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COMPLETE NONCOMPACT SUBMANIFOLDS OF MANIFOLDS WITH NEGATIVE CURVATURE

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Abstract. In this paper, for an m-dimensional ($m \geq 5$) complete noncompact submanifold M immersed in an n-dimensional ($n \geq 6$) simply connected Riemannian manifold N with negative sectional curvature, under suitable constraints on the squared norm of the second fundamental form of M, the norm of its weighted mean curvature vector $|\boldsymbol{H}_f|$ and the weighted real-valued function f, we can obtain:

- several one-end theorems for M;
- two Liouville theorems for harmonic maps from M to complete Riemannian manifolds with nonpositive sectional curvature.

1. Introduction

As we know, Hodge theory is an important and useful tool in the study of the topology of compact Riemannian manifolds. However, the Hodge theory does not work in noncompact manifolds. But, luckily, as revealed by Anderson [1] and Dodziuk [16], the L^2-Hodge theory works well in certain noncompact cases. In the range of this philosophy, interesting results for L^2 harmonic 1-forms on stable minimal hypersurfaces could be expected. In fact, by applying the nonexistence of L^2 harmonic 1-forms, Palmer [38] showed that:

- *If there exists a codimension-one cycle on a complete minimal hypersurface $\mathcal{M}$ in Euclidean space, which does not separate $\mathcal{M}$, then $\mathcal{M}$ is unstable.*

Miyaoka, by mainly using Bochner's vanishing technique, proved that a complete noncompact stable minimal hypersurface in a nonnegatively curved manifold has no nontrivial L^2 harmonic 1-forms (see [37] for details). We prefer to refer [6, 39] to readers for a survey about important conclusions related to L^2 harmonic forms on noncompact manifolds. The L^2 theory is well studied,

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while the L^p theory is less developed in the case $p \neq 2$. So, it is quite natural to ask:

- (**Problem A**). *Is it feasible to improve interesting results about L^2 harmonic forms on noncompact manifolds to the situation of L^p harmonic forms? What is the main difficulty?*

The purpose of this paper is trying to give affirmative answer and nice examples to **Problem A**.

Through a nice combination of an existence theorem for nonconstant bounded harmonic functions with finite energy (see [4]) and a Liouville theorem for harmonic functions in [40] (proven by Schoen and Yau), Cao, Shen and Zhou showed that a complete stable minimal hypersurface in an $(m+1)$-dimensional $(m \geq 3)$ Euclidean space $\mathbb{R}^{m+1}$ has only one end. By using Cao-Shen-Zhu's this idea, Wang and Xia [43, Theorems 1.1 and 1.2] can obtain the nonexistence result of nontrivial L^2 harmonic forms and some one-end theorems for open submanifolds in a simply connected Riemannian manifold with negative sectional curvature. Naturally, one can propose **Problem A** with respect to Wang-Xia's this conclusion, that is, *Could Wang-Xia's nonexistence result of nontrivial L^2 harmonic forms in* [43] *be improved to the L^p situation?* The answer is affirmative.

Before stating our main results, we need to introduce some notations. Let N be an n-dimensional Riemannian manifold, $k \in \{1, \ldots, n-1\}$ and r_0 be a fixed real number. We say that the k-th Ricci curvature of N is greater than or equal to r_0 (resp., greater than r_0), if for any $p_0 \in N$ and any $k+1$ orthonormal vectors $v, e_1, \ldots, e_k \in T_{p_0}N$, we have $\sum_{i=1}^{k} \sec(v \wedge e_i) \geq r_0$ (resp., $> r_0$), where $T_{p_0}N$ is the tangent space (of N) at the point p_0, and for each $i = 1, \ldots, k$, $\sec(v \wedge e_i)$ denotes the sectional curvature of the plane spanned by v and e_i. Denote this fact by $\mathrm{Ric}_{(k)}(N) \geq r_0$ (resp., $> r_0$). It is clear that $\mathrm{Ric}_{(1)}(N) \geq r_0$ (resp., $> r_0$) means $\sec(N) \geq r_0$ (resp., $> r_0$) and $\mathrm{Ric}_{(n-1)}(N) \geq r_0$ (resp., $> r_0$) is equivalent to the fact that the Ricci curvature of N is not less than r_0, i.e., $\mathrm{Ric}(N) \geq r_0$ (resp., $> r_0$) with $\mathrm{Ric}(\cdot)$ the Ricci tensor[1]. We refer readers to [14, 20, 21, 28, 43] for some interesting results about manifolds with positive k-th Ricci curvature. Our main results are:

Theorem 1.1. *Let N be a complete simply connected Riemannian n-manifold $(n \geq 6)$ with sectional curvature $K_N \leq -1$ and let M be a complete noncompact m-dimensional $(m \geq 5)$ submanifold immersed in N. Assume that the $(m-1)$-th Ricci curvature of N is no less than $-(m-1)c$ for some constant $c \in [1, \frac{m}{4})$. Denote by S and $\mathbf{H}_f = \mathbf{H} + \frac{1}{m}(\overline{\nabla} f)^{\perp}$ the squared norm of the second fundamental form and the weighted mean curvature vector of M, respectively, where $\mathbf{H}$ is the mean curvature vector of M, f is a smooth real-valued function defined on M, $(\overline{\nabla} f)^{\perp}$ is the projection of $\overline{\nabla} f$ onto the normal bundle $T^{\perp}M$,*

[1] By abuse of notation, we will also use $\mathrm{Ric}(\cdot)$ to denote the Ricci tensor on submanifolds of N.

and

$$\sup_{x\in M}|\overline{\nabla} f|(x) \le c^+ < \frac{2(m-1)\sqrt{c}}{\sqrt{m}}$$

for some nonnegative constant c^+*, with* $\overline{\nabla}$ *the gradient operator*[2] *of* N*. For any constant* $1 < p < \infty$*, suppose that one of the following conditions was satisfied*:

(i) *There exists a nonnegative constant* $d < \frac{(m-1)(\sqrt{m}-2\sqrt{c})}{m^{3/2}}$ *such that*

$$\sup_{x\in M}|\mathbf{H}_f|(x) \le d \tag{1.1}$$

and

$$\sup_{x\in M} B(S,|\mathbf{H}_f|)(x) < \frac{((p-1)(m-1)+1)\left(m-1-md-c^+\right)^2 - p^2(m-1)^2c}{p^2(m-1)}, \tag{1.2}$$

where

$$\begin{aligned} &B(S,|\mathbf{H}_f|) \\ :=\ & \tfrac{m-1}{m}S - 2(m-1)|\mathbf{H}_f|^2 - \tfrac{2(m-1)}{m^2}|(\overline{\nabla} f)^\perp|^2 + \tfrac{4(m-1)}{m}\langle \mathbf{H}_f, (\overline{\nabla} f)^\perp\rangle \\ &+ (m-2)|\mathbf{H}_f|\sqrt{\tfrac{m-1}{m}\left(S-m|\mathbf{H}_f|^2\right) - \tfrac{m-1}{m^2}|(\overline{\nabla} f)^\perp|^2 + \tfrac{2(m-1)}{m}\langle \mathbf{H}_f, (\overline{\nabla} f)^\perp\rangle} \\ &+ \tfrac{m-2}{m}|(\overline{\nabla} f)^\perp|\sqrt{\tfrac{m-1}{m}\left(S-m|\mathbf{H}_f|^2\right) - \tfrac{m-1}{m^2}|(\overline{\nabla} f)^\perp|^2 + \tfrac{2(m-1)}{m}\langle \mathbf{H}_f, (\overline{\nabla} f)^\perp\rangle}. \end{aligned} \tag{1.3}$$

(ii) *There exists a nonnegative constant* $d < \frac{(m-1)(\sqrt{m}-2\sqrt{c})}{m^{3/2}}$ *such that*

$$\sup_{x\in M}|\mathbf{H}_f|(x) \le d, \tag{1.4}$$

$$\sup_{x\in M} S(x) < \frac{((2p-2)(m-1)+2)\left(m-1-md-c^+\right)^2 - 2p^2(m-1)^2c}{p^2(m-1)^{3/2}}. \tag{1.5}$$

Then there exist no nontrivial L^p *harmonic* 1*-forms on* M *and* M *has only one end.*

Remark 1.2. Naturally, one might define the so-called *weighted mean curvature* (or, f-mean curvature) as $H_f = H - \frac{1}{m}\langle \overline{\nabla} f, \vec{\eta}\rangle$, where H is the regular mean curvature of the submanifold M and $\vec{\eta}$ is its outward unit normal vector. One can find that in some literatures the factor $\frac{1}{m}$ would be removed (i.e., do not directly embody the dimensional information of submanifolds). However, there is no essential difference between keeping the factor $\frac{1}{m}$ and removing it. One can see, e.g., [24, 44, 45] for the notions of weighted mean curvature vector and weighted mean curvature. Besides, (under suitable constraints on weighted mean curvature) some interesting results have been proven therein.

[2] Without specification and in order to avoid repetition, in the sequel $\overline{\nabla}$ denotes the gradient operator on N.

Theorem 1.3. *Let N be a complete simply connected Riemannian n-manifold ($n \geq 6$) with sectional curvature $K_N \leq -1$ and let M be a complete noncompact m-dimensional ($m \geq 5$) submanifold immersed in N. Assume that the ($m-1$)-th Ricci curvature of N is no less than $-(m-1)c$ for some constant $c \in [1, \frac{m}{4})$, and $\sup_{x\in M} |\overline{\nabla} f|(x) \leq c^+ < \frac{2(m-1)\sqrt{c}}{\sqrt{m}}$ for some nonnegative constant c^+. For any $1 < p < \infty$, there exist positive constants c_1, c_2, c_3, c_4 depending only on m such that if*

$$\begin{aligned}
&\sup_{x\in M} |\mathbf{H}_f| \leq c_1, \quad \int_M |\mathbf{H}_f|^m \leq c_2, \\
&\int_M \left(S - m|\mathbf{H}_f|^2 + m|(\overline{\nabla} f)^\perp|^2\right)^{\frac{m}{2}} \leq c_3, \qquad (1.6)\\
&\int_M |\overline{\nabla} f|^m \leq c_4, \quad c_4 \leq m^m c_2,
\end{aligned}$$

then M admits no nontrivial L^p harmonic 1-forms and has only one end.

Remark 1.4. (1) Clearly, if M was taken to be an *f-minimal* submanifold in a hyperbolic space, by Theorems 1.1 and 1.3 one has:

- Let M be a complete noncompact m-dimensional ($m \geq 5$) f-minimal submanifold immersed in $\mathbb{H}^n(-1)$, the n-dimensional ($n \geq 6$) hyperbolic space of constant sectional curvature -1. As in Theorem 1.1, f is a smooth real-valued function defined on M, $(\overline{\nabla} f)^\perp$ is the projection of $\overline{\nabla} f$ onto the normal bundle $T^\perp M$, $\sup_{x\in M} |\overline{\nabla} f|(x) \leq c^+ < \frac{2(m-1)}{\sqrt{m}}$ for some nonnegative constant c^+, and denote by S the squared norm of the second fundamental form of M.

  (i) For any constant $1 < p < \infty$, if

$$\sup_{x\in M}\left[\tfrac{m-1}{m}S - \tfrac{2(m-1)}{m^2}|(\overline{\nabla} f)^\perp|^2 + \tfrac{m-2}{m}|(\overline{\nabla} f)^\perp|\sqrt{\tfrac{m-1}{m}S - \tfrac{m-1}{m^2}|(\overline{\nabla} f)^\perp|^2}\right](x)$$
$$< \frac{((p-1)(m-1)+1)\left(m-1-c^+\right)^2 - p^2(m-1)^2}{p^2(m-1)},$$

  then there exist no nontrivial L^p harmonic 1-forms on M and M has only one end. Especially, when f degenerates into a constant function, c^+ can be chosen to be $c^+ \equiv 0$ and correspondingly the above inequality for the squared norm of the second fundamental form becomes

$$\sup_{x\in M} S(x) < \frac{m\left[((p-1)(m-1)+1) - p^2\right]}{p^2}.$$

  Of course, in this setting, the nonexistence of nontrivial L^p harmonic 1-forms and the existence of only one-end on minimal submanifold M can still be obtained.

(ii) For any constant $1 < p < \infty$, there exist positive constants c_3, c_4 depending only on m such that if

$$\int_M \left(S + m|(\overline{\nabla} f)^{\perp}|^2\right)^{\frac{m}{2}} \le c_3, \qquad \int_M |\overline{\nabla} f|^m \le c_4,$$

then M admits no nontrivial L^p harmonic 1-forms and has only one end. Especially, when f degenerates into a constant function, c^+ can be chosen to be $c^+ \equiv 0$ and the above assumption can be simplified as

$$\int_M S^{\frac{m}{2}} \le c_3$$

for some positive constant c_3 depending only on m.

(2) In (1.6), the condition $\int_M |\overline{\nabla} f|^m \le c_4$ implies that there does not exist any positive constant $c_+ > 0$ such that $|\overline{\nabla} f| \ge c_+$ holds on unbounded domains of M. Speaking roughly, $|\overline{\nabla} f|$ cannot be away from zero on unbounded domains of M.

(3) In the past 10 years, the study of f-minimal submanifolds attracts geometers' attention and many interesting results have been obtained. For instance,

- obviously, an f-minimal surface in the 3-dimensional Euclidean space $\mathbb{R}^3$ implies its mean curvature vector satisfies $\boldsymbol{H} = -\frac{1}{2}(\overline{\nabla} f)^{\perp}$. We know that a self-shrinker of mean curvature flow (MCF) in $\mathbb{R}^3$ is actually an immersed surface in $\mathbb{R}^3$ satisfying $H = \frac{1}{2}\langle x, \vec{n}\rangle$, where x is the position vector in $\mathbb{R}^3$ and $\vec{n}$ is the unit normal vector of the surface. Therefore, a self-shrinker of MCF in $\mathbb{R}^3$ is an f-minimal surface with $f = \frac{|x|^2}{2}$. Self-shrinkers are self-similar solutions to MCF and play an important role in the study of type-I singularities of the MCF. Many interesting classification results for self-shrinkers have been shown – see, e.g., [3, 10, 11, 13, 15] and references therein.
- Li and Wei [32] proved a compactness theorem for closed embedded f-minimal surfaces of fixed topology in a closed 3-manifold with positive Bakry-Émery Ricci curvature. They also gave a Lichnerowicz type lower bound of the first eigenvalue of weighted Laplacian on a compact manifold with positive Bakry-Émery Ricci curvature, and moreover showed that the lower bound is achieved only if the manifold is isometric to the Euclidean sphere of same dimension.

Hence, it is interesting and meaningful to investigate the geometry and topology of f-minimal submanifolds in a prescribed ambient space. This is exactly the reason why we prefer to clearly give a special case of Theorems 1.1 and 1.3 in the first item (1) of this remark – the nonexistence of nontrivial L^p harmonic 1-forms and the existence of only one-end on f-minimal submanifolds in the hyperbolic space.

The approaches for proving Theorems 1.1 and 1.3 can be also used to get Liouville type theorems for harmonic maps from submanifolds in manifolds of negative curvature. Harmonic maps are critical points of the energy functional defined on the space of maps between two Riemannian manifolds, and the Liouville type properties of harmonic maps have been studied extensively (see, e.g., [9,18,19,25,27,42]). In [40], Schoen and Yau proved that a harmonic map of finite energy from a complete Riemannian manifold with nonnegative Ricci curvature to a complete manifold with nonpositive sectional curvature must be constant. Using this Liouville theorem, they can show that:

- *Any smooth map of finite energy from a complete Riemannian manifold with nonnegative Ricci curvature to a compact manifold with nonpositive sectional curvature is homotopic to constant on each compact set.*

Comparing with Schoen-Yau's important result above, we can prove the following Liouville-type theorems:

Theorem 1.5. *Let N be a complete simply connected Riemannian n-manifold ($n \geq 6$) with sectional curvature $K_N \leq -1$ and let Q be a complete s-dimensional Riemannian manifold with nonpositive sectional curvature. Let M be a complete noncompact m-dimensional ($m \geq 5$) submanifold immersed in N. Assume that $(m-1)$-th Ricci curvature of N is no less than $-(m-1)c$ for some constant $c \in \left[1, \frac{(2ms+1)(m-1)}{8ms}\right)$. Denote by S and $\mathbf{H}_f = \mathbf{H} + \frac{1}{m}(\overline{\nabla} f)^{\perp}$ the squared norm of the second fundamental form and the weighted mean curvature vector of M, respectively, where $\mathbf{H}$ is the mean curvature vector of M, f is a smooth real-valued function defined on M, $(\overline{\nabla} f)^{\perp}$ is the projection of $\overline{\nabla} f$ onto the normal bundle $T^{\perp}M$, and*

$$\sup_{x\in M} |\overline{\nabla} f|(x) \leq c^{+} < \sqrt{\frac{8ms(m-1)c}{2ms+1}}$$

for some nonnegative constant c^{+}. Suppose that one of the following items was satisfied:

(i) *There exists a nonnegative constant $d < \frac{m-1}{m} - \sqrt{\frac{8s(m-1)c}{m(2ms+1)}}$ such that*

$$\sup_{x\in M} |\mathbf{H}_f|(x) \leq d \tag{1.7}$$

and

$$\sup_{x\in M} B(S, |\mathbf{H}_f|(x)) < \frac{(2ms+1)\left(m-1-md-c^{+}\right)^2 - 8ms(m-1)c}{8ms}, \tag{1.8}$$

where $B(S, |\mathbf{H}_f|)$ was defined by (1.3).

(ii) *There exists a nonnegative constant $d < \frac{m-1}{m} - \sqrt{\frac{8s(m-1)c}{m(2ms+1)}}$ such that*

$$\sup_{x\in M} |\mathbf{H}_f|(x) \leq d \tag{1.9}$$

and

$$\sup_{x\in M} S(x) < \frac{(2ms+1)\left(m-1-md-c^{+}\right)^2 - 8ms(m-1)c}{4ms(m-1)^{1/2}}. \tag{1.10}$$

Then any harmonic map from M to Q with finite energy is constant.

Theorem 1.6. *Let N be a complete simply connected Riemannian n-manifold ($n \geq 6$) with sectional curvature $K_N \leq -1$ and let Q be a complete s-dimensional Riemannian manifold with nonpositive sectional curvature. Let M be a complete noncompact m-dimensional ($m \geq 5$) submanifold immersed in N. Assume that $(m-1)$-th Ricci curvature of N is no less than $-(m-1)c$ for some constant $c \in \left[1, \frac{(2ms+1)(m-1)}{8ms}\right)$, and $\sup_{x\in M} |\overline{\nabla} f|(x) \leq c^+ < \sqrt{\frac{8ms(m-1)c}{2ms+1}}$ for some nonnegative constant c^+. There exist positive constants d_1, d_2, d_3, d_4 depending only on m and s such that if*

$$\begin{aligned} &\sup_{x\in M} |\mathbf{H}_f| \leq d_1, \ \int_M |\mathbf{H}_f|^m \leq d_2, \\ &\int_M \left(S - m|\mathbf{H}_f|^2 + m|(\overline{\nabla} f)^\perp|^2\right)^{\frac{m}{2}} \leq d_3, \\ &\int_M |\overline{\nabla} f|^m \leq d_4, \quad\quad d_4 \leq m^m d_2, \end{aligned} \tag{1.11}$$

then any harmonic map from M to Q with finite energy is constant.

Remark 1.7. Similar to (2) of Remark 1.4, the assumption $\int_M |\overline{\nabla} f|^m \leq d_4$ in (1.11) implies that $|\overline{\nabla} f|$ cannot be away from zero on unbounded domains of M.

Some useful facts will be mentioned in Section 2, and proofs of the above main results will be shown in Section 3.

2. Preliminaries

In this section, we list some known facts needed for proving our results. Let M be a complete noncompact m-dimensional Riemannian manifold. For a fixed point $x_0 \in M$, we denote by $B(x_0, r)$ the open geodesic ball of radius r with center x_0. Let $\lambda_1(B(x_0, r))$ be the first eigenvalue of the Laplacian of $B(x_0, r)$ with Dirichlet boundary condition. Denote by $\lambda_1(M)$ the first eigenvalue of M which can be defined as

$$\lambda_1(M) := \lim_{R\to+\infty} \lambda_1(B(x_0, R)). \tag{2.1}$$

It is easy to check that by the Domain Monotonicity Principle (see, e.g., [7, pp. 17–18]), the above limit exists and does not depend on the choice of the center x_0. Thus $\lambda_1(M)$ is well defined.

By introducing a geometric constant $c(\Omega)$ for bounded domains Ω on smooth Riemannian manifolds, which depends on vector fields with positive divergence (see [2, Definition 2.1] for details), (and estimating the gradient of a distance function in terms of $c(\Omega)$) Bessa and Montenegro [2, Theorem 4.3] can obtain estimates of eigenvalue of balls inside the cut locus and of domains $\Omega \subset M \cap$

$B_N(p,r)$ in submanifolds $M \subset_\varphi N$ with locally bounded mean curvature[3]. Then by letting the radius of geodesic balls tends to infinity, they have:

Lemma 2.1 ([2, Corollary 4.4]). *Let N be a complete simply connected Riemannian manifold with sectional curvature $K_N \leq -1$ and let M be a m-dimensional complete noncompact submanifold immersed in N. Assume that the mean curvature vector of M satisfies $|\boldsymbol{H}|(x) \leq l < \frac{m-1}{m}$, $\forall x \in M$. Then*

$$\lambda_1(M) \geq \frac{(m-1-ml)^2}{4}. \tag{2.2}$$

Remark 2.2. (1) The *strictly positive* lower bound estimate for $\lambda_1(M)$ in (2.2) generalizes McKean's classical lower bound in [35] and Cheung-Leung's estimate in [12].

(2) As we know, $\lambda_1(\mathbb{R}^n) = 0$ and $\lambda_1(\mathbb{H}^n(-1)) = \frac{(n-1)^2}{4}$, with $n \geq 2$. The latter one was actually shown by Mckean in [35]. By Cheng's eigenvalue comparison theorem [8], it is easy to know that for a complete noncompact n-manifold $\widetilde{M^n}$ with sectional curvature bounded from above by 0 (resp., -1), then $\lambda_1(\widetilde{M^n}) \geq 0$ (resp., $\lambda_1(\widetilde{M^n}) \geq \frac{(n-1)^2}{4}$). Furthermore, by Cheng-type eigenvalue comparison theorem obtained by Mao and his collaborators (see [22, Theorem 4.4]), one knows that this lower bound estimate for $\lambda_1(\widetilde{M^n})$ can be improved to the situation that the complete noncompact n-manifold $\widetilde{M^n}$ only has a *radial sectional curvature* upper bound 0 (resp., -1) with respect to some point in $\widetilde{M^n}$.

(3) By (2.1), one easily has $\lambda_1(M) \geq 0$. However, from examples given in (2) of Remark 2.2, one knows that for a complete noncompact Riemannian manifold M, in some settings, $\lambda_1(M) > 0$. Speaking roughly, the spectral quantity $\lambda_1(M)$ somehow reveals the geometry and topology of the complete noncompact Riemannian manifold M considered. In fact, Schoen-Yau [41, page 106] suggested that it is an important question to find conditions which will imply $\lambda_1(M) > 0$. Speaking in other words, manifolds with $\lambda_1(M) > 0$ might have some special geometric properties. There are many interesting results supporting this. For instance, Cheung-Leung [12] proved that if M is an n-dimensional complete minimal submanifold in the hyperbolic m-space $\mathbb{H}^m(-1)$, then $\lambda_1(M) \geq \frac{(n-1)^2}{4} > 0$, and moreover, M is non-parabolic, i.e., there exists a nonconstant bounded subharmonic function on M. They also showed that if furthermore M has at least two ends, then there exists on M a nonconstant bounded harmonic function with finite Dirichlet energy.

(4) Inspired by Schoen and Yau [41, page 106], Du and Mao [17] firstly showed that one can ask the same question for the weighted Laplacian (or drifting Laplacian) and the nonlinear p-Laplacian ($1 < p < \infty$), and moreover,

[3] Here, $M \subset_\varphi N$ means M is an immersed submanifold of the given complete manifold N by the immersion $\varphi : M \hookrightarrow N$, and $B_N(p,r)$ denotes the geodesic ball in N with center p and radius r. For more details, please check Bessa-Montenegro's article [2].

they successfully gave two interesting (strictly positive) lower bounds for the first eigenvalue of submanifolds with bounded mean curvature in a hyperbolic space, (one of) which generalizes Cheung-Leung's lower bound estimate in [12]. After this, Mao and his collaborators have two continuous works on this topic – see [33, 34] for details.

Lemma 2.1, combining with the discussions in [4], can be used to show the following existence theorem for harmonic functions with finite Dirichlet energy.

Lemma 2.3. *Let M be given as in Lemma 2.1. If M has at least two ends, then there exists on M a nonconstant bounded harmonic function with finite Dirichlet energy.*

We also need the following fact:

Lemma 2.4. ([5]) *Let $z_1, \ldots, z_m$ be m real numbers. Then we have*

$$\sum_{i=2}^{m} z_1 z_i \leq \frac{\sqrt{m-1}}{2} \sum_{i=1}^{m} z_i^2. \tag{2.3}$$

Now we recall some known facts about harmonic maps between Riemannian manifolds.

Let M and Q be complete Riemannian manifolds of dimension m and s, respectively. Denote by $y : M \to Q$ be a harmonic map from M to Q. Let $\{e_i\}_{i=1}^m$ and $\{e'_\alpha\}_{\alpha=1}^s$ be local orthonormal frame fields of M and Q, respectively. Suppose that $\{\omega_i\}_{i=1}^m$ and $\{\theta_\alpha\}_{\alpha=1}^s$ are the dual coframes of $\{e_i\}_{i=1}^m$ and $\{e'_\alpha\}_{\alpha=1}^s$, respectively, and $\{\omega_{ij}\}_{i,j=1}^m$ and $\{\theta_{\alpha\beta}\}_{\alpha,\beta=1}^s$ are the corresponding connection forms. Denote by R_{ijkl} and $K_{\alpha\beta\gamma\delta}$ the curvature tensors of M and Q, respectively. Then we have the structure equations as follows:

$$\begin{cases} d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \\ \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \end{cases}$$

and

$$\begin{cases} d\theta_\alpha = \sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta, \\ \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \\ d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} - \frac{1}{2} \sum_{\gamma,\delta} K_{\alpha\beta\gamma\delta} \theta_\gamma \wedge \theta_\delta. \end{cases}$$

Define $y_{\alpha i},\ 1 \leq \alpha \leq s,\ 1 \leq i \leq m$, by

$$y^*(\theta_\alpha) = \sum_i y_{\alpha i} \omega_i. \tag{2.4}$$

Then the energy density $e(y)$ is given by

$$e(y) = \sum_{\alpha,i} y_{\alpha i}^2.$$

Taking the exterior differentiation of (2.4), we can get

$$y^*(d\theta_\alpha) = \sum_i \left(dy_{\alpha i} \wedge \omega_i + y_{\alpha i} d\omega_i\right),$$

which gives

$$\sum_i \left(dy_{\alpha i} - \sum_j y_{\alpha j}\omega_{ij} - \sum_\beta y^*(\theta_{\alpha\beta}) y_{\beta i}\right) \wedge \omega_i = 0. \tag{2.5}$$

Define $y_{\alpha ij}$ by

$$dy_{\alpha i} + \sum_\beta y_{\beta i} y^*(\theta_{\beta\alpha}) + \sum_j y_{\alpha j}\omega_{ji} := \sum_j y_{\alpha ij}\omega_j. \tag{2.6}$$

Then (2.5) and (2.6) imply that $y_{\alpha ij} = y_{\alpha ji}$ and y is harmonic means

$$\sum_i y_{\alpha ii} = 0, \ \forall \alpha = 1, \dots, s.$$

Exterior differentiating (2.6), we have

$$\begin{aligned}
&\sum_l \left(dy_{\alpha il} + \sum_j \left(y_{\alpha ij}\omega_{jl} + y_{\alpha jl}\omega_{ji}\right) + \sum_\beta y_{\beta il} y^*(\theta_{\beta\alpha})\right) \wedge \omega_l \\
&= \frac{1}{2}\sum_{j,k,l} R_{ijkl} y_{\alpha j}\omega_k \wedge \omega_l + \frac{1}{2}\sum_{\beta,\gamma,\delta,k,l} K_{\alpha\beta\gamma\delta} y_{\beta i} y_{\gamma k} y_{\delta l}\omega_k \wedge \omega_l.
\end{aligned} \tag{2.7}$$

Define

$$\sum_k y_{\alpha ijk}\omega_k := dy_{\alpha ij} + \sum_k \left(y_{\alpha ik}\omega_{kj} + y_{\alpha kj}\omega_{ki}\right) + \sum_\beta y_{\beta ij} y^*(\theta_{\alpha\beta}),$$

and then (2.7) implies that

$$y_{\alpha ikl} - y_{\alpha ilk} = \sum_j R_{ijlk} y_{\alpha j} + \sum_{\beta,\gamma,\delta} K_{\alpha\beta\gamma\delta} y_{\beta i} y_{\gamma l} y_{\delta k}.$$

Set $e = e(y)$ and let Δ be the Laplace operator acting on functions on M. From the above equality, one can easily get the following Bochner type formula for harmonic maps (which was first derived by Eells-Sampson [19]):

$$\frac{1}{2}\Delta e = \sum_{\alpha,i,j} y_{\alpha ij}^2 + \sum_{\alpha,i,j} R_{ij} y_{\alpha i} y_{\alpha j} - \sum_{\alpha,\beta,\gamma,\delta,i,j} K_{\alpha\beta\gamma\delta} y_{\alpha i} y_{\beta j} y_{\gamma i} y_{\delta j}, \tag{2.8}$$

where R_{ij} stands for component of the Ricci tensor of M. Besides, it is also known that (cf. [40])

$$\sum_{\alpha,i,j} y_{\alpha ij}^2 \geq \left(1 + \frac{1}{2ms}\right) |\nabla\sqrt{e}|^2. \tag{2.9}$$

3. Proofs of the main results

All the proofs would be shown clearly in this section.

Proof of Theorem 1.1. Firstly, we consider the case (i). Denote by h the second fundamental form of M in N which is given by

$$h(X,Y) = \overline{\nabla}_X Y - \nabla_X Y, \qquad \forall X, Y \in TM,$$

where ∇ is the Riemannian connection of M. Take an orthonormal frame field $\{e_1, \ldots, e_m\}$ on M. The mean curvature vector of M can be written as

$$\mathbf{H} = \frac{1}{m} \sum_i h(e_i, e_i).$$

Let us deduce a lower bound for the Ricci curvature of M. From the Gauss equation and $\mathrm{Ric}_{(m-1)}(N) \geq -(m-1)c$, we know that the Ricci curvatures of M in the directions e_j, $j = 1, \ldots, m$, satisfy

$$\begin{aligned} &\mathrm{Ric}(e_j, e_j) \\ = &\sum_{k=1, k\neq j}^{m} K(e_j \wedge e_j) + \langle m\mathbf{H}, h(e_j, e_j)\rangle - \sum_{k=1}^{m} \langle h(e_k, e_j), h(e_k, e_j)\rangle \\ \geq &-(m-1)c + \langle m\mathbf{H}, h(e_j, e_j)\rangle - \sum_{k=1}^{m} \langle h(e_k, e_j), h(e_k, e_j)\rangle, \end{aligned} \tag{3.1}$$

where $K(e_j \wedge e_k)$ denotes the sectional curvature of N on the plane $e_j \wedge e_k$ spanned by vectors e_j and e_k, $\langle \cdot, \cdot \rangle$ is the inner product induced by the metric of N.

Set

$$Q = \sum_k |h(e_k, e_k)|^2.$$

For any fixed $j \in \{1, \ldots, m\}$, since

$$\begin{aligned} |m\mathbf{H} - h(e_j, e_j)|^2 &= \left| \sum_{k\neq j} h(e_k, e_k) \right|^2 \\ &\leq \left(\sum_{k\neq j} |h(e_k, e_k)| \right)^2 \\ &\leq (m-1) \sum_{k\neq j} |h(e_k, e_k)|^2 \end{aligned}$$

$$= (m-1)\left(Q - |h(e_j,e_j)|^2\right),$$

we have

$$m^2|\mathbf{H}|^2 - (m-1)Q + m|h(e_j,e_j)|^2 - 2m\langle \mathbf{H}, h(e_j,e_j)\rangle \leq 0. \tag{3.2}$$

It is also easy to see from

$$\sum_i (h(e_i,e_i) - \mathbf{H}) = 0, \quad \sum_i |h(e_i,e_i) - \mathbf{H}|^2 = Q - m|\mathbf{H}|^2,$$

that

$$|h(e_j,e_j) - \mathbf{H}|^2 \leq \frac{m-1}{m}\left(Q - m|\mathbf{H}|^2\right), \tag{3.3}$$

which, combining with (3.2), gives

$$\begin{aligned}
0 \geq\ & m\left(|h(e_j,e_j)|^2 - m\langle \mathbf{H}, h(e_j,e_j)\rangle\right) \\
& + (m-2)m\langle h(e_j,e_j) - \mathbf{H}, \mathbf{H}\rangle + 2(m-1)m|\mathbf{H}|^2 - (m-1)Q \\
\geq\ & m\left(|h(e_j,e_j)|^2 - m\langle \mathbf{H}, h(e_j,e_j)\rangle\right) \\
& - (m-2)m|\mathbf{H}|\sqrt{\frac{m-1}{m}\left(Q - m|\mathbf{H}|^2\right)} + 2(m-1)m|\mathbf{H}|^2 - (m-1)Q.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \langle m\mathbf{H}, h(e_j,e_j)\rangle - |h(e_j,e_j)|^2 \\
\geq\ & -(m-2)|\mathbf{H}|\sqrt{\frac{m-1}{m}\left(Q - m|\mathbf{H}|^2\right)} + 2(m-1)|\mathbf{H}|^2 - \frac{m-1}{m}Q.
\end{aligned}$$

Substituting the above inequality into (3.1) and noticing

$$\begin{aligned}
S &= \sum_{i,j} |h(e_i,e_j)|^2 \\
&= Q + 2\sum_{i<j} |h(e_i,e_j)|^2,
\end{aligned} \tag{3.4}$$

one gets

$$\begin{aligned}
& \mathrm{Ric}(e_j,e_j) \\
\geq\ & -(m-1)c - \tfrac{m-1}{m}S + 2(m-1)|\mathbf{H}|^2 - (m-2)|\mathbf{H}|\sqrt{\tfrac{m-1}{m}\left(S - m|\mathbf{H}|^2\right)} \\
=\ & -(m-1)c - \tfrac{m-1}{m}S + 2(m-1)|\mathbf{H}_f - \tfrac{1}{m}(\overline{\nabla} f)^\perp|^2 \\
& - (m-2)|\mathbf{H}_f - \tfrac{1}{m}(\overline{\nabla} f)^\perp|\sqrt{\tfrac{m-1}{m}\left(S - m|\mathbf{H}_f - \tfrac{1}{m}(\overline{\nabla} f)^\perp|^2\right)} \\
\geq\ & -(m-1)c - \tfrac{m-1}{m}S + 2(m-1)|\boldsymbol{H}_f|^2 + \tfrac{2(m-1)}{m^2}|(\overline{\nabla} f)^\perp|^2 - \tfrac{4(m-1)}{m}\langle \boldsymbol{H}_f, (\overline{\nabla} f)^\perp\rangle \\
& - (m-2)|\boldsymbol{H}_f|\sqrt{\tfrac{m-1}{m}\left(S - m|\boldsymbol{H}_f|^2\right) - \tfrac{m-1}{m^2}|(\overline{\nabla} f)^\perp|^2 + \tfrac{2(m-1)}{m}\langle \boldsymbol{H}_f, (\overline{\nabla} f)^\perp\rangle} \\
& - \tfrac{m-2}{m}|(\overline{\nabla} f)^\perp|\sqrt{\tfrac{m-1}{m}\left(S - m|\boldsymbol{H}_f|^2\right) - \tfrac{m-1}{m^2}|(\overline{\nabla} f)^\perp|^2 + \tfrac{2(m-1)}{m}\langle \boldsymbol{H}_f, (\overline{\nabla} f)^\perp\rangle} \\
=\ & -(m-1)c - B(S, |\boldsymbol{H}_f|).
\end{aligned}$$

Define a quantity $B(S, |\boldsymbol{H}_f|)$ as in (1.3). Since the choice of the frame field $\{e_1, \ldots, e_m\}$ is arbitrary, we conclude that the Ricci curvature of M satisfies

$$\mathrm{Ric}(M) \geq -(m-1)c - B(S, |\mathbf{H}_f|). \tag{3.5}$$

For any L^p-harmonic 1-form ω on M, $1 < p < \infty$, denote by $g = |\omega|$ the length of ω. Let X be the vector field on M dual to ω. The Bochner formula (see, e.g., [29–31]) implies that

$$\frac{1}{2}\Delta g^2 \geq \mathrm{Ric}(X, X) + |\nabla \omega|^2. \tag{3.6}$$

It is also known that (cf. [30, 31])

$$|\nabla \omega|^2 \geq \frac{m}{m-1}|\nabla g|^2. \tag{3.7}$$

Moreover,

$$\frac{1}{2}\Delta g^2 = g\Delta g + |\nabla g|^2. \tag{3.8}$$

For any positive number $1 < p < \infty$, we have

$$\begin{aligned} g^{\frac{p}{2}}\Delta g^{\frac{p}{2}} &= g^{\frac{p}{2}}\mathrm{div}\left(\nabla(g^{\frac{p}{2}})\right) \\ &= g^{\frac{p}{2}}\mathrm{div}\left(\frac{p}{2}g^{\frac{p}{2}-1}\nabla g\right) \\ &= \frac{p}{2}\left(\frac{p}{2}-1\right)g^{p-2}|\nabla g|^2 + \frac{p}{2}g^{p-2}g\Delta g \\ &= \frac{p-2}{p}|\nabla(g^{\frac{p}{2}})|^2 + \frac{p}{2}g^{p-2}g\Delta g. \end{aligned} \tag{3.9}$$

From (3.5)-(3.7) and (3.8)-(3.9), we have

$$\begin{aligned} g^{\frac{p}{2}}\Delta g^{\frac{p}{2}} - \frac{p-2}{p}|\nabla(g^{\frac{p}{2}})|^2 + \frac{p}{2}\left((m-1)c + B(S, |\mathbf{H}_f|)\right)g^p &\geq \frac{p}{2(m-1)}g^{p-2}|\nabla g|^2 \\ &= \frac{2}{p(m-1)}|\nabla(g^{\frac{p}{2}})|^2, \end{aligned}$$

that is,

$$g^{\frac{p}{2}}\Delta g^{\frac{p}{2}} + \frac{p}{2}\left((m-1)c + B(S, |\mathbf{H}_f|)\right)g^p \geq \left(1 - \frac{2}{p} + \frac{2}{p(m-1)}\right)|\nabla(g^{\frac{p}{2}})|^2. \tag{3.10}$$

Fix a point $x_0 \in M$ and choose ϕ to be a nonnegative cut-off function with the properties

$$|\nabla \phi| \leq \frac{1}{r},\ \phi = \begin{cases} 1 & \text{on } B(x_0, r), \\ 0 & \text{on } M \setminus B(x_0, 3r). \end{cases} \tag{3.11}$$

Set

$$q = \sup_{x \in M} B(S, |\mathbf{H}_f|)(x),$$

and then (1.2) gives

$$\frac{2p((m-1)c+q)}{(m-1-md-c^+)^2} < 2 - \frac{2}{p} + \frac{2}{p(m-1)}. \tag{3.12}$$

Multiplying (3.10) by ϕ^2 and integrating over M, one gets

$$\begin{aligned} &\left(1-\frac{2}{p}+\frac{2}{p(m-1)}\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 \\ \leq &\int_M g^{\frac{p}{2}}\Delta g^{\frac{p}{2}}\phi^2 + \frac{p}{2}\left((m-1)c+q\right)\int_M g^p\phi^2. \end{aligned} \tag{3.13}$$

Besides, the divergence theorem gives us

$$\int_M g^{\frac{p}{2}}\triangle g^{\frac{p}{2}}\phi^2 = -2\int_M g^{\frac{p}{2}}\phi\langle\nabla\phi,\nabla(g^{\frac{p}{2}})\rangle - \int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2. \tag{3.14}$$

Substituting (3.14) into (3.13) yields

$$\begin{aligned} &\left(1-\frac{2}{p}+\frac{2}{p(m-1)}\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 \\ \leq &-2\int_M g^{\frac{p}{2}}\phi\langle\nabla\phi,\nabla(g^{\frac{p}{2}})\rangle - \int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 \\ &+\frac{p}{2}\left((m-1)c+q\right)\int_M g^p\phi^2. \end{aligned} \tag{3.15}$$

Since $\sup_{x\in M}|\mathbf{H}_f| \leq d$, $\sup_{x\in M}|\overline{\nabla} f|(x) \leq c^+ < \frac{2(m-1)\sqrt{c}}{\sqrt{m}}$, we know from Lemma 2.1 that

$$\lambda_1(M) \geq \frac{(m-1-md-c^+)^2}{4},$$

which implies

$$\int_M g^p\phi^2 \leq \frac{4}{(m-1-md-c^+)^2}\int_M |\nabla(\phi g^{\frac{p}{2}})|^2. \tag{3.16}$$

Set

$$l = \frac{2p\left((m-1)c+q\right)}{(m-1-md-c^+)^2}$$

and take an $\epsilon > 0$ so that

$$l + \epsilon|l-1| < 2 - \frac{2}{p} + \frac{2}{p(m-1)}.$$

Since

$$2\left|\int_M g^{\frac{p}{2}}\phi\langle\nabla\phi,\nabla(g^{\frac{p}{2}})\rangle\right| \leq \epsilon\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 + \frac{1}{\epsilon}\int_M |\nabla\phi|^2 g^p,$$

it follows from (3.15) and (3.16) that

$$\begin{aligned}
&\left(1-\frac{2}{p}+\frac{2}{p(m-1)}\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2\\
\le\ & l\int_M |\nabla(g^{\frac{p}{2}}\phi)|^2 - 2\int_M g^{\frac{p}{2}}\phi\langle\nabla\phi,\nabla(g^{\frac{p}{2}})\rangle - \int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2\\
=\ & l\int_M |\nabla\phi|^2 g^p + (l-1)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 + 2(l-1)\int_M \phi g^{\frac{p}{2}}\langle\nabla\phi,\nabla(g^{\frac{p}{2}})\rangle\\
\le\ & (l-1+\epsilon|l-1|)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 + \left(l+\frac{|l-1|}{\epsilon}\right)\int_M |\nabla\phi|^2 g^p,
\end{aligned}$$

that is,

$$\left(2-\frac{2}{p}+\frac{2}{p(m-1)}-(l+\epsilon|l-1|)\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 \le \left(l+\frac{|l-1|}{\epsilon}\right)\int_M |\nabla\phi|^2 g^p,$$

which implies

$$\begin{aligned}
&\left(2-\frac{2}{p}+\frac{2}{p(m-1)}-(l+\epsilon|l-1|)\right)\int_{B(x_0,r)} |\nabla(g^{\frac{p}{2}})|^2\\
\le\ & \left(2-\frac{2}{p}+\frac{2}{p(m-1)}-(l+\epsilon|l-1|)\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2\\
\le\ & \left(l+\frac{|l-1|}{\epsilon}\right)\int_M |\nabla\phi|^2 g^p\\
\le\ & \frac{1}{r^2}\left(l+\frac{|l-1|}{\epsilon}\right)\int_{B(x_0,3r)\setminus B(x_0,r)} g^p.
\end{aligned}$$

Since $|\omega| \in L^p(M)$, the RHS tends to 0 when $r \to \infty$, so g is constant. Since M is a complete noncompact submanifold in a Hadamard manifold, it has infinite volume (cf. [23]). We know from $g \in L^p(M)$ that $g = 0$, $1 < p < \infty$. Thus M admits no nontrivial L^p harmonic 1-form, and there exists no nonconstant harmonic function with finite Dirichlet energy on M which, combining with Lemma 2.3, implies that M has only one end.

The case (ii) can be proven by using almost the same argument as the case (i) except only one thing. That is, in the case (ii), the following lower bound

$$\mathrm{Ric}(M) \ge -(m-1)c - \frac{\sqrt{m-1}}{2}S \tag{3.17}$$

for the Ricci curvature (cf. [5, Lemma 3.1]) should be used to instead of (3.5). In fact, by (3.1), one gets

$$\begin{aligned}
\mathrm{Ric}(e_j,e_j) \ge\ & -(m-1)c + \sum_{k=1,k\neq j}^{m}\langle h(e_k,e_k),h(e_j,e_j)\rangle\\
& - \sum_{k=1,k\neq j}^{m}\langle h(e_k,e_j),h(e_k,e_j)\rangle,
\end{aligned}$$

so we know from Schwarz inequality and Lemma 2.4 that

$$
\begin{aligned}
\mathrm{Ric}(e_j,e_j) &\geq -(m-1)c-\sum_{k=1,k\neq j}^{m}|h(e_j,e_j)||h(e_k,e_k)|-\sum_{k=1,k\neq j}^{m}|h(e_k,e_j)|^2\\
&\geq -(m-1)c-\frac{\sqrt{m-1}}{2}\sum_{k=1}^{m}|h(e_k,e_k)|^2-\sum_{k=1,k\neq j}^{m}|h(e_k,e_j)|^2\\
&\geq -(m-1)c-\frac{\sqrt{m-1}}{2}\left(\sum_{k=1}^{m}|h(e_k,e_k)|^2+2\sum_{k=1,k\neq j}^{m}|h(e_k,e_j)|^2\right)\\
&\geq -(m-1)c-\frac{\sqrt{m-1}}{2}\sum_{i,k=1}^{m}|h(e_i,e_k)|^2\\
&= -(m-1)c-\frac{\sqrt{m-1}}{2}S.
\end{aligned}
$$

Summing up the above arguments, the conclusion of Theorem 1.1 follows naturally. $\square$

Proof of Theorem 1.3. Since M is a complete submanifold in a Hadamard manifold, by [26, 36], there exists a positive constant a which depends only on m such that

$$\left(\int_M|\psi|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}\leq a\left(\int_M(|\nabla\psi|+|\mathbf{H}||\psi|)\right)$$

for any compactly supported $\psi\in H_{1,2}(M)$. So we can get

$$
\begin{aligned}
\left(\int_M|\psi|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} &\leq a\int_M\left(|\nabla\psi|+|\mathbf{H}_f-\frac{1}{m}(\overline{\nabla}f)^{\perp}||\psi|\right)\\
&\leq a\int_M|\nabla\psi|+a\int_M\left(\left(|\mathbf{H}_f|+\frac{1}{m}|(\overline{\nabla}f)^{\perp}|\right)|\psi|\right)\\
&= a\int_M(|\nabla\psi|+|\mathbf{H}_f||\psi|)+\frac{a}{m}\int_M|(\overline{\nabla}f)^{\perp}||\psi|.
\end{aligned}
$$

By the Hölder's inequality, one has

$$\int_M|\mathbf{H}_f||\psi|\leq\left(\int_M|\mathbf{H}_f|^m\right)^{\frac{1}{m}}\left(\int_M|\psi|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}},$$

$$\int_M|(\overline{\nabla}f)^{\perp}||\psi|\leq\left(\int_M|(\overline{\nabla}f)^{\perp}|^m\right)^{\frac{1}{m}}\left(\int_M|\psi|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}.$$

Let $c_2=\frac{1}{(4a)^m}$, and then the assumptions

$$\int_M|\mathbf{H}_f|^m\leq c_2,\ \int_M|\overline{\nabla}f|^m\leq c_4,\ c_4\leq m^m c_2,$$

give

$$\frac{1}{2a}\left(\int_M |\psi|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \leq \int_M |\nabla \psi|.$$

Replacing ψ by $\psi^{\frac{2(m-1)}{m-2}}$ and using the Hölder's inequality, we can obtain

$$\begin{aligned}\frac{1}{2a}\left(\int_M |\psi|^{\frac{2m}{m-2}}\right)^{\frac{m-1}{m}} &\leq \int_M |\nabla \psi^{\frac{2(m-1)}{m-2}}| \\ &= \frac{2(m-1)}{m-2}\int_M |\nabla\psi||\psi^{\frac{m}{m-2}}| \\ &\leq \frac{2(m-1)}{m-2}\left(\int_M |\psi|^{\frac{2m}{m-2}}\right)^{\frac{1}{2}}\left(\int_M |\nabla\psi|^2\right)^{\frac{1}{2}},\end{aligned}$$

which implies

$$\begin{aligned}\left(\int_M |\psi|^{\frac{2m}{m-1}}\right)^{\frac{m-2}{m}} &\leq \frac{16a^2(m-1)^2}{(m-2)^2}\int_M |\nabla\psi|^2 \\ &=: a'\int_M |\nabla\psi|^2.\end{aligned} \tag{3.18}$$

Here, of course, $a' = \frac{16a^2(m-1)^2}{(m-2)^2}$. We take the constant c_1 in our Theorem 1.3 to be a fixed positive number less than $\frac{(m-1)(\sqrt{m}-2\sqrt{c})}{m^{3/2}}$ and set

$$c_3 = \left(\frac{5}{3mpa'}\left(2 - \frac{2}{p} + \frac{2}{p(m-1)} - \frac{2p(m-1)c}{(m-1-mc_1-c^+)^2}\right)\right)^{\frac{m}{2}}. \tag{3.19}$$

Since $\sup_{x\in M}|\mathbf{H}_f|(x) \leq c_1$, $\sup_{x\in M}|\overline{\nabla} f|(x) \leq c^+ < \frac{2(m-1)\sqrt{c}}{\sqrt{m}}$, Lemma 2.1 implies that

$$\lambda_1(M) \geq \frac{(m-1-mc_1-c^+)^2}{4}.$$

Thus

$$\int_M (g^{\frac{p}{2}}\phi)^2 \leq \frac{4}{(m-1-mc_1-c^+)^2}\int_M |\nabla(g^{\frac{p}{2}}\phi)|^2. \tag{3.20}$$

Let

$$A_0 = \left(\int_M (S - m|\mathbf{H}_f|^2 + m|(\overline{\nabla} f)^{\perp}|^2)^{\frac{m}{2}}\right)^{\frac{2}{m}},$$

and then it follows from

$$\int_M (S - m|\mathbf{H}_f|^2 + m|(\overline{\nabla} f)^{\perp}|^2)^{\frac{m}{2}} \leq c_3$$

that

$$l_1 = \frac{3pmA_0a'}{8} + \frac{2p(m-1)c}{(m-1-mc_1-c^+)^2} < 2 - \frac{2}{p} + \frac{2}{p(m-1)}. \tag{3.21}$$

One gets from Hölder's inequality and (3.18) that

$$\begin{aligned}
&\int_M \phi^2(S-m|\mathbf{H}_f|^2+m|(\overline{\nabla} f)^\perp|^2)g^p \\
&\le \left(\int_M (S-m|\mathbf{H}_f|^2+m|(\overline{\nabla} f)^\perp|^2)^{\frac{m}{2}}\right)^{\frac{2}{m}} \left(\int_M (\phi g^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \\
&\le A_0 a' \int_M |\nabla(\phi g^{\frac{p}{2}})|^2.
\end{aligned} \tag{3.22}$$

Take a positive $\epsilon_1 > 0$ such that

$$l_1 + \epsilon_1|l_1-1| < 2 - \frac{2}{p} + \frac{2}{p(m-1)}.$$

It follows from (3.5) and the definition of $B(S,|\mathbf{H}_f|)$ that

$$\mathrm{Ric}(M) \ge -(m-1)c - \frac{3m}{4}\left(S-m|\mathbf{H}_f|^2+m|(\overline{\nabla} f)^\perp|^2\right). \tag{3.23}$$

Thus we get from (3.10) that

$$\begin{aligned}
&g^{\frac{p}{2}}\Delta g^{\frac{p}{2}} + \frac{p}{2}\left((m-1)c + \frac{3m}{4}\left(S-m|\mathbf{H}_f|^2+m|(\overline{\nabla} f)^\perp|^2\right)\right)g^p \\
&\ge \left(1-\frac{2}{p}+\frac{2}{p(m-1)}\right)|\nabla(g^{\frac{p}{2}})|^2.
\end{aligned} \tag{3.24}$$

Multiplying (3.24) by ϕ^2 and integrating over M, by using (3.19)-(3.23) we have

$$\begin{aligned}
&\left(1-\frac{2}{p}+\frac{2}{p(m-1)}\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 \\
&\le \frac{p}{2}(m-1)c\int_M g^p\phi^2 + \frac{3pm}{8}\int_M \phi^2(S-m|\mathbf{H}_f|^2+m|(\overline{\nabla} f)^\perp|^2)g^p \\
&\quad - 2\int_M g^{\frac{p}{2}}\phi\langle\nabla\phi,\nabla(g^{\frac{p}{2}})\rangle - \int_M \phi^2|\nabla(g^{\frac{p}{2}})|^2 \\
&\le (l_1 - 1 + \epsilon_1|l_1-1|)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 + \left(l_1 + \frac{|l_1-1|}{\epsilon_1}\right)\int_M |\nabla\phi|^2 g^p.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left(2-\frac{2}{p}+\frac{2}{p(m-1)} - (l_1+\epsilon_1|l_1-1|)\right)\int_M |\nabla(g^{\frac{p}{2}})|^2\phi^2 \\
&\le \left(l_1 + \frac{|l_1-1|}{\epsilon_1}\right)\int_M |\nabla\phi|^2 g^p,
\end{aligned}$$

which implies

$$\left(2-\frac{2}{p}+\frac{2}{p(m-1)} - (l_1+\epsilon_1|l_1-1|)\right)\int_{B(x_0,r)} |\nabla(g^{\frac{p}{2}})|^2$$

$$\leq \frac{1}{r^2}\left(l_1 + \frac{|l_1 - 1|}{\epsilon_1}\right)\int_{B(x_0,3r)\setminus B(x_0,r)} g^p.$$

Taking $r \to \infty$, we conclude that $g = 0$ and so M has only one end. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.5. Let $g : M \to Q$ be a harmonic map with finite energy. Denote by e the energy density of g. It follows from (2.8), (2.9), (3.5) and the non-positivity of the sectional curvature of Q that

$$\begin{aligned}\frac{1}{2}\Delta e &\geq \left(1 + \frac{1}{2ms}\right)|\nabla\sqrt{e}|^2 - ((m-1)c + B(S, |\mathbf{H}_f|))e \\ &\geq \left(1 + \frac{1}{2ms}\right)|\nabla\sqrt{e}|^2 - ((m-1)c + q))e,\end{aligned} \tag{3.25}$$

where

$$q = \sup_{x\in M} B(S, |\mathbf{H}_f|)(x).$$

Multiplying (3.25) by ϕ^2 defined in (3.11) and integrating over M result in

$$\left(1+\frac{1}{2ms}\right)\int_M |\nabla\sqrt{e}|^2\phi^2 \leq ((m-1)c+q)\int_M e\phi^2 - 2\int_M \sqrt{e}\phi\langle\nabla\sqrt{e}, \nabla\phi\rangle. \tag{3.26}$$

Assume that the item (i) holds. Since

$$\sup_{x\in M}|\mathbf{H}_f| \leq d < \frac{m-1}{m} - \sqrt{\frac{8s(m-1)c}{m(2ms+1)}}, \tag{3.27}$$

and

$$\sup_{x\in M}|\overline{\nabla} f|(x) \leq c^+ < \sqrt{\frac{8ms(m-1)c}{2ms+1}},$$

we know from Lemma 2.1 that

$$\lambda_1(M) \geq \frac{(m-1-md-c^+)^2}{4},$$

and so

$$\int_M e\phi^2 \leq \frac{4}{(m-1-md-c^+)^2}\int_M |\nabla(\sqrt{e}\phi)|^2. \tag{3.28}$$

By (3.27) and the fact

$$q < \frac{(2ms+1)(m-1-md-c^+)^2 - 8ms(m-1)c}{8ms},$$

one has

$$\frac{4((m-1)c+q)}{(m-1-md-c^+)^2} < 1 + \frac{1}{2ms}.$$

Let

$$l_2 = \frac{4((m-1)c+q)}{(m-1-md-c^+)^2},$$

and take an $\epsilon_2 > 0$ so that

$$l_2 + \epsilon_2 |l_2 - 1| < 1 + \frac{1}{2ms}.$$

Since

$$2\left|\int_M \sqrt{e}\phi\langle\nabla\sqrt{e}, \nabla\phi\rangle\right| \le \epsilon_2 \int_M |\nabla\sqrt{e}|^2\phi^2 + \frac{1}{\epsilon_2}\int_M e|\nabla\phi|^2,$$

we have from (3.26) and (3.28) that

$$\left(1 + \frac{1}{2ms} - (l_2 + \epsilon_2|l_2 - 1|)\right)\int_M |\nabla\sqrt{e}|^2\phi^2 \le \left(l_2 + \frac{|l_2-1|}{\epsilon_2}\right)\int_M |\nabla\phi|^2 e,$$

which implies

$$\begin{aligned}&\left(1 + \frac{1}{2ms} - (l_2 + \epsilon_2|l_2 - 1|)\right)\int_{B(x_0,r)} |\nabla\sqrt{e}|^2 \\ &\le \frac{1}{r^2}\left(l_2 + \frac{|l_2-1|}{\epsilon_2}\right)\int_{B(x_0,3r)\setminus B(x_0,r)} e.\end{aligned}$$

When $r \to \infty$, the RHS tends to 0 since g has finite energy. Thus, e is a constant. We then conclude from $E(g) < \infty$ and the infinity of the volume of M that $e = 0$. Consequently, g is a constant.

In the case of item (ii), the conclusion of Theorem 1.5 can also be proven by using almost the same argument as in the case of item (i) except the replacement of Ricci curvature bound estimate (3.17) to (3.5). This completes the proof of Theorem 1.5. □

It is not hard to see that by using similar arguments to those in the proofs of Theorems 1.3 and 1.5, one can easily prove Theorem 1.6 and we prefer to omit the details here.

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