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GEOMETRY OF THE MODULI SPACE OF HIGGS PAIRS ON AN IRREDUCIBLE NODAL CURVE OF ARITHMETIC GENUS ONE

Sang-Bum Yoo

Abstract. We describe the moduli space of Higgs pairs on an irreducible nodal curve of arithmetic genus one and its geometric structures in terms of the Hitchin map and a flat degeneration of the moduli space of Higgs bundles on an elliptic curve.

1. Introduction

1.1. Motivations and results

The moduli space of Higgs bundles on a smooth curve has been intensively studied in view of mirror symmetry that was first raised over an elliptic curve by M. Thaddeus in [18] and over any smooth curve by T. Hausel and M. Thaddeus in [11]. Our work in this paper provides concrete ingredients to be useful later when we prove or disprove a mirror symmetry phenomenon over a singular curve.

The purpose of this paper is to describe the moduli space of Higgs pairs on an irreducible nodal curve of arithmetic genus one and its geometric structures in terms of the Hitchin map and a flat degeneration of the moduli space of Higgs bundles on an elliptic curve explicitly.

Throughout this paper, Y denotes a reduced irreducible projective curve of arithmetic genus one, with only one ordinary node p, defined over $\mathbb{C}$, let $\nu : X \to Y$ be the normalization map and let $\nu^{-1}(p) = \{p_1, p_2\}$. Note that $X \cong \mathbb{P}^1$.

There are two ways to study Higgs pairs on Y like torsion-free sheaves on Y (see [9,15]). One is to compare them with generalized parabolic Higgs bundles on X in order to give the Hitchin map on the moduli space of Higgs pairs on Y (see [6]). Another is to compare them with Gieseker-Hitchin pairs on X

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attached with a chain of projective lines in order to give a flat degeneration of the moduli space of Higgs bundles on an elliptic curve (see [3]).

For a positive integer n and an arbitrary integer d, we consider the following moduli spaces.

- Let $U_Y(n,d)$ be the moduli space of semistable torsion-free sheaves of rank n and degree d on Y.
- Let $U_X^{\mathrm{GPB}}(n,d)$ be the moduli space of semistable generalized parabolic vector bundles of rank n and degree d on X.
- Let $\mathcal{M}_Y(n,d)$ be the moduli space of semistable Higgs pairs of rank n and degree d on Y.
- Let $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ be the moduli space of semistable good generalized parabolic Higgs bundles of rank n and degree d on X.

We describe $\mathcal{M}_Y(n,d)$ and $\mathcal{M}_Y^{\mathrm{GGPH}}(n,d)$ as follows.

Theorem 1.1 (Theorem 3.2)**.** *If* $\gcd(n,d)=1$*, then* $\mathcal{M}_Y(n,d)\cong U_Y(n,d)\times\mathbb{C}\cong Y\times\mathbb{C}$.

Theorem 1.2 (Theorem 3.3)**.** *If* $\gcd(n,d)>1$*, then there is no stable Higgs pairs of rank* n *and degree* d *on* Y.

Theorem 1.3 (Theorem 3.11)**.** *Let* $\gcd(n,d)=h$.

(1) *There exists a bijective morphism*

$$\mathrm{Sym}^h(Y\times\mathbb{C})\to\mathcal{M}_Y(n,d).$$

(2) $\mathcal{M}_Y(n,d)$ *is irreducible.*

Theorem 1.1 follows from a simplified stability of Higgs pairs on Y and a description of $U_Y(n,d)$ by [7] in the case $\gcd(n,d)=1$. The proof of Theorem 1.2 is given by considering a degeneration of the moduli space of semistable Higgs bundles of rank n and degree d on an elliptic curve and using the nonexistence of stable Higgs bundles of rank n and degree d on an elliptic curve in the case $\gcd(n,d)>1$. Theorem 1.3 follows from a family of semistable Higgs pairs of rank n and degree d on Y parametrized by $(Y\times\mathbb{C})\times\overset{h}{\cdots}\times(Y\times\mathbb{C})$ induced from the universal family of stable Higgs pairs of rank $\frac{n}{h}$ and degree $\frac{d}{h}$ on Y parametrized by $Y\times\mathbb{C}$ in the case $\gcd(n,d)=h$.

Theorem 1.4 (Theorem 3.7)**.** (1) *If* $\gcd(n,d)=1$*, then* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)\cong U_X^{\mathrm{GPB}}(n,d)\times\mathbb{C}\cong\mathbb{P}^1\times\mathbb{C}$.

(2) *If* $\gcd(n,d)>1$*, then the stable locus* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)^s$ *of* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ *is empty.*

Theorem 1.5 (Theorem 3.14)**.** *Let* $\gcd(n,d)=h$.

(1) *There exists a bijective morphism*

$$\mathrm{Sym}^h(\mathbb{P}^1\times\mathbb{C})\to\mathcal{M}_X^{\mathrm{GGPH}}(n,d).$$

(2) $\mathrm{Sym}^h(\mathbb{P}^1\times\mathbb{C})$ *is the normalization of* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$.

(3) $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ *is irreducible.*

Theorem 1.4(1) follows from a simplified stability of generalized parabolic Higgs bundles on X and the known result that $U_X^{\mathrm{GPB}}(n,d)$ is the normalization of $U_Y(n,d)$ in the case $\gcd(n,d)=1$. Theorem 1.4(2) follows from the nonexistence of stable Higgs bundles of rank n and degree d on an elliptic curve in the case $\gcd(n,d)>1$. Theorem 1.5 follows from the normality of $\mathrm{Sym}^h(\mathbb{P}^1\times\mathbb{C})$ and a family of semistable generalized parabolic Higgs bundles of rank n and degree d on X parametrized by $(\mathbb{P}^1\times\mathbb{C})\times\overset{h}{\cdots}\times(\mathbb{P}^1\times\mathbb{C})$ induced from the universal family of stable generalized parabolic Higgs bundles of rank $\frac{n}{h}$ and degree $\frac{d}{h}$ on X parametrized by $\mathbb{P}^1\times\mathbb{C}$ in the case $\gcd(n,d)=h$.

We also describe all fibers of the Hitchin maps H on $\mathcal{M}_Y(n,d)$ and H^{GGPH} on $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ as follows.

Theorem 1.6 (Corollary 4.7)**.** *Let* $\gcd(n,d)=h$. *The generic fiber of the Hitchin map* H^{GGPH} *on* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ *is set-theoretically isomorphic to* $\mathbb{P}^1\times\overset{h}{\cdots}\times\mathbb{P}^1$. *The fiber over an arbitrary point of the base is set-theoretically isomorphic to* $\mathbb{P}^{m_1}\times\overset{h}{\cdots}\times\mathbb{P}^{m_l}$, *where* $h=m_1+\cdots+m_l$. *The fiber over an arbitrary point of the base is isomorphic to* $\mathbb{P}^1$ *for the case* $\gcd(n,d)=1$.

Theorem 1.7 (Corollary 4.8)**.** *Let* $\gcd(n,d)=h$. *The generic fiber of the Hitchin map* H *on* $\mathcal{M}_Y(n,d)$ *is set-theoretically isomorphic to* $Y\times\overset{h}{\cdots}\times Y$. *The fiber over an arbitrary point of the base is set-theoretically isomorphic to* $\mathrm{Sym}^{m_1}Y\times\overset{l}{\cdots}\times\mathrm{Sym}^{m_l}Y$, *where* $h=m_1+\cdots+m_l$. *The fiber over an arbitrary point of the base is isomorphic to* Y *for the case* $\gcd(n,d)=1$.

We define the Hitchin map H^{GGPH} on $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ and then it induces the Hitchin map H on $\mathcal{M}_Y(n,d)$ by a surjective birational morphism $f:\mathcal{M}_X^{\mathrm{GGPH}}(n,d)\to\mathcal{M}_Y(n,d)$. H^{GGPH} is set-theoretically identified with the projection

$$\pi_h:\mathrm{Sym}^h(\mathbb{P}^1\times\mathbb{C})\to\mathrm{Sym}^h\mathbb{C},$$
$$[(x_1,t_1),\dots,(x_h,t_h)]_{\mathfrak{S}_h}\mapsto[t_1,\dots,t_h]_{\mathfrak{S}_h}.$$

Then the identification of fibers of H^{GGPH} and π_h gives Theorem 1.6. Theorem 1.7 follows from Theorem 1.6 and the surjective birational morphism $f:\mathcal{M}_X^{\mathrm{GGPH}}(n,d)\to\mathcal{M}_Y(n,d)$.

We finally describe a flat degeneration of the moduli space of stable Higgs bundles of rank n and degree d on an elliptic curve for the case $\gcd(n,d)=1$. Let $Z\to T$ be a flat family of irreducible complex projective curves of arithmetic genus one parametrized by a smooth curve T such that for all $t\neq t_0$, the fiber Z_t is an elliptic curve and $Z_{t_0}\cong Y$. Let $\mathcal{G}_{Z/T}(n,d)\to T$ be a flat family of the moduli spaces of stable Gieseker-Hitchin pairs of rank n and degree d over Z parametrized by T, which is constructed in [3, Proposition 5.13]. Let H^{GH} be the Hitchin map on $\mathcal{G}_{Z/T}(n,d)$.

Theorem 1.8 (Theorem 5.8)**.** (1) *$\mathcal{G}_{Z/T}(n,d) \cong Z \times \mathbb{C}$ as T-schemes.*

(2) *$\nu_*^{\mathrm{GH}} : \mathcal{G}_{Z/T}(n,d) \to \mathcal{M}_{Z/T}(n,d)$ is identified with the identity map $\mathrm{id}_{Z\times\mathbb{C}} : Z \times \mathbb{C} \to Z \times \mathbb{C}$, where $\mathcal{M}_{Z/T}(n,d) \to T$ is the relative moduli space of stable Higgs pairs of rank n and degree d on Z.*

(3) *The fiber of the Hitchin map H^{GH} on $\mathcal{G}_{Z/T}(n,d)$ is isomorphic to Z.*

1.2. Organization of the paper

In Section 2, we describe $U_Y(n,d)$ and $U_X^{\mathrm{GPB}}(n,d)$ explicitly. In Section 3, we describe $\mathcal{M}_Y(n,d)$ and $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ explicitly. In Section 4, we give descriptions of all fibers of the Hitchin maps on $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ and $\mathcal{M}_Y(n,d)$, respectively. In Section 5, we prove that $\mathcal{G}_{Z/T}(n,d) \cong Z \times \mathbb{C}$ as T-schemes and the fiber of the Hitchin map on $\mathcal{G}_{Z/T}(n,d)$ is isomorphic to Z.

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2. Torsion-free sheaves and generalized parabolic vector bundles

In this section we aim to describe the moduli space of torsion-free sheaves on Y and that of generalized parabolic vector bundles on X explicitly by relating these moduli spaces.

Definition ([5])**.** A *generalized parabolic vector bundle* (*GPB*) of rank n and degree d on X is a pair $(E, F_1(E))$, where E is a vector bundle of rank n and degree d on X and $F_1(E)$ is an n-dimensional subspace of $E_{p_1} \oplus E_{p_2}$.

Definition ([5])**.** A GPB $(E, F_1(E))$ is *semistable* (respectively, *stable*) if for every proper subbundle $N \subset E$,

$$\frac{\deg N + \dim(F_1(E) \cap (N_{p_1} \oplus N_{p_2}))}{\operatorname{rank} N} \leq (<) \frac{\deg E + \dim F_1(E)}{\operatorname{rank} E}.$$

Consider a GPB $(E, F_1(E))$ of rank n and degree d on X. To $(E, F_1(E))$, we associate a torsion-free sheaf F of rank n and degree d on Y by the following short exact sequence:

$$0 \to F \to \nu_* E \to \nu_*(E) \otimes \mathbb{C}(p)/F_1(E) \to 0.$$

Proposition 2.1 (Proposition 4.2 of [5])**.** *$(E, F_1(E))$ is semistable* (*respectively, stable*) *of rank n and degree d if and only if F is semistable* (*respectively, stable*) *of rank n and degree d.*

Let $U_Y(n,d)$ be the moduli space of semistable torsion-free sheaves of rank n and degree d on Y and let $U_X^{\mathrm{GPB}}(n,d)$ be the moduli space of generalized parabolic vector bundles of rank n and degree d on X. Denote the stable loci of $U_Y(n,d)$ and $U_X^{\mathrm{GPB}}(n,d)$ by $U_Y(n,d)^s$ and $U_X^{\mathrm{GPB}}(n,d)^s$, respectively.

We begin with referring to the results of a classification of stable torsion-free sheaves on Y in [7].

Proposition 2.2 (Lemma 2.2 and Theorem 2.5 of [7])**.** (1) *If* $\gcd(n,d) = 1$*, then* $U_Y(n,d) \cong Y$*.*

(2) *If* $\gcd(n,d) > 1$*, then there is no stable torsion-free sheaf of rank* n *and degree* d *over* Y*, that is,* $U_Y(n,d)^s$ *is empty.*

Let us denote $Y \xrightarrow{\cong} U_Y(n,d)$ of Proposition 2.2(1) by $\zeta_{n,d}$.

Next we classify stable GPBs on X. Combining Proposition 2.1 with Proposition 2.2(2), we have the following statement.

Lemma 2.3. *If* $\gcd(n,d) > 1$*, then there is no stable GPB of rank* n *and degree* d *over* X*.*

The following result relates $U_Y(n,d)$ to $U_X^{\mathrm{GPB}}(n,d)$.

Proposition 2.4 (Theorem 1 and Theorem 3 of [5], Proposition 2.1 of [17])**.** $U_X^{\mathrm{GPB}}(n,d)$ *is the normalization of* $U_Y(n,d)$*.*

The following classification is immediately obtained from Proposition 2.2(1), Lemma 2.3 and Proposition 2.4.

Proposition 2.5. (1) *If* $\gcd(n,d) = 1$*, then* $U_X^{\mathrm{GPB}}(n,d) \cong \mathbb{P}^1$*.*

(2) *If* $\gcd(n,d) > 1$*, then* $U_X^{\mathrm{GPB}}(n,d)^s$ *is empty.*

Denote $\mathbb{P}^1 \xrightarrow{\cong} U_X^{GPB}(n,d)$ of Proposition 2.5(1) by $\zeta_{n,d}^{\mathrm{GPB}}$.

Now we classify semistable torsion-free sheaves on Y and semistable GPBs on X. By Proposition 2.2(2) and Lemma 2.3, the Jordan-Hölder filtrations for torsion-free sheaves and GPBs imply the following observations.

Lemma 2.6. *Assume* $\gcd(n,d) = h$*.*

(1) *Any semistable torsion-free sheaf* F *of rank* n *and degree* d *over* Y *is S-equivalent to* $F_1 \oplus \cdots \oplus F_h$*, where each* F_i *is stable of rank* $\frac{n}{h}$ *and degree* $\frac{d}{h}$*.*

(2) *Any semistable GPB* $(E, F_1(E))$ *of rank* n *and degree* d *over* X *is S-equivalent to* $(E_1, F_1(E_1)) \oplus \cdots \oplus (E_h, F_1(E_h))$*, where each* $(E_i, F_1(E_i))$ *is stable of rank* $\frac{n}{h}$ *and degree* $\frac{d}{h}$*.*

When $\gcd(n,d) = 1$, there exist universal families of stable torsion-free sheaves and stable GPBs of rank n and degree d.

Lemma 2.7. *Assume that* $\gcd(n,d) = 1$*. Then there exists a universal family* $\mathcal{V}_{n,d}$ *of stable torsion-free sheaves of rank* n *and degree* d *parametrized by* Y *such that for every* $y \in Y$*,*

$$\zeta_{n,d}(y) = [(\mathcal{V}_{n,d})_y]_S,$$

where $[(\mathcal{V}_{n,d})_y]_S$ *is the S-equivalence class of* $(\mathcal{V}_{n,d})_y$*.*

Proof. Since $\chi(E(m)) = nm + d$ for any $E \in U_Y(n,d)$ and $\gcd(n,d) = 1$, by [12, Corollary 4.6.6] and Proposition 2.2(1), we prove the statement. □

Lemma 2.8. *Assume that* $\gcd(n,d)=1$. *Then there exists a universal family* $(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))$ *of stable GPBs of rank* n *and degree* d *parametrized by* X *such that for every* $x \in X$,

$$\zeta_{n,d}^{\mathrm{GPB}}(x) = [(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))_x]_S,$$

where $[(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))_x]_S$ *is the* S*-equivalence class of* $(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))_x$.

Proof. By Proposition 3.16 of [5] and Proposition 2.5(1), we prove the statement. □

If $\gcd(n,d)=h>1$, $n'=\frac{n}{h}$ and $d'=\frac{d}{h}$, then we can consider the families

$$\mathcal{V}_{n,d} = \mathcal{V}_{n',d'} \underset{Y}{\times} \overset{h}{\cdots} \underset{Y}{\times} \mathcal{V}_{n',d'}$$

of polystable torsion-free sheaves parametrized by $Y \times \overset{h}{\cdots} \times Y$ and

$$(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d})) = (\mathcal{W}_{n',d'}, F_1(\mathcal{W}_{n',d'})) \underset{X}{\times} \overset{h}{\cdots} \underset{X}{\times} (\mathcal{W}_{n',d'}, F_1(\mathcal{W}_{n',d'}))$$

of polystable GPBs parametrized by $\mathbb{P}^1 \times \overset{h}{\cdots} \times \mathbb{P}^1$.

The following maps induced by $\mathcal{V}_{n,d}$ and $(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))$,

$$\nu_{\mathcal{V}_{n,d}} : Y \times \overset{h}{\cdots} \times Y \to U_Y(n,d)$$

and

$$\nu_{(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))} : \mathbb{P}^1 \times \overset{h}{\cdots} \times \mathbb{P}^1 \to U_X^{\mathrm{GPB}}(n,d)$$

are surjective by Lemma 2.6 and factor through $\mathrm{Sym}^h Y$ and $\mathrm{Sym}^h \mathbb{P}^1$. Now we complete the classification as follows.

Proposition 2.9. (1) *For* $\gcd(n,d)=h$, *there exists a bijective morphism*

$$\mathrm{Sym}^h Y \to U_Y(n,d).$$

(2) *For* $\gcd(n,d)=h$, $U_X^{\mathrm{GPB}}(n,d) \cong \mathrm{Sym}^h \mathbb{P}^1 \cong \mathbb{P}^h$.

Proof. (1) $\nu_{\mathcal{V}_{n,d}} : Y \times \overset{h}{\cdots} \times Y \to U_Y(n,d)$ induces a bijective morphism

$$\mathrm{Sym}^h Y \to U_Y(n,d).$$

(2) $\nu_{(\mathcal{W}_{n,d}, F_1(\mathcal{W}_{n,d}))} : \mathbb{P}^1 \times \overset{h}{\cdots} \times \mathbb{P}^1 \to U_X^{\mathrm{GPB}}(n,d)$ induces a bijective morphism

$$\zeta_{n,d}^{\mathrm{GPB}} : \mathrm{Sym}^h \mathbb{P}^1 \to U_X^{\mathrm{GPB}}(n,d).$$

Since $\mathrm{Sym}^h \mathbb{P}^1 \cong \mathbb{P}^h$ is connected and $U_X^{\mathrm{GPB}}(n,d)$ is normal, $\zeta_{n,d}^{\mathrm{GPB}}$ is an isomorphism by Zariski's main theorem. □

3. Higgs pairs and generalized parabolic Higgs bundles

In this section we describe the moduli space of Higgs pairs on Y and that of generalized parabolic Higgs bundles on X explicitly. Note that the dualizing sheaf ω_Y is trivial.

Definition ([6])**.** A *Higgs pair* of rank n and degree d on Y is a pair (E, ϕ_E), where E is a torsion-free sheaf of rank n and degree d on Y and ϕ_E is a global section of $\mathcal{E}nd E$.

Definition ([6])**.** A Higgs pair (E, ϕ_E) is *semistable* (respectively, *stable*) if for every proper ϕ_E-invariant subsheaf $N \subset E$,

$$\mu(N) \leq (<) \mu(E),$$

where $\mu(E) = \frac{\deg E}{\operatorname{rank} E}$ is the slope of E.

Definition ([6])**.** A *generalized parabolic Higgs bundle* (*GPH*) of rank n and degree d on X is a triple $(E, \phi_E, F_1(E))$, where E is a vector bundle of rank n and degree d on X, ϕ_E is a global section of $\mathcal{E}nd E$ and $F_1(E)$ is an n-dimensional subspace of $E_{p_1} \oplus E_{p_2}$. A GPH $(E, \phi_E, F_1(E))$ is *good* if $\phi_E|_{p_1+p_2}(F_1(E)) \subset F_1(E)$.

Definition ([6])**.** A GPH $(E, \phi_E, F_1(E))$ is *semistable* (respectively, *stable*) if for every proper ϕ_E-invariant subbundle $N \subset E$,

$$\frac{\deg N + \dim(F_1(E) \cap (N_{p_1} \oplus N_{p_2}))}{\operatorname{rank} N} \leq (<) \frac{\deg E + \dim F_1(E)}{\operatorname{rank} E}.$$

Let $\mathcal{M}_Y(n, d)$ be the moduli space of semistable Higgs pairs of rank n and degree d on Y and $\mathcal{M}_Y(n, d)^s$ denotes the stable locus. Let $\mathcal{M}_X^{\mathrm{GPH}}(n, d)$ be the moduli space of semistable GPHs of rank n and degree d on X and $\mathcal{M}_X^{\mathrm{GPH}}(n, d)^s$ denotes the stable locus. Let $\mathcal{M}_X^{\mathrm{GGPH}}(n, d)$ be the moduli space of semistable good GPHs of rank n and degree d on X and $\mathcal{M}_X^{\mathrm{GGPH}}(n, d)^s$ denotes the stable locus. Note that $\mathcal{M}_X^{\mathrm{GGPH}}(n, d)$ is a closed subscheme of $\mathcal{M}_X^{\mathrm{GPH}}(n, d)$ by Theorem 4.8 of [6].

The triviality of ω_Y gives a simplicity of the study of the semistability of Higgs pairs on Y.

Lemma 3.1. (1) *A Higgs pair (E, ϕ) is semistable if and only if E is semistable.*

(2) *If $\gcd(n, d) = 1$ and $(E, \phi) \in \mathcal{M}_Y(n, d)$, then (E, ϕ) is stable if and only if E is stable.*

Proof. Since the dualizing sheaf ω_Y is trivial, the proof is same as that of Proposition 4.1 of [8]. □

We start with classifying stable Higgs pairs on Y.

Theorem 3.2. *If $\gcd(n, d) = 1$, then $\mathcal{M}_Y(n, d) \cong U_Y(n, d) \times \mathbb{C} \cong Y \times \mathbb{C}$.*

Proof. Since $\mathcal{M}_Y(n,d) \to \mathcal{M}_Y(n, d+n\deg M)$, $(E,\phi_E) \mapsto (E\otimes M, \phi_E \otimes \mathrm{id}_M)$ is an isomorphism for some fixed line bundle M on Y, we may assume that $d > n$ as [14, page 281].

Let U be the restriction of the universal sheaf to $Y \times R^s$, where R is the open subset of the quot scheme parametrizing quotient sheaves of rank n and degree d in [14, Section 3]. Applying [14, Lemma 3.5] with $\mathcal{F} = \mathcal{E}nd\, U$, we get a linear scheme $F \to R^s$ given by

$$F = \mathbf{Spec}\,\mathrm{Sym}_{\mathcal{O}_{R^s}}(\pi_{R^s *}\mathcal{E}nd\, U)^{\vee},$$

where $\pi_{R^s} : Y \times R^s \to R^s$ is the projection onto R^s. Since $\pi_{R^s *}\mathcal{E}nd\, U \cong \mathcal{O}_{R^s}$ by [12, Lemma 4.6.3], we have $F \cong R^s \times \mathbb{C}$. It follows from Lemma 3.1 that $F^s \cong R^s \times \mathbb{C}$.

Hence by the construction of [14, Section 3] and Proposition 2.2(1),

$$\begin{aligned}\mathcal{M}_Y(n,d) &\cong F^s//\mathrm{PGL}(d)\\ &\cong (R^s//\mathrm{PGL}(d)) \times \mathbb{C}\\ &\cong U_Y(n,d) \times \mathbb{C}\\ &\cong Y \times \mathbb{C}.\end{aligned}$$

□

Let us denote $Y \times \mathbb{C} \xrightarrow{\cong} \mathcal{M}_Y(n,d)$ of Theorem 3.2 by $\eta_{n,d}$.

Theorem 3.3. *If* $\gcd(n,d) > 1$, *then there is no stable Higgs pairs of rank* n *and degree* d *over* Y. *In other words* $\mathcal{M}_Y(n,d)^s$ *is empty.*

Proof. Let $Z \to S$ be a flat family of irreducible complex projective curves of arithmetic genus one parametrized by a smooth curve S, and let $s_0 \in S$ be a base point such that for all $s \neq s_0$, the fiber Z_s is an elliptic curve and $Z_{s_0} \cong Y$. Let $M \to S$ be the relative moduli variety over S such that M_s is the moduli space of semistable Higgs bundles of rank n and degree d on Z_s for all $s \neq s_0$ and $M_{s_0} = \mathcal{M}_Y(n,d)$ (see [16] for the existence of M).

Let $M^s \subset M$ denote the subset corresponding to stable Higgs pairs. By the openness of the stability condition, M^s is an open subset of M (see Proposition 3.1 of [14]). Assume that there is a stable Higgs pair on Y of rank n and degree d in an irreducible component M_{irr} of M. Then $M^s \cap M_{\mathrm{irr}} \cap M_{s_0}$ is nonempty, hence $M^s \cap M_{\mathrm{irr}}$ is nonempty. Then $M^s \cap (M_{\mathrm{irr}} \setminus M_{s_0})$ is a nonempty open subset of M_{irr}. Consequently, M_s is nonempty for some $s \neq s_0$. This contradicts the fact that there are no stable Higgs bundles of rank n and degree d on an elliptic curve ([8, Proposition 4.3] and FACT of [19]). This completes the proof. □

Next we classify stable good GPHs on X. We first recall an equivalence of semistabilities between good GPHs on X and Higgs pairs on Y. Consider a good GPH $(E, \phi_E, F_1(E))$ of rank n and degree d on X. To $(E, \phi_E, F_1(E))$, we associate a Higgs pair (E, ϕ_E) of rank n and degree d on Y by the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \nu_*E & \longrightarrow & \nu_*E\otimes\mathbb{C}(p)/F_1(E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_F} & & \downarrow{\scriptstyle \nu_*\phi_E} & & \downarrow{\scriptstyle (\nu_*\phi_E)_p} & & \\ 0 & \longrightarrow & F & \longrightarrow & \nu_*E & \longrightarrow & \nu_*E\otimes\mathbb{C}(p)/F_1(E) & \longrightarrow & 0. \end{array} \tag{3.1}$$

Proposition 3.4 (Theorem 2.9 of [6])**.** *A good GPH $(E, \phi_E, F_1(E))$ is semistable if and only if (F, ϕ_F) is semistable.*

Indeed we also have an equivalence of stabilities between good GPHs on X and Higgs pairs on Y.

Proposition 3.5. *A good GPH $(E, \phi_E, F_1(E))$ is stable if and only if (F, ϕ_F) is stable.*

Proof. Assume that $(E, \phi_E, F_1(E))$ is stable of rank n and degree d. Then

$$\deg F = \deg \nu_*E - \dim(\nu_*E \otimes \mathbb{C}(p)/F_1(E)) = \deg E + n - (2n - n) = \deg E$$

and $\phi_E = \nu^*\phi_F$. Let K_1 be a ϕ_F-invariant subsheaf of F of rank r. Let K be the subbundle of E generated by the image of $\nu^*K_1/\text{torsion}$ in E. Then K is a ϕ_E-invariant subbundle of E of rank r and K_1 is obtained by

$$0 \to K_1 \to \nu_*K \to \nu_*K \otimes \mathbb{C}(p)/F_1(K) \to 0,$$

where $F_1(K) = F_1(E) \cap (K_{p_1} \oplus K_{p_2})$. Let $s = \dim F_1(K)$. Then we have

$$\deg K_1 = \deg \nu_*K - \dim(\nu_*K \otimes \mathbb{C}(p)/F_1(K)) = \deg K + r - (2r - s) = \deg K + s - r.$$

Hence $\frac{\deg K + s}{r} < \mu(E) + 1$ if and only if $\frac{\deg K_1}{r} < \mu(F)$. □

Combining Proposition 2.1 and Lemma 3.1 with Proposition 3.4, the semistability of good GPHs is simplified.

Proposition 3.6. *A good GPH $(E, \phi_E, F_1(E))$ is semistable if and only if $(E, F_1(E))$ is semistable.*

We complete the classification as follows.

Theorem 3.7. (1) *If $\gcd(n, d) = 1$, then $\mathcal{M}_X^{\mathrm{GGPH}}(n, d) \cong U_X^{\mathrm{GPB}}(n, d) \times \mathbb{C} \cong \mathbb{P}^1 \times \mathbb{C}$.*

(2) *If $\gcd(n, d) > 1$, then $\mathcal{M}_X^{\mathrm{GGPH}}(n, d)^s$ is empty.*

Proof. (1) Note that if $(E, F_1(E))$ is stable, then any endomorphism of $(E, F_1(E))$ is a scalar (see Corollary 3.9 of [5]). The proof is similar to that of Theorem 3.2. Proposition 3.6 and the construction of $\mathcal{M}_X^{\mathrm{GGPH}}(n, d)$ of [6] are applied to the proof.

(2) The statement is an immediate consequence of Theorem 3.3 and Proposition 3.5. □

Denote $\mathbb{P}^1 \times \mathbb{C} \xrightarrow{\cong} \mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ of Theorem 3.7(1) by $\eta_{n,d}^{\mathrm{GGPH}}$.

Now we classify semistable Higgs pairs on Y. By Theorem 3.3 and using Jordan-Hölder filtration for Higgs pairs, we have the following observation.

Lemma 3.8. *Assume* $\gcd(n,d) = h$. *Then any semistable Higgs pair* (F, ϕ_F) *of rank* n *and degree* d *over* Y *is S-equivalent to* $(F_1, \phi_{F_1}) \oplus \cdots \oplus (F_h, \phi_{F_h})$, *where each* (F_i, ϕ_{F_i}) *is stable of rank* $\frac{n}{h}$ *and degree* $\frac{d}{h}$.

When $\gcd(n,d) = 1$, there exists a universal family of stable Higgs pairs.

Lemma 3.9. *Let* $\gcd(n,d) = 1$. *There exists a universal family* $\mathcal{E}_{n,d} = (\mathcal{V}_{n,d}, \Phi_{n,d})$ *of stable Higgs pairs of rank* n *and degree* d *parametrized by* $Y \times \mathbb{C}$.

Proof. Consider the family of stable Higgs pairs $\mathcal{E}_{n,d} = (\mathcal{V}_{n,d}, \Phi_{n,d})$ over $Y \times \mathbb{C}$ such that

$$\mathcal{E}_{n,d}|_{Y \times (y,t)} \cong ((\mathcal{V}_{n,d})_y, \frac{t}{n} \otimes \mathrm{id}_{(\mathcal{V}_{n,d})_y}).$$

Then for any $(y,t) \in Y \times \mathbb{C}$, we have

$$\eta_{n,d}((y,t)) = [\mathcal{E}_{n,d}|_{Y \times (y,t)}]_S,$$

where $[\mathcal{E}_{n,d}|_{Y \times (y,t)}]_S$ is the S-equivalence class of $\mathcal{E}_{n,d}|_{Y \times (y,t)}$.

Any family $\mathcal{F} \to Y \times T$ induces canonically a morphism

$$\nu_{\mathcal{F}} : T \to \mathcal{M}_Y(n,d).$$

Then $f = (\eta_{n,d})^{-1} \circ \nu_{\mathcal{F}}$ is a morphism $T \to Y \times \mathbb{C}$ such that $\mathcal{F}$ is S-equivalent to $f^* \mathcal{E}_{n,d}$. □

If $\gcd(n,d) = h > 1$, $n' = \frac{n}{h}$ and $d' = \frac{d}{h}$, then we can consider the family

$$\mathcal{E}_{n,d} = \mathcal{E}_{n',d'} \underset{Y}{\times} \overset{h}{\cdots} \underset{Y}{\times} \mathcal{E}_{n',d'}$$

of polystable Higgs pairs parametrized by $(Y \times \mathbb{C}) \times \overset{h}{\cdots} \times (Y \times \mathbb{C})$.

Remark 3.10. The action of the symmetric group $\mathfrak{S}_h$ on $Y \times \overset{h}{\cdots} \times Y$ induces an action of $\mathfrak{S}_h$ on $(Y \times \mathbb{C}) \times \overset{h}{\cdots} \times (Y \times \mathbb{C})$. If ω_1 and ω_2 are two points of $(Y \times \mathbb{C}) \times \overset{h}{\cdots} \times (Y \times \mathbb{C})$, the Higgs pairs $(\mathcal{E}_{n,d})_{\omega_1}$ and $(\mathcal{E}_{n,d})_{\omega_2}$ are S-equivalent if and only if $\omega_2 = \gamma \cdot \omega_1$ for some $\gamma \in \mathfrak{S}_h$.

The following map induced by $\mathcal{E}_{n,d}$,

$$\nu_{\mathcal{E}_{n,d}} : (Y \times \mathbb{C}) \times \overset{h}{\cdots} \times (Y \times \mathbb{C}) \to \mathcal{M}_Y(n,d),$$

is surjective by Lemma 3.8 and factors through $\mathrm{Sym}^h(Y \times \mathbb{C})$ by Remark 3.10. Now we complete the classification of semistable Higgs pairs as follows.

Theorem 3.11. *Let* $\gcd(n,d) = h$.

(1) *There exists a bijective morphism*

$$\eta_{n,d} : \mathrm{Sym}^h(Y \times \mathbb{C}) \to \mathcal{M}_Y(n,d).$$

(2) $\mathcal{M}_Y(n,d)$ *is irreducible.*

Proof. (1) $\nu_{\mathcal{E}_{n,d}}$ induces a bijective morphism

$$\eta_{n,d} : \mathrm{Sym}^h(Y \times \mathbb{C}) \to \mathcal{M}_Y(n,d).$$

(2) Since $\nu_{\mathcal{E}_{n,d}}$ is continuous and $(Y \times \mathbb{C}) \times \overset{h}{\cdots} \times (Y \times \mathbb{C})$ is irreducible, we get the result. $\square$

Finally we classify semistable good GPHs on X. By Theorem 3.7(2) and using Jordan-Hölder filtration of GPHs, we have the following observation.

Lemma 3.12. *Assume* $\gcd(n,d) = h$. *Then any semistable good GPH* $(E, \phi_E, F_1(E))$ *of rank* n *and degree* d *over* X *is S-equivalent to* $(E_1, \phi_{E_1}, F_1(E_1)) \oplus \cdots \oplus (E_h, \phi_{E_h}, F_1(E_h))$*, where each* $(E_i, \phi_{E_i}, F_1(E_i))$ *is stable of rank* $\frac{n}{h}$ *and degree* $\frac{d}{h}$*.*

When $\gcd(n,d) = 1$, there exists a universal family of stable good GPHs.

Lemma 3.13. *Let* $\gcd(n,d) = 1$*. There exists a universal family* $\mathcal{E}^{\mathrm{GGPH}}_{n,d} = (\mathcal{W}_{n,d}, \Phi_{n,d}, F_1(\mathcal{W}_{n,d}))$ *of stable good GPHs of rank* n *and degree* d *parametrized by* $X \times \mathbb{C}$*.*

Proof. Consider the family of stable good GPHs $\mathcal{E}^{\mathrm{GGPH}}_{n,d} = (\mathcal{W}_{n,d}, \Phi_{n,d}, F_1(\mathcal{W}_{n,d}))$ over $X \times \mathbb{C}$ such that

$$\mathcal{E}^{\mathrm{GGPH}}_{n,d}|_{X \times (x,t)} \cong ((\mathcal{W}_{n,d})_x, \frac{t}{n} \otimes \mathrm{id}_{(\mathcal{W}_{n,d})_x}, (F_1(\mathcal{W}_{n,d}))_x).$$

Then for any $(x,t) \in X \times \mathbb{C}$, we have

$$\eta^{\mathrm{GGPH}}_{n,d}((x,t)) = [\mathcal{E}^{\mathrm{GGPH}}_{n,d}|_{X \times (x,t)}]_S,$$

where $[\mathcal{E}^{\mathrm{GGPH}}_{n,d}|_{X \times (x,t)}]_S$ is the S-equivalence class of $\mathcal{E}^{\mathrm{GGPH}}_{n,d}|_{X \times (x,t)}$.

Any family $\mathcal{F} \to X \times T$ induces canonically a morphism

$$\nu_{\mathcal{F}} : T \to \mathcal{M}^{\mathrm{GGPH}}_X(n,d).$$

Then $f = (\eta^{\mathrm{GGPH}}_{n,d})^{-1} \circ \nu_{\mathcal{F}}$ is a morphism $T \to X \times \mathbb{C}$ such that $\mathcal{F}$ is S-equivalent to $f^* \mathcal{E}^{\mathrm{GGPH}}_{n,d}$. $\square$

If $\gcd(n,d) = h > 1$, $n' = \frac{n}{h}$ and $d' = \frac{d}{h}$, then we can consider the family

$$\mathcal{E}^{\mathrm{GGPH}}_{n,d} = \mathcal{E}^{\mathrm{GGPH}}_{n',d'} \underset{X}{\times} \overset{h}{\cdots} \underset{X}{\times} \mathcal{E}^{\mathrm{GGPH}}_{n',d'}$$

of polystable good GPHs parametrized by $(\mathbb{P}^1 \times \mathbb{C}) \times \overset{h}{\cdots} \times (\mathbb{P}^1 \times \mathbb{C})$.

The following map induced by $\mathcal{E}^{\mathrm{GGPH}}_{n,d}$,

$$\nu_{\mathcal{E}^{\mathrm{GGPH}}_{n,d}} : (\mathbb{P}^1 \times \mathbb{C}) \times \overset{h}{\cdots} \times (\mathbb{P}^1 \times \mathbb{C}) \to \mathcal{M}^{\mathrm{GGPH}}_X(n,d),$$

is surjective by Lemma 3.12 and factors through $\mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C})$. We complete the classification of semistable good GPHs as follows.

Theorem 3.14. *Let* $\gcd(n,d)=h$.

(1) *There exists a bijective morphism*

$$\eta_{n,d}^{\mathrm{GGPH}} : \mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C}) \to \mathcal{M}_X^{\mathrm{GGPH}}(n,d).$$

(2) $\mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C})$ *is the normalization of* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$.

(3) $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ *is irreducible.*

Proof. (1) $\nu_{\mathcal{E}_{n,d}^{\mathrm{GGPH}}}$ induces a bijective morphism

$$\eta_{n,d}^{\mathrm{GGPH}} : \mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C}) \to \mathcal{M}_X^{\mathrm{GGPH}}(n,d).$$

(2) Since $\mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C})$ is normal, the result follows by Zariski's main theorem.

(3) Since $\nu_{\mathcal{E}_{n,d}}^{\mathrm{GGPH}}$ is continuous and $(\mathbb{P}^1 \times \mathbb{C}) \times \overset{h}{\cdots} \times (\mathbb{P}^1 \times \mathbb{C})$ is irreducible, we get the result. □

4. Hitchin map

In this section, we give descriptions of all fibers of the Hitchin maps on $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ and $\mathcal{M}_Y(n,d)$, respectively.

Definition (Section 5 of [6])**.** The *Hitchin map on* $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$ is defined by

$$H^{\mathrm{GGPH}} : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to A := \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}_X),$$

$$(E, \phi_E, F_1(E)) \mapsto (a_1(\phi_E), \dots, a_n(\phi_E)),$$

where the characteristic polynomial $\det(\lambda - \phi_E)$ of $(E, \phi_E, F_1(E))$ is $\lambda^n + a_1(\phi_E)\lambda^{n-1} + \cdots + a_n(\phi_E)$.

There is a result relating $\mathcal{M}_Y(n,d)$ to $\mathcal{M}_X^{\mathrm{GGPH}}(n,d)$.

Proposition 4.1 (Theorem 4.9 of [6])**.** *There exists a birational morphism*

$$f : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to \mathcal{M}_Y(n,d), (E, \phi_E, F_1(E)) \mapsto (F, \phi_F),$$

where (F, ϕ_F) *is given by* (3.1).

Remark 4.2. If $\gcd(n,d)=1$, then f is surjective (see Theorem 3 of [5]).

Indeed f is surjective in any case.

Proposition 4.3. f *is surjective.*

Proof. For each $(F, \phi_F) \in \mathcal{M}_Y(n,d)$, it follows from Lemma 3.8 that

$$(F_1, \phi_{F_1}) \oplus \cdots \oplus (F_h, \phi_{F_h}),$$

where each (F_i, ϕ_{F_i}) is stable of rank $\frac{n}{h}$ and degree $\frac{d}{h}$. By Remark 4.2, there exists $(E_i, \phi_{E_i}, F_1(E_i)) \in \mathcal{M}_X^{\mathrm{GGPH}}(\frac{n}{h}, \frac{d}{h})$ such that $f((E_i, \phi_{E_i}, F_1(E_i))) = (F_i, \phi_{F_i})$. Then we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^h F_i & \longrightarrow & \nu_*\left(\bigoplus_{i=1}^h E_i\right) & \longrightarrow & \nu_*\left(\bigoplus_{i=1}^h E_i\right)\otimes \mathbb{C}(p)/\bigoplus_{i=1}^h F_1(E_i) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \bigoplus_{i=1}^h \phi_{F_i}} & & \downarrow {\scriptstyle \bigoplus_{i=1}^h \nu_*\phi_{E_i}} & & \downarrow {\scriptstyle \left(\bigoplus_{i=1}^h \nu_*\phi_{E_i}\right)_p} & & \\ 0 & \longrightarrow & \bigoplus_{i=1}^h F_i & \longrightarrow & \nu_*\left(\bigoplus_{i=1}^h E_i\right) & \longrightarrow & \nu_*\left(\bigoplus_{i=1}^h E_i\right)\otimes \mathbb{C}(p)/\bigoplus_{i=1}^h F_1(E_i) & \longrightarrow & 0. \end{array}$$

Hence

$$f((E_1,\phi_{E_1},F_1(E_1))\oplus\cdots\oplus(E_h,\phi_{E_h},F_1(E_h))) = (F_1,\phi_{F_1})\oplus\cdots\oplus(F_h,\phi_{F_h}).$$

□

Proposition 4.4 (Corollary 5.2 of [6]). (1) *$H^{\mathrm{GGPH}} : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to A$ is proper.*

(2) *H^{GGPH} defines a proper morphism*

$$H : f(\mathcal{M}_X^{\mathrm{GGPH}}(n,d)) \to A,$$

where f is given in Proposition 4.1.

Proposition 4.3 and Proposition 4.4(2) imply the following statement.

Corollary 4.5. *H^{GGPH} defines a proper morphism*

$$H : \mathcal{M}_Y(n,d) \to A.$$

This $H : \mathcal{M}_Y(n,d) \to A$ is the *Hitchin map on $\mathcal{M}_Y(n,d)$*.

Now we describe all fibers of $H^{\mathrm{GGPH}} : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to A$ and $H : \mathcal{M}_Y(n,d) \to A$. To describe all fibers of $H^{\mathrm{GGPH}} : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to A$, we follow the arguments of [8, Section 5].

Set $h = \gcd(n,d)$, $n' = \frac{n}{h}$ and $d' = \frac{d}{h}$. If $(E,\phi_E,F_1(E))$ is a polystable good GPH of rank n and degree d, then we have

$$(E,\phi_E,F_1(E)) \cong (E_1,\phi_{E_1},F_1(E_1))\oplus\cdots\oplus(E_h,\phi_{E_h},F_1(E_h)),$$

where each $(E_i,\phi_{E_i},F_1(E_i))$ is stable of rank n' and degree d' by Lemma 3.12. Then $(E_i,F_1(E_i))$ is stable by Proposition 3.6 and $\phi_{E_i} = \frac{t_i}{n'}\mathrm{id}_{E_i}$, where $\eta_{n',d'}^{\mathrm{GGPH}}((x_i,t_i)) = [(E_i,\phi_{E_i},F_1(E_i))]_S$. Then

$$\begin{aligned} a_1(\phi_E) &= \sum_{i=1}^h t_i, \\ &\vdots \\ a_n(\phi_E) &= \left(\frac{t_1}{n'}\right)^{n'}\cdots\left(\frac{t_h}{n'}\right)^{n'}. \end{aligned}$$

Denote the image of H^{GGPH} by $A_{n,d}$. If $D_{\bar\lambda}$ is the diagonal matrix with eigenvalues $\bar\lambda = (\lambda_1,\ldots,\lambda_h)$, then the following morphism

$$\begin{aligned} \alpha_{n,d} &: \mathrm{Sym}^h\mathbb{C} \to A_{n,d}, \\ [\bar t]_{\mathfrak{S}_h} = [t_1,\ldots,t_h]_{\mathfrak{S}_h} &\mapsto (a_1(D_{\frac{1}{n'}\bar t}),\ldots,a_n(D_{\frac{1}{n'}\bar t})) \end{aligned}$$

is bijective.

Define the projection

$$\pi_h : \mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C}) \to \mathrm{Sym}^h\mathbb{C},$$
$$[(x_1, t_1), \dots, (x_h, t_h)]_{\mathfrak{S}_h} \mapsto [t_1, \dots, t_h]_{\mathfrak{S}_h}.$$

Then the following diagram

$$\begin{array}{ccc} \mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C}) & \xrightarrow{\pi_h} & \mathrm{Sym}^h\mathbb{C} \\ {\scriptstyle \eta_{n,d}^{\mathrm{GGPH}}}\downarrow{\scriptstyle 1:1} & & {\scriptstyle 1:1}\downarrow{\scriptstyle \alpha_{n,d}} \\ \mathcal{M}_X^{\mathrm{GGPH}}(n,d) & \xrightarrow{H^{\mathrm{GGPH}}} & A_{n,d} \end{array}$$

commutes.

The set $G = \{(t_1, \dots, t_h) \in \mathbb{C}^h \,|\, t_i \neq t_j \text{ if } i \neq j\}$ forms an open dense subset of $\mathbb{C}^h$. A point of $A_{n,d}$ is called *generic* if it is the image under $\alpha_{n,d}$ of the $\mathfrak{S}_h$-orbit of some $\bar{t}_g \in G$. An arbirtrary point of $A_{n,d}$ is the image under $\alpha_{n,d}$ of a h-tuple of the form

$$\bar{t}_a = (t_1, \overset{m_1}{\dots}, t_1, \dots, t_l, \overset{m_l}{\dots}, t_l),$$

where $h = m_1 + \cdots + m_l$.

Proposition 4.6.

$$\pi_h^{-1}([\bar{t}_g]_{\mathfrak{S}_h}) \cong \mathbb{P}^1 \times \overset{h}{\cdots} \times \mathbb{P}^1$$

and

$$\pi_h^{-1}([\bar{t}_a]_{\mathfrak{S}_h}) \cong \mathrm{Sym}^{m_1}\mathbb{P}^1 \times \overset{l}{\cdots} \times \mathrm{Sym}^{m_l}\mathbb{P}^1 \cong \mathbb{P}^{m_1} \times \overset{h}{\cdots} \times \mathbb{P}^{m_l}.$$

Proof. The proof is same as that of Proposition 5.1 of [8]. □

Corollary 4.7. *The generic fiber of $H^{\mathrm{GGPH}} : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to A_{n,d}$ is set-theoretically isomorphic to $\mathbb{P}^1 \times \overset{h}{\cdots} \times \mathbb{P}^1$. The fiber over an arbitrary point of the base is set-theoretically isomorphic to $\mathbb{P}^{m_1} \times \overset{h}{\cdots} \times \mathbb{P}^{m_l}$. The fiber over an arbitrary point of the base is isomorphic to $\mathbb{P}^1$ for the case* $\gcd(n,d) = 1$.

Proof. By Theorem 3.14, there exists a bijective morphism from the fiber of π_h to the fiber of H^{GGPH}. So the first and the second statements follow from Proposition 4.6. The last statement follows from Theorem 3.7(1). □

Corollary 4.8. *The generic fiber of $H : \mathcal{M}_Y(n,d) \to A_{n,d}$ is set-theoretically isomorphic to $Y \times \overset{h}{\cdots} \times Y$. The fiber over an arbitrary point of the base is set-theoretically isomorphic to $\mathrm{Sym}^{m_1}Y \times \overset{l}{\cdots} \times \mathrm{Sym}^{m_l}Y$. The fiber over an arbitrary point of the base is isomorphic to Y for the case* $\gcd(n,d) = 1$.

Proof. Note that the normalization map $\nu : \mathbb{P}^1 \to Y$ induces the surjective map

$$g : \mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C}) \to \mathrm{Sym}^h(Y \times \mathbb{C}),$$
$$[(x_1, t_1), \dots, (x_h, t_h)]_{\mathfrak{S}_h} \mapsto [(\nu(x_1), t_1), \dots, (\nu(x_h), t_h)]_{\mathfrak{S}_h}.$$

By Theorem 3.11(1) and Theorem 3.14(1), set-theoretically $f : \mathcal{M}_X^{\mathrm{GGPH}}(n,d) \to \mathcal{M}_Y(n,d)$ and g coincide. The map π_h and g induce

$$\bar{\pi}_h : g(\mathrm{Sym}^h(\mathbb{P}^1 \times \mathbb{C})) = \mathrm{Sym}^h(Y \times \mathbb{C}) \to \mathrm{Sym}^h\mathbb{C},$$

$$[(y_1, t_1), \ldots, (y_h, t_h)]_{\mathfrak{S}_h} \mapsto [t_1, \ldots, t_h]_{\mathfrak{S}_h}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^h(Y \times \mathbb{C}) & \xrightarrow{\quad \bar{\pi}_h \quad} & \mathrm{Sym}^h\mathbb{C} \\ {\scriptstyle \eta_{n,d}} \Big\downarrow {\scriptstyle 1:1} & & {\scriptstyle 1:1} \Big\downarrow {\scriptstyle \alpha_{n,d}} \\ \mathcal{M}_Y(n,d) & \xrightarrow{\quad H \quad} & A_{n,d} \end{array}$$

So $f(\mathbb{P}^1 \times \overset{h}{\cdots} \times \mathbb{P}^1) = Y \times \overset{h}{\cdots} \times Y$ and $f(\mathrm{Sym}^{m_1}\mathbb{P}^1 \times \overset{l}{\cdots} \times \mathrm{Sym}^{m_l}\mathbb{P}^1) = \mathrm{Sym}^{m_1}Y \times \overset{l}{\cdots} \times \mathrm{Sym}^{m_l}Y$. By Theorem 3.11 and Corollary 4.7, there exists a bijective morphism from the fiber of $\bar{\pi}_h$ to the fiber of H. The last statement follows from Theorem 3.2. □

5. A flat degeneration

Assume that $\gcd(n,d) = 1$. In this section we consider a flat degeneration of the moduli space of stable Higgs bundles of rank n and degree d on an elliptic curve, which was constructed in [3].

We show that the moduli space of stable Higgs bundles of rank n and degree d on an elliptic curve degenerates to $Y \times \mathbb{C}$ in a flat family. We also show that the fiber of the Hitchin map on the moduli space of stable Higgs bundles of rank n and degree d on an elliptic curve degenerates to Y via the same flat family.

Let $Y^{(m)}$ be the curves which are semistably equivalent to Y, i.e., X is a component of $Y^{(m)}$ and if $\nu : Y^{(m)} \to Y$ is the canonical morphism, $\nu^{-1}(p)$ is a chain R of projective lines of length m, passing through p_1 and p_2.

Definition ([13])**.** A *Gieseker vector bundle* of rank n and degree d on $Y^{(m)}$ is a vector bundle E of rank n and degree d on $Y^{(m)}$ such that

- If $m \geq 1$, then $E|_R$ is strictly standard,
- $\nu_* E$ is a torsion-free sheaf on Y.

A Gieseker vector bundle E on $Y^{(m)}$ is *stable* if $\nu_* E$ is a stable torsion-free sheaf on Y.

Definition ([3])**.** A *Gieseker-Hitchin pair* of rank n and degree d on $Y^{(m)}$ is a pair (E, ϕ) such that E is a Gieseker vector bundle of rank n and degree d on $Y^{(m)}$, ϕ is a global section of $\mathcal{E}nd\, E$ and $\nu_*(E, \phi)$ is a Higgs pair on Y. A Gieseker-Hitchin pair (E, ϕ) on $Y^{(m)}$ is *stable* if $\nu_*(E, \phi)$ is a stable Higgs pair on Y.

Let $G_Y(n,d)$ be the moduli space of stable Gieseker vector bundles of rank n and degree d on $Y^{(m)}$ for some $0 \le m \le n$ (see [13, Theorem 1]) and let $\mathcal{G}_Y(n,d)$ the moduli space of stable Gieseker-Hitchin pairs of rank n and degree d on $Y^{(m)}$ for some $0 \le m \le n$ (see [3, Proposition 5.13]).

By Lemma 3.1, the stability of Gieseker-Hitchin pairs can be simplified as follows.

Lemma 5.1. *A Gieseker-Hitchin pair (E,ϕ) on $Y^{(m)}$ is stable if and only if the underlying Gieseker vector bundle E on $Y^{(m)}$ is stable.*

We can see that any endomorphism of a stable Gieseker-Hitchin pair is a scalar.

Lemma 5.2. *For a stable Gieseker vector bundle E on $Y^{(m)}$,* $\operatorname{End} E = \mathbb{C}$.

Proof. Since any finite dimensional division algebra over $\mathbb{C}$ is $\mathbb{C}$, it suffices to show that any nonzero $\varphi \in \operatorname{End} E$ is an isomorphism.

Consider a nonzero $\varphi \in \operatorname{End} E$. If $\varphi|_X : E|_X \to E|_X$ is zero, then φ is zero at p_1 and p_2. Then $\varphi|_R : E|_R \to E|_R$ is zero, which is a contradiction. So $\varphi|_X : E|_X \to E|_X$ is nonzero. Since $\operatorname{End}(E|_X) \cong \operatorname{End}(\nu_* E)$ by [13, Remark 4], $\varphi|_X$ is an isomorphism. Then φ is an isomorphism at p_1 and p_2. Since $(\det(E|_R))^{-1} \otimes (\det(E|_R))$ is trivial, $\det \varphi$ is nowhere zero and then φ is an isomorphism (see [9, The proof of Proposition 3.1]). □

Now we define a flat family of Gieseker vector bundles, that of Gieseker-Hitchin pairs and their stabilities. Let R be a discrete valuation ring with quotient field K and residue field $\mathbb{C}$. Let $T = \operatorname{Spec} R$, $\operatorname{Spec} K$ the generic point and t_0 the closed point of T. Let $Z \to T$ be a proper flat family such that the generic fiber $(Z_K, \mathbf{0} : \operatorname{Spec} K \to Z_K)$ is an elliptic curve and the closed fiber $Z_{t_0} \cong Y$.

Definition ([3])**.** Let $Z^{(\mathrm{mod})} \to T$ be a flat morphism such that there exists a nonnegative integer m satisfying that $(Z^{(\mathrm{mod})})_t = Y^{(m)}$ for each $t \in T$ with the commutative diagram:

$$\begin{array}{ccc} Z^{(\mathrm{mod})} & \xrightarrow{\ \nu\ } & Z \\ & {\scriptstyle p_T}\searrow \quad \swarrow & \\ & T & \end{array}$$

where ν restricts to the morphism which contracts the chain R of $\mathbb{P}^1$'s on $Y^{(m)}$.

(1) A *Gieseker vector bundle on* $Z^{(\mathrm{mod})}$ is a vector bundle $\mathcal{E}_T$ on $Z^{(\mathrm{mod})}$ such that its restriction to the fiber $(Z^{(\mathrm{mod})})_t$ over $t \in T$ is a Gieseker vector bundle on $Y^{(m)}$ for some m.

(2) A Gieseker vector bundle $\mathcal{E}_T$ on $Z^{(\mathrm{mod})}$ is *stable* if $\nu_*\mathcal{E}_T$ is a family of stable torsion-free sheaves on $Z \to T$.

(3) A *Gieseker-Hitchin pair on* $Z^{(\mathrm{mod})}$ is a pair $(\mathcal{E}_T, \varphi_T)$ on $Z^{(\mathrm{mod})}$ such that $\mathcal{E}_T$ is a vector bundle on $Z^{(\mathrm{mod})}$, φ_T is a global section of $(p_T)_*\mathcal{E}nd\,\mathcal{E}_T$ and its restriction to the fiber $(Z^{(\mathrm{mod})})_t$ over $t \in T$ is a Gieseker-Hitchin pair on $Y^{(m)}$ for some m.
(4) A Gieseker-Hitchin pair $(\mathcal{E}_T, \varphi_T)$ on $Z^{(\mathrm{mod})}$ is *stable* if $\nu_*(\mathcal{E}_T, \varphi_T)$ is a family of stable Higgs pairs on $Z \to T$.

Let $G_{Z/T}(n,d) \to T$ be the relative moduli space of stable Gieseker vector bundles of rank n and degree d on $Z^{(\mathrm{mod})}$ (see [13, Theorem 2]) and let $\mathcal{G}_{Z/T}(n,d) \to T$ the relative moduli space of stable Gieseker-Hitchin pairs of rank n and degree d on $Z^{(\mathrm{mod})}$ (see [3, Proposition 5.13]). Note that $G_{Z/T}(n,d) \to T$ and $\mathcal{G}_{Z/T}(n,d) \to T$ are flat over T (see [3, Proposition 5.15] and [13, Theorem 2]). Moreover $G_{Z/T}(n,d)_t$ is the moduli space of vector bundles on the elliptic curve Z_t for all $t \neq t_0$, $G_{Z/T}(n,d)_{t_0} \cong G_Y(n,d)$, $\mathcal{G}_{Z/T}(n,d)_t$ is the moduli space of Higgs bundles on the elliptic curve Z_t for all $t \neq t_0$ and $\mathcal{G}_{Z/T}(n,d)_{t_0} \cong \mathcal{G}_Y(n,d)$.

Now we describe $G_Y(n,d)$ and $\mathcal{G}_Y(n,d)$ explicitly.

Lemma 5.3. $G_Y(n,d) \cong Y$.

Proof. Let $G = G_{Z/T}(n,d)$. Then $G \to T$ is flat over T such that $G_t \cong Z_t$ for all $t \neq t_0$ (see [2]) and $G_{t_0} \cong G_Y(n,d)$.

By [13, Theorem 1 and Theorem 2], $G_{t_0} \cong G_Y(n,d)$ is a singular curve of arithmetic genus one.

If $\tilde{G}_{t_0}$ is the normalization of G_{t_0} with the normalization morphism $\pi : \tilde{G}_{t_0} \to G_{t_0}$, then it follows from [10, Corollary V.3.7 and Proposition V.3.8] that

$$p_a(\tilde{G}_{t_0}) = p_a(G_{t_0}) - \sum_{i=1}^{r} \frac{e_i(e_i-1)}{2},$$

where p_a denotes the arithmetic genus and e_i are the multiplicities of the infinitesimally near points of the singular points of G_{t_0}. Since $p_a(G_{t_0}) = 1$, then $p_a(\tilde{G}_{t_0}) = 0$, $r = 1$ and $e_i = 2$, which implies that G_{t_0} has a unique ordinary double point, that is, a node or a cusp. Since $\tilde{G}_{t_0}$ is nonsingular and $p_a(\tilde{G}_{t_0}) = 0$, $\tilde{G}_{t_0} \cong \mathbb{P}^1$.

By [4, Lemma 3.1], the natural stratification of $G_{t_0} \cong G_Y(n,d)$

$$(G_{t_0})^0 = G_{t_0} \supset (G_{t_0})^1 \supset \cdots \supset (G_{t_0})^n \supset (G_{t_0})^{n+1} = \emptyset,$$

where $(G_{t_0})^{r+1}$ is the singular locus of $(G_{t_0})^r$ for every $0 \leq r \leq n$ has the following property:

- $(G_{t_0})^i = \{x \in (G_{t_0})^0 \,|\, \text{cardinality of the set } \pi^{-1}(x) \geq i+1\}$ for every $0 \leq i \leq n$.
- $(G_{t_0})^{i+1}$ is a Zariski-closed subvariety of $(G_{t_0})^i$ of pure codimension 1, if non-empty.

Since G_{t_0} is a curve, $(G_{t_0})^2 = \emptyset$ and

$$(G_{t_0})^1 = \{x \in (G_{t_0})^0 \,|\, \text{cardinality of the set } \pi^{-1}(x) = 2\}.$$

Hence the unique singular point of G_{t_0} is a node and then $G_{t_0} \cong Y$. □

Lemma 5.4. $\mathcal{G}_Y(n,d) \cong G_Y(n,d) \times \mathbb{C} \cong Y \times \mathbb{C}$.

Proof. The proof is same as that of Theorem 3.2 and Theorem 5.8(1). We use Lemma 5.1, Lemma 5.2 and Lemma 5.3. □

The Hitchin map on $\mathcal{G}_{Z/T}(n,d)$ is defined as follows.

Definition (Definition 6.2 of [3])**.** The *Hitchin map on* $\mathcal{G}_{Z/T}(n,d)$ is defined by

$$H^{\mathrm{GH}} : \mathcal{G}_{Z/T}(n,d) \to A_T,$$
$$(E_T, \varphi_T) \mapsto (a_1(\varphi_T), \ldots, a_n(\varphi_T)),$$

where $A_T \to T$ is the affine T-scheme representing the functor

$$S \mapsto \bigoplus_{i=1}^{n} H^0(Z \times_T S, \mathcal{O}_{Z \times_T S})$$

and the characteristic polynomial $\det(\lambda - \varphi_T)$ of (E_T, φ_T) is

$$\lambda^n + a_1(\varphi_T)\lambda^{n-1} + \cdots + a_n(\varphi_T).$$

Proposition 5.5 (Theorem 6.6 of [3])**.** $H^{\mathrm{GH}} : \mathcal{G}_{Z/T}(n,d) \to A_T$ *is proper over* T.

By [13, Theorem 1 and Theorem 2], there exist proper and birational morphisms

$$\nu_*^{\mathrm{GV}} : G_Y(n,d) \to U_Y(n,d),\ E \mapsto \nu_* E$$

and

$$\nu_*^{\mathrm{GV}} : G_{Z/T}(n,d) \to U_{Z/T}(n,d),\ \mathcal{E}_T \mapsto \nu_* \mathcal{E}_T,$$

where $U_{Z/T}(n,d)$ denotes the relative moduli space of stable torsion-free sheaves of rank n and degree d on Z. By [3, Corollary 5.14], there exist proper and birational morphisms

$$\nu_*^{\mathrm{GH}} : \mathcal{G}_Y(n,d) \to \mathcal{M}_Y(n,d),\ (E,\phi) \mapsto \nu_*(E,\phi)$$

and

$$\nu_*^{\mathrm{GH}} : \mathcal{G}_{Z/T}(n,d) \to \mathcal{M}_{Z/T}(n,d),\ (\mathcal{E}_T, \varphi_T) \mapsto \nu_*(\mathcal{E}_T, \varphi_T),$$

where $\mathcal{M}_{Z/T}(n,d)$ denotes the relative moduli space of stable Higgs pairs of rank n and degree d on Z.

Indeed we have the following observation.

Proposition 5.6. (1) $\nu_*^{\mathrm{GV}} : G_Y(n,d) \to U_Y(n,d)$ *is identified with the identity map* $\mathrm{id}_Y : Y \to Y$.

(2) $\nu_*^{\mathrm{GH}} : \mathcal{G}_Y(n,d) \to \mathcal{M}_Y(n,d)$ *is identified with the identity map* $\mathrm{id}_{Y \times \mathbb{C}} : Y \times \mathbb{C} \to Y \times \mathbb{C}$.

Proof. By [4, Proposition 3.1] and Lemma 5.3, the singular locus of $G_Y(n,d)$ consists of stable Gieseker vector bundles on $Y^{(1)}$ and it corresponds to the single node p of Y. Moreover the image of the singular locus of $G_Y(n,d)$ under ν_*^{GV} is exactly the singular locus of $U_Y(n,d)$ by [7, Remark 2.3]. Thus $\nu_*^{\mathrm{GV}} : G_Y(n,d) \to U_Y(n,d)$ is identified with the identity map $\mathrm{id}_Y : Y \to Y$. Moreover $\nu_*^{\mathrm{GH}} : \mathcal{G}_Y(n,d) \to \mathcal{M}_Y(n,d)$ is identified with the identity map $\mathrm{id}_{Y\times\mathbb{C}} : Y \times \mathbb{C} \to Y \times \mathbb{C}$ by Theorem 3.2 and Lemma 5.4. □

Lemma 5.3 is relativized as follows.

Lemma 5.7. *$G_{Z/T}(n,d) \cong Z$ as T-schemes.*

Proof. We first see that $U_{Z/T}(1,d) \cong Z$ as T-schemes. Let $T' = T \setminus \{t_0\}$. For any T'-scheme S, we have the following isomorphism

$$Z|_{T'}(S) \to U_{Z/T}(1,d)|_{T'}(S),\ \sigma \mapsto \mathcal{O}_{Z|_{T'}}(\sigma) \otimes \mathcal{O}_{Z|_{T'}}(\mathbf{0})^{d-1}$$

on S-valued points, where $\mathbf{0} : S \to Z|_{T'}$ is a section which makes $(Z|_{T'}, \mathbf{0})$ a flat family of elliptic curves. Then we have an isomorphism $\gamma : Z|_{T'} \to U_{Z/T}(1,d)|_{T'}$ by [1, Lemma 5.7]. By [1, Corollary 5.4], γ extends uniquely to an isomorphism $Z \to U_{Z/T}(1,d)$ over T.

Next we see that $U_{Z/T}(n,d) \cong Z$ as T-schemes. Consider the relative determinant morphism $\det_{Z/T} : U_{Z/T}(n,d) \to U_{Z/T}(1,d)$. Since $(\det_{Z/T})_t : U_{Z/T}(n,d)_t \to U_{Z/T}(1,d)_t$ is an isomorphism for all $t \in T'$, $\det_{Z/T}|_{T'} : U_{Z/T}(n,d)|_{T'} \to U_{Z/T}(1,d)|_{T'}$ is also an isomorphism by [1, Lemma 5.7]. By [1, Corollary 5.4], $\det_{Z/T}|_{T'}$ extends uniquely to an isomorphism $\det_{Z/T} : U_{Z/T}(n,d) \to U_{Z/T}(1,d)$. Since we have seen that $U_{Z/T}(1,d) \cong Z$, we get an isomorphism $U_{Z/T}(n,d) \to Z$ over T.

Now we claim that $G_{Z/T}(n,d) \cong Z$ as T-schemes. Consider the morphism

$$\nu_*^{\mathrm{GV}} : G_{Z/T}(n,d) \to U_{Z/T}(n,d),\ \mathcal{E}_T \mapsto \nu_*\mathcal{E}_T.$$

For all $t \neq t_0$, $(\nu_*^{\mathrm{GV}})_t : G_{Z/T}(n,d)_t \to U_{Z/T}(n,d)_t$ is the identity map $\mathrm{id}_{Z_t} : Z_t \to Z_t$. By Proposition 5.6(1), $(\nu_*^{\mathrm{GV}})_{t_0} : G_{Z/T}(n,d)_{t_0} \to U_{Z/T}(n,d)_{t_0}$ is also the identity map $\mathrm{id}_Y : Y \to Y$. Thus $\nu_*^{\mathrm{GV}} : G_{Z/T}(n,d) \to U_{Z/T}(n,d)$ is the identity map $\mathrm{id}_Z : Z \to Z$ by [1, Corollary 5.4 and Lemma 5.7]. Since we have seen that $U_{Z/T}(n,d) \cong Z$ as T-schemes, we get $G_{Z/T}(n,d) \cong Z$ as T-schemes. □

We have a conclusion as follows.

Theorem 5.8. (1) *$\mathcal{G}_{Z/T}(n,d) \cong G_{Z/T}(n,d) \times \mathbb{C} \cong Z \times \mathbb{C}$ as T-schemes.*

(2) *$\nu_*^{\mathrm{GH}} : \mathcal{G}_{Z/T}(n,d) \to \mathcal{M}_{Z/T}(n,d)$ is identified with the identity map $\mathrm{id}_{Z\times\mathbb{C}} : Z \times \mathbb{C} \to Z \times \mathbb{C}$.*

(3) *The fiber of the Hitchin map H^{GH} on $\mathcal{G}_{Z/T}(n,d)$ is isomorphic to Z.*

Proof. (1) By using the construction of $\mathcal{G}_{Z/T}(n,d)$ in [3, Section 5], we show that $\mathcal{G}_{Z/T}(n,d) \cong G_{Z/T}(n,d) \times \mathbb{C}$.

Note that $U_{Z/T}(n,d) \cong R_T^s//\mathrm{PGL}(d)$ for some T-scheme R_T. The Gieseker functor $\mathcal{G}_{R_T}$ is represented by a $\mathrm{PGL}(d)$-invariant open subscheme $\mathcal{Y}$ of the T-scheme $\mathrm{Hilb}^{P_1}(Z \times_T \mathrm{Gr}(d,n))$ for some Hilbert polynomial P_1 and $G_{Z/T}(n,d) \cong \mathcal{Y}^s//\mathrm{PGL}(d)$. Let $\Delta_{\mathcal{Y}} \subset Z \times_T \mathcal{Y} \times_T \mathrm{Gr}(d,n)$ be the universal object defining the functor $\mathcal{G}_{R_T}$. The embedding $\Delta_{\mathcal{Y}} \subset Z \times_T \mathcal{Y} \times_T \mathrm{Gr}(d,n)$ gives the natural projection of T-schemes:

$$\begin{array}{ccc} \Delta_{\mathcal{Y}} & \xrightarrow{\;q\;} & \mathcal{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle r} & \\ & T & \end{array}$$

Let $\mathcal{U}$ be the universal vector bundle on $\Delta_{\mathcal{Y}}|_{\mathcal{Y}^s}$ obtained from the tautological quotient bundle on $\mathrm{Gr}(d,n)$. Applying [14, Lemma 3.5] with $\mathcal{F} = \mathcal{E}nd\,\mathcal{U}$, we get a linear scheme $\mathcal{Y}^{\mathrm{GH}} \to \mathcal{Y}^s$ given by

$$\mathcal{Y}^{\mathrm{GH}} = \mathbf{Spec}\,\mathrm{Sym}_{\mathcal{O}_{\mathcal{Y}^s}}(q_*\mathcal{E}nd\,\mathcal{U})^\vee.$$

Since $q_*\mathcal{E}nd\,\mathcal{U} \cong \mathcal{O}_{\mathcal{Y}^s}$ by Lemma 5.2 and the same argument of the proof of [12, Lemma 4.6.3], we have $\mathcal{Y}^{\mathrm{GH}} \cong \mathcal{Y}^s \times \mathbb{C}$. Lemma 5.1 implies that $(\mathcal{Y}^{\mathrm{GH}})^s \cong \mathcal{Y}^s \times \mathbb{C}$.

Hence by the construction of [3, Section 5] and Lemma 5.7,

$$\mathcal{G}_{Z/T}(n,d) \cong (\mathcal{Y}^{\mathrm{GH}})^s//\mathrm{PGL}(d) \cong (\mathcal{Y}^s//\mathrm{PGL}(d)) \times \mathbb{C} \cong G_{Z/T}(n,d) \times \mathbb{C} \cong Z \times \mathbb{C}.$$

(2) We first show that $\mathcal{M}_{Z/T}(n,d) \cong U_{Z/T}(n,d) \times \mathbb{C}$ by the same argument as in the proof of item (1). Note that $U_{Z/T}(n,d) \cong R_T^s//\mathrm{PGL}(d)$ for some T-scheme R_T as mentioned. Let $\mathcal{U}$ be the restriction of the universal sheaf to $Z \times_T R_T^s$. [14, Lemma 3.5] with $\mathcal{F} = \mathcal{E}nd\,\mathcal{U}$, the fact $\pi_{R_T^s *}\mathcal{E}nd\,\mathcal{U} \cong \mathcal{O}_{R_T^s}$ from [12, Lemma 4.6.3], Lemma 3.1 and the proof of Lemma 5.7 implies that

$$\mathcal{M}_{Z/T}(n,d) \cong U_{Z/T}(n,d) \times \mathbb{C} \cong Z \times \mathbb{C}.$$

Now we describe $\nu_*^{\mathrm{GH}} : \mathcal{G}_{Z/T}(n,d) \to \mathcal{M}_{Z/T}(n,d)$. Since

$$\mathcal{G}_{Z/T}(n,d) \cong G_{Z/T}(n,d) \times \mathbb{C}$$

in item (1) and $\mathcal{M}_{Z/T}(n,d) \cong U_{Z/T}(n,d) \times \mathbb{C}$ as shown previously, ν_*^{GH} is exactly $\nu_*^{\mathrm{GV}} \times \mathrm{id}_{\mathbb{C}} : G_{Z/T}(n,d) \times \mathbb{C} \to U_{Z/T}(n,d) \times \mathbb{C}$. In the proof of Lemma 5.7, we have seen that $\nu_*^{\mathrm{GV}} : G_{Z/T}(n,d) \to U_{Z/T}(n,d)$ is the identity map $\mathrm{id}_Z : Z \to Z$. Thus we get the statement.

(3) By the definition of H^{GH} and $\mathcal{G}_{Z/T}(n,d) \cong Z \times \mathbb{C}$ in item (1), H^{GH} is identified with the projection $Z \times \mathbb{C} \to \mathbb{C}$ onto the second factor. Thus we get the result. □

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Sang-Bum Yoo
Department of Mathematics Education
Gongju National University of Education
Gongju-si 32553, Korea
Email address: sbyoo@gjue.ac.kr