x). \quad (9)$$

6. If one takes $\beta=\gamma=1,\alpha=\delta=s$ where $s\in[0,1]$, in (8), then one has the $SVN-$Ostrowski inequality for $r-$convex functions in 2^{nd} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\kappa(x). \quad (9)$$

7. If one takes $\alpha=\delta=0$ and $\beta=\gamma=1$ in (8), then one has the SVN −Ostrowski inequality for P −convex functions:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\kappa(x). \quad (10)$$

8. If one takes $\alpha=\beta=\gamma=\delta=1$, in (8), then one has the SVN −Ostrowski inequality for convex functions:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M}{2}\kappa(x).$$

Theorem 2.8 *Suppose all the assumptions of Lemma* **2.3** *hold. Additionally, assume that* $[D(\varphi',\tilde{0})]^q$ *is* $(\alpha,\beta,\gamma,\delta)-$ *convex function on* $[\rho_a,\rho_b], q\geq 1$ *and* $D(\varphi'(x),\tilde{0})\leq M$*. Then* $\forall x\in(\rho_a,\rho_b)$ *the following inequality holds:*

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot (SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq M\left[\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{1}{6+5\alpha\gamma+\alpha\gamma^2}+\frac{\Gamma\left[\frac{2+\beta}{\beta}\right]\Gamma[2+\delta]}{2\Gamma\left[2+\frac{2}{\beta}+\delta\right]}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{1}{3+\alpha\gamma}+\frac{\Gamma\left[\frac{3+\beta}{\beta}\right]\Gamma[1+\delta]}{3\Gamma\left[1+\frac{3}{\beta}+\delta\right]}\right)^{\frac{1}{q}}\Bigg]\kappa(x). \quad (12)$$

Proof. From the Inequality (5) and improved power mean inequality

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\leq\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}.$$

Since $[D(\varphi',\tilde{0})]^q$ be $(\alpha,\beta,\gamma,\delta)$ −convex function in mixed kind and $D(\varphi'(x),\tilde{0})\leq M$, we have

$$\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q\leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q+\left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_a),\tilde{0}\right)\right]^q\leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right], \quad (14)$$

$$\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q\leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q+\left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_b),\tilde{0}\right)\right]^q\leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]. \quad (15)$$

Now using (14) and (15) in (13) we get,

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\leq M\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}$$

$$+M\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}$$

$$+M\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}$$

$$+M\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}.$$

Therefore

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\leq M\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left[\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\gamma+1}(1-t)dt+\int_0^1 t\left(1-t^\beta\right)^{\delta+1}dt\right)^{\frac{1}{q}}+\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\gamma+2}dt+\int_0^1 t^2\left(1-t^\beta\right)^\delta dt\right)^{\frac{1}{q}}\right]$$

$$+M\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left[\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\gamma+1}(1-t)dt+\int_0^1 t\left(1-t^\beta\right)^{\delta+1}dt\right)^{\frac{1}{q}}+\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\gamma+2}dt+\int_0^1 t^2\left(1-t^\beta\right)^\delta dt\right)^{\frac{1}{q}}\right].$$

Hence,

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \quad (13)$$

$$\leq M\left[\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{1}{6+5\alpha\gamma+\alpha\gamma^2}+\frac{\Gamma\left[\frac{2+\beta}{\beta}\right]\Gamma[2+\delta]}{2\Gamma\left[2+\frac{2}{\beta}+\delta\right]}\right)^{\frac{1}{q}}+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{1}{3+\alpha\gamma}+\frac{\Gamma\left[\frac{3+\beta}{\beta}\right]\Gamma[1+\delta]}{3\Gamma\left[1+\frac{3}{\beta}+\delta\right]}\right)^{\frac{1}{q}}\right]\kappa(x).$$

Theorem 2.10 *Suppose all the assumptions of Lemma* **2.3** *hold. Additionally, assume that* $[D(\varphi',\tilde{0})]^q$ *is* $(\alpha,\beta,\gamma,\delta)-$ *convex function on* $[\rho_a,\rho_b], q>1$ *and* $D(\varphi'(x),\tilde{0})\leq M$. *Then for each* $x\in(\rho_a,\rho_b)$, *the following inequality holds:*

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \quad (14)$$

$$\leq\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\kappa(x), \quad (16)$$

where $p^{-1}+q^{-1}=1$.

Proof. From the Inequality (5) and Hölder's inequality [37]

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\leq\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}.$$

Since $[D(\varphi',\tilde{0})]^q$ be $(\alpha,\beta,\gamma,\delta)$ −convex function in mixed kind and $D(\varphi'(x),\tilde{0})\leq M$, we have

$$\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q\leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q+\left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_a),\tilde{0}\right)\right]^q\leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right], \quad (18)$$

$$\begin{aligned}&[D(\varphi'(tx+(1-t)\rho_b),\tilde{0})]^q\\ &\quad\le t^{\alpha\gamma}[D(\varphi'(x),\tilde{0})]^q\\ &\quad+(1-t^\beta)^\delta[D(\varphi'(\rho_b),\tilde{0})]^q\\ &\quad\le M^q\left[t^{\alpha\gamma}+(1-t^\beta)^\delta\right].\end{aligned}\tag{19}$$

Now using (18) and (19) in (17) we get (16).

Corollary 2.11 *In Theorem* 2.10, *one can see the following.*

1. If one takes $\gamma=\delta=1,\alpha\in[0,1]$ and $\beta\in(0,1]$, in (16), one has the SVN −Ostrowski inequality for (α,β) −convex functions in 1^{st} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}\kappa(x).$$

2. If one takes $\beta=\gamma=1,\alpha\in[0,1]$ and $\delta\in[0,1]$, in (16), then one has the $SVN-$Ostrowski inequality for $(\alpha,\delta)-$convex functions in 2^{nd} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{1}{\delta+1}\right)^{\frac{1}{q}}\kappa(x).$$

3. If one takes $\alpha=\delta=s,\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (16), then one has the SVN −Ostrowski inequality for (s,r) −convex functions in mixed kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\kappa(x).$$

4. If one takes $\alpha=\beta=r$ and $\gamma=\delta=1$, where $r\in(0,1]$ in (16), then one has the $SVN-$ Ostrowski inequality for r −convex functions in 1^{st} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{r+1}+\frac{B\left(\frac{1}{r},2\right)}{r}\right)^{\frac{1}{q}}\kappa(x).$$

5. If one takes $\beta=\gamma=1,\alpha=\delta=s$ where $s\in[0,1]$ in (16), then one has the $SVN-$Ostrowski inequality for $s-$convex functions in 2^{nd} kind:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\kappa(x).$$

6. If one takes $\alpha=\delta=0$ and $\beta=\gamma=1$ in (16), then one has the SVN −Ostrowski inequality for P −convex functions:

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{(2)^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}}\kappa(x).$$

7. If one takes $\alpha=\beta=\gamma=\delta=1$ in (16), then one has the SVN −Ostrowski inequality for convex functions: (17)

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le\frac{M}{(p+1)^{\frac{1}{p}}}\kappa(x).$$

Theorem 2.12 *Suppose all the assumptions of Lemma* **2.3** *hold. Additionally, assume that* $[D(\varphi',\tilde{0})]^q$ *is* $(\alpha,\beta,\gamma,\delta)-$ *convex function on* $[\rho_a,\rho_b],q>1$ *and* $D(\varphi'(x),\tilde{0})\le M$. *Then for each* $x\in(\rho_a,\rho_b)$, *the following inequality holds:*

$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$

$$\le M\left[\left(\frac{1}{2+3p+p^2}\right)^{\frac{1}{p}}\left(\begin{array}{c}\frac{1}{2+3\alpha\gamma+\alpha\gamma^2}\\ +\Gamma\left[\frac{1+\delta}{2}\right]\left(\frac{2\Gamma\left[1+\frac{1}{\beta}\right]}{\Gamma\left[1+\frac{1}{\beta}+\delta\right]}-\frac{\Gamma\left[\frac{2+\beta}{\beta}\right]}{\Gamma\left[1+\frac{2}{b}+\delta\right]}\right)\end{array}\right)^{\frac{1}{q}}+\left(\frac{1}{2+p}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha\gamma+2}+\frac{\Gamma\left[\frac{2+\beta}{\beta}\right]\Gamma[1+\delta]}{2\Gamma\left[1+\frac{2}{\beta}+\delta\right]}\right)^{\frac{1}{q}}\right]\kappa(x).$$

where $p^{-1}+q^{-1}=1$. (20)

Proof. From the Inequality (5) and Hölder's İşcan inequality

$$\begin{aligned}&D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\ &\le\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1(1-t)t^pdt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)[D(\varphi'(tx+(1-t)\rho_a),\tilde{0})]^qdt\right)^{\frac{1}{q}}\\ &+\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1t[D(\varphi'(tx+(1-t)\rho_a),\tilde{0})]^qdt\right)^{\frac{1}{q}}\\ &+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1t[D(\varphi'(tx+(1-t)\rho_b),\tilde{0})]^qdt\right)^{\frac{1}{q}}\\ &+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1(1-t)t^pdt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)[D(\varphi'(tx+(1-t)\rho_b),\tilde{0})]^qdt\right)^{\frac{1}{q}}.\end{aligned}\tag{21}$$

This gives

$$\begin{aligned}&D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\ &\le\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left[\left(\int_0^1(1-t)t^pdt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)[D(\varphi'(tx+(1-t)\rho_a),\tilde{0})]^qdt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1t[D(\varphi'(tx+(1-t)\rho_a),\tilde{0})]^qdt\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left[\left(\int_0^1 t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 (1-t)t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}\right]. \quad (22)$$

Since $[D(\varphi',\tilde{0})]^q$ be $(\alpha,\beta,\gamma,\delta)$ −convex function in mixed kind and $D(\varphi'(x),\tilde{0}) \leq M$, we have

$$\begin{aligned}\left[D\left(\varphi'(tx+(1-t)\rho_a),\tilde{0}\right)\right]^q &\leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q + \left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_a),\tilde{0}\right)\right]^q \\ &\leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right], \end{aligned} \quad (23)$$

$$\begin{aligned}\left[D\left(\varphi'(tx+(1-t)\rho_b),\tilde{0}\right)\right]^q &\leq t^{\alpha\gamma}\left[D\left(\varphi'(x),\tilde{0}\right)\right]^q + \left(1-t^\beta\right)^\delta\left[D\left(\varphi'(\rho_b),\tilde{0}\right)\right]^q \\ &\leq M^q\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]. \end{aligned} \quad (24)$$

Now using (23) and (24) in (22) we get,

$$\begin{aligned} D&\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \\ &\leq M\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left[\left(\int_0^1(1-t)t^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}} + \left(\int_0^1 t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}\right] \\ &+M\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left[\left(\int_0^1 t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}} + \left(\int_0^1(1-t)t^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}\right]. \end{aligned} \quad (25)$$

Therefore

$$\begin{aligned} D&\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \\ &\leq M\left[\left(\int_0^1(1-t)t^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}} + \left(\int_0^1 t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[t^{\alpha\gamma}+\left(1-t^\beta\right)^\delta\right]dt\right)^{\frac{1}{q}}\right]\kappa(x). \end{aligned} \quad (26)$$

Finally we get

$$\begin{aligned} D&\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \\ &\leq M\left[\left(\frac{1}{2+3p+p^2}\right)^{\frac{1}{p}}\left(\frac{1}{2+3\alpha\gamma+\alpha\gamma^2}+\Gamma\left[\frac{1+\delta}{2}\right]\left(\frac{2\Gamma\left[1+\frac{1}{\beta}\right]}{\Gamma\left[1+\frac{1}{\beta}+\delta\right]}-\frac{\Gamma\left[\frac{2+\beta}{\beta}\right]}{\Gamma\left[1+\frac{2}{b}+\delta\right]}\right)\right)^{\frac{1}{q}} \right. \\ &\left. +\left(\frac{1}{2+p}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha\gamma+2}+\frac{\Gamma\left[\frac{2+\beta}{\beta}\right]\Gamma[1+\delta]}{2\Gamma\left[1+\frac{2}{\beta}+\delta\right]}\right)^{\frac{1}{q}}\right]\kappa(x). \end{aligned} \quad (27)$$

(21)

3. SVN-Ostrowski type midpoint Inequalities via $(\alpha,\beta,\gamma,\delta)-$ convex functions in mixed kind

Remark 3.1*In Theorem* 2.6, *one can see the following.*

(22)

1. If one takes $x=\frac{\rho_a+\rho_b}{2}$ in (8), one has the SVN −Ostrowski Midpoint inequality for $(\alpha,\beta,\gamma,\delta)-$ convex functions in Mixed kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

2. If one takes $x=\frac{\rho_a+\rho_b}{2},\gamma=\delta=1,\alpha\in[0,1]$ and $\beta\in(0,1]$ in (8), one has the $SVN-$ Ostrowski Midpoint inequality for (α,β) −convex functions in 1^{st} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}.$$

3. If one takes $x=\frac{\rho_a+\rho_b}{2},\beta=\gamma=1,\alpha\in[0,1]$ and $\delta\in[0,1]$ in (8), then one has the SVN −Ostrowski Midpoint inequality for (α,δ) −convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{(\alpha+2)}+\frac{1}{(\delta+1)(\delta+2)}\right)^{\frac{1}{q}}.$$

4. If one takes $x=\frac{\rho_a+\rho_b}{2},\alpha=\delta=s,\beta=\gamma=r,$ where $s\in[0,1]$ and $r\in(0,1]$ in (8), then one has the $SVN-$Ostrowski Midpoint inequality for (s,r) −convex functions in mixed kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

5. If one takes $x = \frac{\rho_a+\rho_b}{2}, \alpha = \beta = r$ and $\gamma = \delta = 1$, where $r \in (0,1]$ in (8), then one has the $SVN-$ Ostrowski Midpoint inequality for r −convex functions in 1^{st} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{r+2}+\frac{B\left(\frac{2}{r},2\right)}{r}\right)^{\frac{1}{q}}.$$

6. If one takes $x = \frac{\rho_a+\rho_b}{2}, \beta = \gamma = 1, \alpha = \delta = s$ where $s \in [0,1]$, in (8), then one has the $SVN-$ Ostrowski Midpoint inequality for r −convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}.$$

7. If one takes $x = \frac{\rho_a+\rho_b}{2}, \alpha = \delta = 0$ and $\beta = \gamma = 1$ in (8), then one has the SVN −Ostrowski Midpoint inequality for P −convex functions:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{(2)^{2-\frac{1}{q}}}.$$

8. If one takes $x = \frac{\rho_a+\rho_b}{2}, \alpha = \beta = \gamma = \delta = 1$, in (8), then one has the $SVN-$ Ostrowski Midpoint inequality for convex functions:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{4}.$$

Remark 3.2 *In Theorem* 2.10, *one can see the following.*

1. If one takes $x = \frac{\rho_a+\rho_b}{2}$ in (16), one has the SVN −Ostrowski Midpoint inequality for $(\alpha,\beta,\gamma,\delta)-$ convex functions in Mixed kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

2. If one takes $x = \frac{\rho_a+\rho_b}{2}, \gamma = \delta = 1, \alpha \in [0,1]$ and $\beta \in (0,1]$, in (16), one has the $SVN-$ Ostrowski Midpoint inequality for (α,β) −convex functions in 1^{st} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}.$$

3. If one takes $x = \frac{\rho_a+\rho_b}{2}, \beta = \gamma = 1, \alpha \in [0,1]$ and $\delta \in [0,1]$, in (16), then one has the SVN −Ostrowski Midpoint inequality for (α,δ) −convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{1}{\delta+1}\right)^{\frac{1}{q}}.$$

4. If one takes $x = \frac{\rho_a+\rho_b}{2}, \alpha = \delta = s, \beta = \gamma = r$, where $s \in [0,1]$ and $r \in (0,1]$ in (16), then one has the SVN −Ostrowski Midpoint inequality for (s,r) −convex functions in mixed kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

5. If one takes $x = \frac{\rho_a+\rho_b}{2}, \alpha = \beta = r$ and $\gamma = \delta = 1$, where $r \in (0,1]$ in (16), then one has the $SVN-$ Ostrowski Midpoint inequality for r −convex functions in 1^{st} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{r+1}+\frac{B\left(\frac{1}{r},2\right)}{r}\right)^{\frac{1}{q}}.$$

6. If one takes $x = \frac{\rho_a+\rho_b}{2}, \beta = \gamma = 1, \alpha = \delta = s$ where $s \in [0,1]$ in (16), then one has the SVN −Ostrowski Midpoint inequality for s −convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{(2)^{\frac{1}{q}-1}M(\rho_b-\rho_a)}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}.$$

7. If one takes $x = \frac{\rho_a+\rho_b}{2}, \alpha = \delta = 0$ and $\beta = \gamma = 1$ in (16), then one has the $SVN-$ Ostrowski Midpoint inequality for P −convex functions:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{(2)^{\frac{1}{q}-1}M(\rho_b-\rho_a)}{(p+1)^{\frac{1}{p}}}.$$

8. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $\alpha = \beta = \gamma = \delta = 1$ in (16), then one has the SVN −Ostrowski Midpoint inequality for convex functions:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2(p+1)^{\frac{1}{p}}}.$$

4. Conclusion and Remarks:

Ostrowski inequality is one of the most celebrated inequalities, In this paper, we presented the generalized notion of $(\alpha,\beta,\gamma,\delta)-$convex (concave) functions in mixed kind. This class of functions contains many important classes including class of $(\alpha,\beta)-$convex (concave) functions in 1^{st} and 2^{nd} kind [19], $(s,r)-$convex (concave) functions in mixed kind [2], $s-$convex (concave) functions in 1^{st} and 2^{nd} kind [5], $P-$convex (concave) functions [18], quasi convex(concave) functions [21] and the class of convex (concave) functions[3]). We have stated our first main result in section 2, the generalization of Ostrowski inequality [33] via SVN-Riemann integrals with $(\alpha,\beta,\gamma,\delta)-$convex (concave) functions in mixed kind. Further, we used different techniques including Hölder's and Hölder's Iscan inequality[37] and power mean and improved power mean inequality[36] for generalization of SVN-Ostrowski inequality.

5. Future Ideas:

1. One may do similar work to generalize all results stated in this article by applying weights.
2. One may also do similar work by using various different classes of functions.
3. One may also generalize this work in fractional integral form.
4. One may try to state all results stated in this article for fractional integral with respect to another function.
5. One may also state all results stated in this article for higher order derivatives.
6. One may also state all results stated in this article for multivariable real valued functions.
7. One may also state all results stated in this article for quantum Calculus.
8. One may also state all results stated in this article in time scale domain.

6. References

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