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EXISTENCE OF SOLUTIONS TO A GENERALIZED SELF-DUAL CHERN-SIMONS EQUATION ON FINITE GRAPHS

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Abstract. Let $G = (V, E)$ be a connected finite graph. We study the existence of solutions for the following generalized Chern-Simons equation on G

$$\Delta u = \lambda \mathrm{e}^u \left(\mathrm{e}^u - 1\right)^5 + 4\pi \sum_{s=1}^{N} \delta_{p_s},$$

where $\lambda > 0$, δ_{p_s} is the Dirac mass at the vertex p_s, and $p_1, p_2, \ldots, p_N$ are arbitrarily chosen distinct vertices on the graph. We show that there exists a critical value $\hat{\lambda}$ such that when $\lambda > \hat{\lambda}$, the generalized Chern-Simons equation has at least two solutions, when $\lambda = \hat{\lambda}$, the generalized Chern-Simons equation has a solution, and when $\lambda < \hat{\lambda}$, the generalized Chern-Simons equation has no solution.

1. Introduction

In recent years, increasing efforts have been devoted to the development of partial differential equations on graphs. For heat equations on graphs, various methods and techniques have been used to study the existence and qualitative properties of solutions. Lin and Wu [15, 22] established the existence and nonexistence of global solutions for semilinear heat equations on finite or locally finite connected weighted graphs. Bauer et al. [2] established the Li-Yau gradient estimate for the heat kernel on graphs. Horn et al. [12] proved Li-Yau-type estimates for bounded and positive solutions of the heat equation on graphs. For elliptic equations on graphs, there have been extensive literature; see, for example, [4, 7, 9, 10, 13] and the references therein. Bendito et al. [4] constructed solutions of self-adjoint boundary value problems on finite graphs. Ge, Hua and Jiang [7] studied the Liouville equation $-\Delta u = e^u$ on a graph satisfying a certain isoperimetric inequality. In [9], Ge and Jiang studied the 1-Yamabe equation $\Delta_1 u + g\,\mathrm{Sgn}(u) = h|u|^{\alpha-1}\,\mathrm{Sgn}(u)$ on connected finite

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graphs. Grigor'yan, Lin, and Yang [10] studied the Kazdan-Warner equation on a finite graph. Ge and Jiang [8] used a heat flow method to study the Kazdan-Warner equation on an infinite graph. By the method of Brouwer degree, Sun and Wang [19] and Liu [16] established the existence of solutions to the Kazdan-Warner equation and the mean field equation respectively on finite graphs.

Abrikosov [1] considered configurations of magnetic vortices in the context of the Ginzburg-Landau theory of superconductivity. Subsequently, Nielsen and Olesen pointed out the relevance to high energy physics of vortex line solutions of the Abelian Higgs model in the context of string dual models [18]. Since then, interest in vortices has continued to grow in both condensed matter and particle physics. Self-dual Chern-Simons models now play an important role in various areas of physics. Recently, there has been a growing amount of effort devoted to the existence of vortices to the self-dual Chern-Simons models. See, for example, [3, 6, 11, 21, 23] and the references therein. Caffarelli and Yang [5] established the existence of condensates or periodic multivortices in the Abelian Chern-Simons-Higgs model. A generalized Chern-Simons model was proposed by Bazeia et al. [3], who obtained a generalized self-dual Chern-Simons equation. The existence of doubly periodic multi-vortex solutions to a generalized self-dual Chern-Simons model was subsequently established by Han [11].

Recently, there have been many studies of Chern-Simons equations on graphs. See, for example, [13,14,17] and the references therein. In [14], Huang, Lin, and Yau studied the mean field equation

$$\Delta u = \lambda e^u \left(e^u - 1\right) + 4\pi \sum_{j=1}^{M} \delta_{p_j} \tag{1}$$

on a connected finite graph $G = (V, E)$, and proved an existence result to (1). In [13], Hou and Sun established the existence of solutions to a generalized Chern-Simons-Higgs equation. In [17], Lü and Zhong considered a generalized self-dual Chern-Simons equation on a finite graph.

Motivated by these works, we are concerned in this article the following generalized self-dual Chern-Simons equation derived from Han [11], i.e.,

$$\Delta u = \lambda \mathrm{e}^u \left(\mathrm{e}^u - 1\right)^5 + 4\pi \sum_{s=1}^{N} \delta_{p_s} \tag{2}$$

on a finite connected graph G, where $\lambda > 0$, δ_{p_s} satisfies

$$\delta_{p_j} = \begin{cases} \frac{1}{\mu(p_j)}, & \text{at } p_j, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

and $p_1, p_2, \dots, p_N$ are arbitrarily chosen distinct vertices on the graph.

The rest of the paper is arranged as below. In Section 2, we present some results that we will use in the following pages and state the main theorem. In Section 3, we prove Theorem 2.5.

2. Preliminary results

Let $G = (V, E)$ be a connected finite graph, where V denotes the set of vertices and E denotes the edge set. Throughout this paper, all graphs are assumed to be connected. For each edge $xy \in E$, we suppose that its weight $w_{xy} > 0$ and that $w_{xy} = w_{yx}$. Set $\mu : V \to (0, +\infty)$ be a finite measure. For any function $u : V \to \mathbb{R}$, the Laplacian of u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{yx}(u(y) - u(x)), \tag{4}$$

where $y \sim x$ means $xy \in E$. The gradient form of u and v reads

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{5}$$

We denote the length of the gradient of u by

$$|\nabla u|(x) = \sqrt{\Gamma(u, u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}.$$

Denote, for any function $u : V \to \mathbb{R}$, an integral of u on V by

$$\int_V u d\mu = \sum_{x \in V} \mu(x) u(x).$$

Denote

$$|V| = \mathrm{Vol}(V) = \sum_{x \in V} \mu(x)$$

the volume of V. For $p \geq 1$, denote $||u||_p := (\int_V |u|^p d\mu)^{\frac{1}{p}}$. Define a Sobolev space and a norm on it by

$$W^{1,2}(V) = \left\{ u : V \to \mathbb{R} : \int_V \left(|\nabla u|^2 + u^2 \right) d\mu < +\infty \right\},$$

and

$$\|u\|_{H^1(V)} = \|u\|_{W^{1,2}(V)} = \left(\int_V \left(|\nabla u|^2 + u^2 \right) d\mu \right)^{1/2}.$$

To prove our main result, we need the following Sobolev embedding, Poincare inequality and maximum principle on graphs.

Lemma 2.1 ([10, Lemma 5]). *Let $G = (V, E)$ be a finite graph. The Sobolev space $W^{1,2}(V)$ is precompact. Namely, if $\{u_j\}$ is bounded in $W^{1,2}(V)$, then there exists some $u \in W^{1,2}(V)$ such that up to a subsequence, $u_j \to u$ in $W^{1,2}(V)$.*

Lemma 2.2 ([10, Lemma 6]). *Let $G=(V,E)$ be a finite graph. For all functions $u: V \to \mathbb{R}$ with $\int_V u d\mu = 0$, there exists some constant C depending only on G such that*

$$\int_V u^2 d\mu \leq C \int_V |\nabla u|^2 d\mu.$$

Lemma 2.3 ([14, Lemma 4.1]). *Let $G=(V,E)$, where V is a finite set, and $K \geq 0$ is a constant. Suppose a real function $u: V \to \mathbb{R}$ satisfies*

$$(\Delta - K)u(x) \geq 0$$

for all $x \in V$. Then $u(x) \leq 0$ for all $x \in V$.

Lemma 2.4. *Let $G=(V,E)$ be a finite graph. For all functions $u: V \to \mathbb{R}$ with $\int_V u d\mu = 0$ and $p \geq 1$, there exists a constant $C = C(G,p)$ such that*

$$||u||_p \leq C||\nabla u||_2.$$

Proof. By Lemma 2.2, there exists $C = C(G)$ such that

$$M := \max_{x\in V} |u(x)| \leq (\frac{C}{\min\limits_V \mu} \int_V |\nabla u|^2 d\mu)^{\frac{1}{2}}.$$

Thus we have

$$||u||_p \leq (\frac{C}{\min\limits_V \mu} \int_V |\nabla u|^2 d\mu)^{\frac{1}{2}} |V|^{1/p} := C_2 ||\nabla u||_2.$$

□

We are now ready to delineate the major result of this article.

Theorem 2.5. *Let $G=(V,E)$ be a finite connected graph. Then there exists a critical value*

$$\hat{\lambda} \geq \frac{6^6}{5^5} \frac{4\pi N}{Vol(V)}$$

such that if $\lambda > \hat{\lambda}$, then (2) has at least two solutions, if $\lambda = \hat{\lambda}$, then (2) has a solution, and if $\lambda < \hat{\lambda}$, then (2) admits no solution.

3. The proof of Theorem 2.5

Since $\int_V -\frac{4\pi N}{\mathrm{Vol}(V)} + 4\pi \sum_{j=1}^N \delta_{p_j} d\mu = 0$, we can choose a solution u_0 of the equation

$$\Delta u_0 = -\frac{4\pi N}{\mathrm{Vol(V)}} + 4\pi \sum_{j=1}^N \delta_{p_j}. \tag{6}$$

Letting $u = u_0 + v$, the equation (2) can be reduced to the following question

$$\Delta v = \lambda e^{u_0+v}\left(e^{u_0+v}-1\right)^5 + \frac{4\pi N}{\mathrm{Vol}(V)}. \tag{7}$$

Set $F(y) := (e^y - 1)^5 e^y$ on $\mathbb{R}$, it is clear that F has a unique minimal value $-\frac{5^5}{6^6}$. Thus, it follows from (7) that

$$0 = \int_V \Delta v d\mu \geq \lambda \int_V -\frac{5^5}{6^6} d\mu + 4\pi N = -\frac{5^5}{6^6}\lambda \mathrm{Vol}(V) + 4\pi N.$$

This implies that

$$\lambda \geq \frac{6^6}{5^5}\frac{4\pi N}{|V|},$$

which is a necessary condition for the existence of solutions to (2).

To solve (7), for a constant $K \geq \lambda$, we define a sequence $\{w_n\}$ by a monotone iterative scheme:

$$\begin{aligned}(\Delta - K)W_n = {} & \lambda e^{u_0+W_{n-1}}\left(e^{u_0+W_{n-1}} - 1\right)^5 \\ & - KW_{n-1} + \frac{4\pi N}{\mathrm{Vol}(V)},\ n = 1, 2, \ldots, \\ W_0 = {} & -u_0.\end{aligned} \tag{8}$$

Next, we establish a solution to (7) by a supersolution and subsolution method.

Definition. A function u on V is called a subsolution of (7) if

$$\Delta u \geq \lambda e^{u_0+u}\left(e^{u_0+u} - 1\right)^5 + \frac{4\pi N}{\mathrm{Vol}(V)}.$$

Lemma 3.1. *Let W_n be a sequence defined by scheme (8) with $K \geq \lambda$. Then*

$$W_- \leq \cdots \leq W_n \leq \cdots \leq W_2 \leq W_1 \leq W_0 \tag{9}$$

for any subsolution W_- of (7).

Proof. By (6) and (8), we have

$$(\Delta - K)(W_1 - W_0) = 4\pi \sum_{s=1}^{N} \delta_{p_s} > 0,\ x \in V. \tag{10}$$

By Lemma 2.3, we deduce that

$$(W_1 - W_0)(x) \leq 0$$

for all $x \in V$. Suppose that

$$W_0 \geq W_1 \geq \cdots \geq W_k.$$

By (8), we conclude that

$$\begin{aligned}&(\Delta - K)\left(W_{k+1} - W_k\right) \\ = {} & \lambda \mathrm{e}^{u_0+W_k}\left(\mathrm{e}^{u_0+W_k} - 1\right)^5 - \lambda \mathrm{e}^{u_0+W_{k-1}}\left(\mathrm{e}^{u_0+W_{k-1}} - 1\right)^5 - K\left(W_k - W_{k-1}\right) \\ = {} & \left[\lambda \mathrm{e}^{u_0+\xi}\left(\mathrm{e}^{u_0+\xi} - 1\right)^4\left(6\mathrm{e}^{u_0+\xi} - 1\right) - K\right]\left(W_k - W_{k-1}\right) \\ \geq {} & (\lambda - K)\left(W_k - W_{k-1}\right) \geq 0,\end{aligned}$$

where $W_k \le \xi \le W_{k-1}$. By Lemma 2.3, we see that $W_{k+1} \le W_k$ on V.

Set $(W_- - W_0)(x_0) = \max_{x\in V}(W_- - W_0)(x)$, we claim that

$$(W_- - W_0)(x_0) \le 0.$$

Otherwise, $(W_- - W_0)(x_0) > 0$. It follows that

$$\Delta(W_- - W_0)(x_0) \ge \lambda e^{u_0+W_-(x_0)}(e^{u_0+W_-(x_0)} - 1)^5 + \frac{4\pi N}{\mathrm{Vol}(V)} + \Delta u_0 > 0.$$

By the definition of Laplace operator, we obtain

$$\Delta(W_- - W_0)(x_0) \le 0.$$

This is a contradiction. Thus we have

$$(W_- - W_0)(x_0) \le 0,$$

which implies that

$$W_- - W_0 \le 0$$

on V. Assume that $W_- - W_k \le 0$ for some integer $k \ge 0$. Thanks to W_- is a subsolution of (7) and $K \ge \lambda$, we deduce that

$$\begin{aligned}
&(\Delta - K)\left(W_- - W_{k+1}\right)\\
&\ge \lambda\left[\mathrm{e}^{u_0+W_-}\left(\mathrm{e}^{u_0+W_-} - 1\right)^5 - \mathrm{e}^{u_0+W_k}\left(\mathrm{e}^{u_0+W_k} - 1\right)^5\right] - K\left(W_- - W_k\right)\\
&\ge \left[\lambda\mathrm{e}^{u_0+\eta}\left(\mathrm{e}^{u_0+\eta} - 1\right)^4\left(6\mathrm{e}^{u_0+\eta} - 1\right) - K\right]\left(W_- - W_k\right)\\
&\ge (\lambda - K)\left(W_- - W_k\right) \ge 0,
\end{aligned}$$

where $W_- \le \eta \le W_k$. By Lemma 2.3, we have $W_- \le W_{k+1}$ on V.

We now complete the proof. □

Lemma 3.2. *If $\lambda > 0$ is sufficiently large, then there exists a solution of* (7) *on V.*

Proof. Assume that u_0 is a solution of (7). Select a constant Q_0 such that $u_0 < Q_0$. Let $\hat{W}_- \equiv -Q_0$ on V, then for sufficiently large λ, we have

$$0 = \Delta\hat{W}_- > \lambda e^{u_0+\hat{W}_-}(e^{u_0+\hat{W}_-} - 1)^5 + \frac{4\pi N}{\mathrm{Vol}(V)}.$$

Thus $\hat{W}_-$ is a subsolution of (7). By Lemma 3.1, we get a sequence $\{W_n\}$ satisfying

$$\hat{W}_- \le \cdots \le W_n \le \cdots \le W_2 \le W_1 \le -u_0.$$

Thus we can define

$$w(x) := \lim_{n\to+\infty} W_n(x).$$

Letting $n \to +\infty$ in (8), then we know that w is a solution of (7). □

In order to prove Lemma 3.4, we need the following proposition.

Lemma 3.3. *If u is a solution of equation* (2) *in V, then $u < 0$ on V.*

Proof. Suppose that

$$u(x_0) = \max_{x \in V} u(x),$$

we claim that $u(x_0) < 0$. Suppose by way of contradiction that $u(x_0) \geq 0$, then

$$e^{u(x_0)} - 1 \geq 0,$$

which implies that $\Delta u(x_0) > 0$. By (4), we have

$$0 \geq \Delta u(x_0).$$

This is a contradiction. □

Lemma 3.4. *There exists $\hat{\lambda} \geq \frac{4\pi N}{Vol(V)} \frac{6^6}{5^5}$ such that when $\lambda \geq \hat{\lambda}$, (2) admits a solution, and when $\lambda < \hat{\lambda}$, (2) admits no solutions.*

Proof. Denote $A := \{\lambda > 0 \,|\, \lambda \text{ is such that (2) admits a solution}\}$. We claim that A is an interval. If $\lambda_0 \in A$, let v' be the solution of (2) with $\lambda = \lambda_0$. By Lemma 3.3, we have

$$v' < 0 \text{ on } V.$$

Set $u' = v' - u_0$, then

$$u' + u_0 < 0 \text{ on } V.$$

It is easy to check that u' is a subsolution of (7) for $\lambda \geq \lambda_0$. It follows from Lemma 3.1 that $\lambda \in A$ for $\lambda \geq \lambda_0$. Thus A is an interval. Clearly,

$$\lambda_i := \inf A$$

is well defined. We can choose a sequence $\{\lambda_n\} \subset A$ such that $\lambda_n \to \lambda_i$. On account of $\lambda_n \geq \frac{4\pi N}{\mathrm{Vol(V)}} \frac{6^6}{5^5}$, we obtain

$$\lambda_i \geq \frac{4\pi N}{\mathrm{Vol}(V)} \frac{6^6}{5^5}.$$

For any $\lambda > \hat{\lambda}$, we can find a solution of (7) denoted by $u_\lambda(x)$. We next prove that if $\lambda_1 > \lambda_2 > \hat{\lambda}$, then $u_{\lambda_1} \geq u_{\lambda_2}$ on V. By Lemma 3.3, $u_0 + u_{\lambda_2} < 0$. Thus we deduce that

$$\begin{aligned} \Delta u_{\lambda_2} &= \lambda_2 \mathrm{e}^{u_0 + u_{\lambda_2}} \left(\mathrm{e}^{u_0 + u_{\lambda_2}} - 1\right)^5 + \frac{4\pi N}{\mathrm{Vol}(V)} \\ &> \lambda_1 \mathrm{e}^{u_0 + u_{\lambda_2}} \left(\mathrm{e}^{u_0 + u_{\lambda_2}} - 1\right)^5 + \frac{4\pi N}{\mathrm{Vol}(V)}, \end{aligned}$$

and hence that u_{λ_2} is a subsolution of (7) with $\lambda = \lambda_1$. By a similar argument as Lemma 3.1, we can show that

$$u_{\lambda_2} \leq u_{\lambda_1} \quad \text{on} \quad V. \tag{11}$$

Thus we can define $U(x) := \lim_{\lambda \to \hat{\lambda}^+} u_\lambda(x) \in [-\infty, -u_0)$.

We claim that

$$U(x) > -\infty \ \forall x \in V. \tag{12}$$

Suppose that $\lim_{\lambda\to\hat{\lambda}^+} u_\lambda(x) = -\infty$ for all $x \in V$. Integrating

$$\Delta u_\lambda = \lambda e^{u_0+u_\lambda}\left(e^{u_0+u_\lambda}-1\right)^5 + \frac{4\pi N}{\mathrm{Vol}(V)} \tag{13}$$

on V, we obtain

$$\begin{aligned} 0 = \int_V \Delta u_\lambda d\mu &= \int_V \lambda e^{u_0+u_\lambda}\left(e^{u_0+u_\lambda}-1\right)^5 d\mu + 4\pi N \\ &= \lambda \sum_{x\in V} \mu(x) e^{u_0+u_\lambda}\left(e^{u_0+u_\lambda}-1\right)^5 d\mu + 4\pi N. \end{aligned} \tag{14}$$

Letting $\lambda \to \hat{\lambda}^+$ in (14), we see that $0 = 4\pi N$, which is a contradiction. Define

$$V_1 := \left\{ x \in V | \lim_{\lambda\to\hat{\lambda}^+} u_\lambda = -\infty \right\} \tag{15}$$

and

$$V_2 := \left\{ x \in V \mid \lim_{\lambda\to\hat{\lambda}^+} u_\lambda \text{ exists in } (-\infty, -u_0) \right\}. \tag{16}$$

If $V_1 = \emptyset$, then (12) holds. Next, we suppose that $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. Choose $y_2 \in V_2$, then

$$\begin{aligned} &\Delta u_\lambda(y_2) \\ &= \frac{1}{\mu(y_2)} \sum_{x\sim y_2} w_{xy_2}\left(u_\lambda(x) - u_\lambda(y_2)\right) \\ &= \frac{1}{\mu(y_2)} \sum_{y\sim y_2, y\in V_1} w_{yx_2}\left(u_\lambda(y) - u_\lambda(y_2)\right) + \frac{1}{\mu(y_2)} \sum_{y\sim y_2, y\in V_2} w_{yx_2}\left(u_\lambda(y) - u_\lambda(y_2)\right) \\ &=: I_1(\lambda) + I_2(\lambda). \end{aligned}$$

Clearly, $\lim_{\lambda\to\hat{\lambda}^+} I_1(\lambda) = -\infty$ and $\lim_{\lambda\to\hat{\lambda}^+} I_2(\lambda)$ exists in $\mathbb{R}$. By (13), we have

$$\Delta u_\lambda(y_2) \geq \lambda\left(-\frac{5^5}{6^6}\right) + \frac{4\pi N}{\mathrm{Vol}(V)}.$$

This is impossible. Thus we have $V_1 = \emptyset$.

Letting $\lambda \to \hat{\lambda}^+$ in (13), we can deduce that U is a solution of (7) with $\lambda = \hat{\lambda}$. □

Define

$$I_\lambda(v) := \int_V \frac{1}{2}|\nabla v|^2 + \frac{\lambda}{6}\left(e^{u_0+v}-1\right)^6 + \frac{4\pi N}{\mathrm{Vol}(V)} v d\mu. \tag{17}$$

We may give a sufficient condition under which the problem (7) admits a solution and $I_\lambda(v)$ has a minimizer.

Lemma 3.5. *If $\lambda > \hat{\lambda}$, then there exists a solution v_λ of* (7) *and it is a local minimum of the functional $I_\lambda(v)$ defined by* (17).

Proof. Thanks to $u_0 + U(x) < 0$, we conclude that $U(x)$ is a subsolution of (7) for $\lambda > \hat{\lambda}$. We define

$$A = \left\{ v \in W^{1,2} \mid v \geq U \text{ in } V \right\}.$$

Clearly, I_λ is bounded from below on V. Thus we can define

$$\eta_0 := \inf_{v \in A} I_\lambda(v).$$

Set $\{v_n\}$ be a minimizing sequence and $v_n = v_n' + c_n$, $n = 1, 2, \dots$, where

$$c_n = \frac{\int_V v_n d\mu}{\mathrm{Vol(V)}}.$$

It is easy to see that

$$c_n \geq \frac{\int_V U d\mu}{\mathrm{Vol(V)}}.$$

Thus, we get

$$I_\lambda(v_n) \geq \int_V \frac{1}{2} |\nabla v_n|^2 d\mu + \frac{4\pi N}{\mathrm{Vol(V)}} \int_V U d\mu, \tag{18}$$

which implies that $\{\|\nabla v_n\|_2\}_{n=1}^{\infty}$ is bounded. By (17), we have

$$I_\lambda(v_n) \geq \int_V \frac{4\pi N}{\mathrm{Vol(V)}} c_n d\mu,$$

which implies that

$$c_n \leq \frac{I_\lambda(v_n)}{4\pi N}.$$

Thus, $\{v_n\}$ is bounded in $W^{1,2}(V)$. Since V is a finite graph, by passing to a subsequence, there exists $v_\lambda(x)$ such that

$$v_n(x) \to v_\lambda(x)$$

as $n \to +\infty$, for every $x \in V$. Thus

$$I_\lambda(v_\lambda) = \eta_0.$$

By a similar argument as the appendix of [20], we can deduce that v_λ is a solution of (7).

We next show that $v_\lambda > U$ in V. It is easy to check that

$$\Delta(U - v_\lambda) > \hat{\lambda}(U - v_\lambda). \tag{19}$$

By Lemma 2.3, we have $W := U - v_\lambda \leq 0$ on V. We claim that

$$W(x_0) := \max_V W < 0.$$

Otherwise, $W(x_0) = 0$. Clearly, $\Delta W(x_0) \leq 0$. By (19), we obtain $W(x_0) < 0$, which is a contradiction. Thus we have

$$U < v_\lambda \text{ on } V.$$

We claim that v_λ is a local minimum of $I_\lambda(v)$ in A. For any integer $n \geq 1$, we see that

$$\inf \left\{ I_\lambda(w) \mid w \in W^{1,2}(V), \|w - v_\lambda\|_{W^{1,2}(V)} \leq \frac{1}{n} \right\} = \varepsilon_n < I_\lambda(v_\lambda). \tag{20}$$

By a similar argument as above, we can deduce that there exists a sequence

$$\{v_n\}_{n=1}^{\infty} \subset W^{1,2}(V)$$

satisfying

$$\|v_n - v_\lambda\|_{W^{1,2}(V)} \leq \frac{1}{n}$$

and

$$I_\lambda(v_n) = \epsilon_n.$$

Thus we conclude that, by passing to a subsequence, $v_n \to v_\lambda$ in V as $n \to +\infty$, and hence that $v_n > U$ for sufficiently large n. Therefore, we obtain

$$I_\lambda(v_n) \geq I_\lambda(v_\lambda).$$

This is a contradiction. □

We now prove that $I_\lambda(v)$ satisfies the Palais-Smale condition.

Lemma 3.6. *Every sequence $\{v_n\} \subset W^{1,2}(V)$ satisfying*

$$I_\lambda(v_n) \to \alpha \text{ and } \|I_\lambda'(v_n)\| \to 0 \text{ as } n \to +\infty \tag{21}$$

has a convergent subsequence.

Proof. From (21), we deduce that

$$\begin{aligned} &\frac{1}{2} \|\nabla v_n\|_2^2 + \frac{\lambda}{6} \int_V \left(e^{u_0+v_n} - 1\right)^6 \, \mathrm{d}x + \frac{4\pi N}{|V|} \int_V v_n \, \mathrm{d}x \\ &= \alpha + o(1) \text{ as } n \to +\infty \end{aligned} \tag{22}$$

and that there exists $\{\epsilon_n\}_{n=1}^{\infty}$ satisfying $\epsilon_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned} &\left| \int_V \Gamma(v_n, \varphi) \mathrm{d}x + \lambda \int_V e^{u_0+v_n} \left(e^{u_0+v_n} - 1\right)^5 \varphi \mathrm{d}x + \frac{4\pi N}{|V|} \int_V \varphi \mathrm{d}x \right| \\ &\leq \varepsilon_n \|\varphi\|_{W^{1,2}(V)} \end{aligned} \tag{23}$$

as $n \to +\infty$, for any $\varphi \in H^1(V)$. By taking $\varphi \equiv 1$ in (23), we have

$$\lambda \int_V e^{u_0+v_n} \left(e^{u_0+v_n} - 1\right)^5 \, \mathrm{d}x + 4\pi N \leq \varepsilon_n |V|^{1/2},$$

from which we deduce that

$$\begin{aligned} \frac{\varepsilon_n |V|^{1/2}}{\lambda} &\geq \frac{4\pi N}{\lambda} + \int_V e^{u_0+v_n} \left(e^{u_0+v_n} - 1\right)^5 \, \mathrm{d}\mu \\ &= \frac{4\pi N}{\lambda} + \int_V \left(e^{u_0+v_n} - 1\right)^6 \, \mathrm{d}\mu + \int_V \left(e^{u_0+v_n} - 1\right)^5 \, \mathrm{d}\mu \\ &\geq \frac{4\pi N}{\lambda} - \frac{1}{6}|V| + \frac{1}{6} \int_V \left(e^{u_0+v_n} - 1\right)^6 \, \mathrm{d}\mu. \end{aligned}$$

This implies that there exists a constant $C = C(\epsilon_n, \lambda, |V|) > 0$ such that

$$\int_V \left(e^{u_0+v_n} - 1\right)^6 \, \mathrm{d}\mu \leq C. \tag{24}$$

Hence, we can find $C_2 > 0$ such that

$$\begin{aligned}\int_V e^{6(u_0+v_n)}\mathrm{d}\mu &= \int_V \left[\left(e^{u_0+v_n} - 1\right) + 1\right]^6 \, \mathrm{d}\mu \\ &\leq 2^6 \left[\int_V \left(e^{u_0+v_n} - 1\right)^6 \, \mathrm{d}\mu + |V|\right] \\ &\leq C_2.\end{aligned} \tag{25}$$

Then by Hölder inequality, there exists $C_3 > 0$ such that

$$\int_V e^{2(u_0+v_n)}\mathrm{d}\mu \leq \left(\int_V e^{6(u_0+v_n)}\mathrm{d}\mu\right)^{\frac{1}{3}} |V|^{\frac{2}{3}} \leq C_3. \tag{26}$$

Similarly, $\int_V e^{4(u_0+v_n)}\mathrm{d}\mu \leq C_4$ for a suitable constant $C_4 > 0$. Decompose $v_n = v_n' + c_n$, where $\int_V v_n' d\mu = 0$ and $c_n \in \mathbb{R}$ for $n = 1, 2, \ldots$. Substituting it in (22), we conclude that

$$\frac{1}{2} \|\nabla v_n'\|_2^2 + \frac{\lambda}{6} \int_V \left(e^{u_0+v_n'+c_n} - 1\right)^6 \, \mathrm{d}\mu + 4\pi N c_n \to \alpha \tag{27}$$

as $n \to +\infty$, and hence that c_n is bounded from above. By (22), we see that there exists an integer N such that

$$\alpha - 1 < I_\lambda(v_n) < \alpha + 1$$

for $n \geq N$. This implies that

$$\alpha - 1 < \frac{1}{2} \|\nabla v_n'\|_2^2 + \frac{\lambda}{6} \int_V \left(e^{u_0+v_n'+c_n} - 1\right)^6 \, \mathrm{d}\mu + 4\pi N c_n < \alpha + 1. \tag{28}$$

From (24) and (28), we conclude that

$$\alpha - 1 + \frac{4\lambda\pi N}{5} - \left(\frac{\lambda}{6} + \frac{\varepsilon_n}{5}\right)|V| < \frac{1}{2} \|\nabla v_n'\|_2^2 + 4\pi N c_n < \alpha + 1. \tag{29}$$

Next we show that c_n is bounded from below. Taking v_n' in (23), by Lemma 2.2, we can find a constant C_5 such that

$$\begin{aligned}\|\nabla v_n'\|_2^2 + \lambda \int_V e^{u_0+v_n} \left(e^{u_0+v_n} - 1\right)^5 v_n' \mathrm{d}\mu &\leq \varepsilon_n \|v_n'\|_{W^{1,2}(V)} \\ &\leq C_5 \varepsilon_n \|\nabla v_n'\|_2 .\end{aligned} \tag{30}$$

This implies that

$$\begin{aligned}&\|\nabla v_n'\|_2^2+\lambda\int_V e^{6(u_0+c_n)}\left(e^{6v_n'}-1\right)v_n'\mathrm{d}\mu\\ &\le \lambda\int_V e^{6(u_0+c_n)}v_n'\mathrm{d}\mu + C_5\varepsilon_n\|\nabla v_n'\|_2\\ &\quad + C_6\int_V e^{u_0+v_n}\left(e^{4(u_0+v_n)}+e^{3(u_0+v_n)}+e^{2(u_0+v_n)}+e^{u_0+v_n}+1\right)|v_n'|\,\mathrm{d}\mu.\end{aligned}\tag{31}$$

By Lemma 2.2, Lemma 2.4 and Hölder inequality, we deduce that

$$\int_V e^{6(u_0+c_n)}v_n'\mathrm{d}\mu \le C_7\|v_n'\|_2 \le C_8\|\nabla v_n'\|_2\,,\tag{32}$$

and

$$\begin{aligned}\int_V e^{5(u_0+v_n)}|v_n'|\,\mathrm{d}\mu &\le \left(\int_V e^{6(u_0+v_n)}\mathrm{d}\mu\right)^{\frac{5}{6}}\left(\int_V |v_n'|^6\ \mathrm{d}\mu\right)^{\frac{1}{6}}\\ &\le C_9\|v_n'\|_6\\ &\le C_{10}\|\nabla v_n'\|_2\end{aligned}\tag{33}$$

for suitable positive constants $C_7,\dots,C_{10}$. Similarly, we can get all the other terms on the right hand side of (31) can be bounded by $\hat{C}||\nabla v_n'||_2$, where $\hat{C}>0$ is a constant. Thus, there exists a constant $C_{11}>0$ such that

$$\|\nabla v_n'\|_2^2+\lambda\int_V e^{6(u_0+c_n)}\left(e^{6v_n'}-1\right)v_n'\mathrm{d}\mu \le C_{11}\|\nabla v_n'\|_2\,.\tag{34}$$

Clearly,

$$\int_V e^{6(u_0+c_n)}\left(e^{6v_n'}-1\right)v_n'\mathrm{d}\mu \ge 0.\tag{35}$$

Hence by (34), we have

$$||\nabla v_n'||_2 \le C_{12}$$

for a suitable constant $C_{12}>0$. Therefore, by (29), we deduce that c_n is bounded from below.

Thus $\{v_n\}$ is bounded in $H^1(V)$. Thus, there exists $v\in H^1(V)$ such that, by passing to a subsequence, $v_n(x)\to v(x)$ for all $x\in V$. □

Next, we find the second solution of (7). From now on, we suppose that v_λ is the local minimum as defined by Lemma 3.5 (if not, we could have already found our second solution). Thus there exists $\rho_0>0$ such that

$$I_\lambda(v_\lambda)\le I_\lambda(v)$$

for all $v:||v-v_\lambda||_{H^1(V)}\le\rho_0$. For $c>0$, we have

$$\begin{aligned}I_\lambda\left(v_\lambda-c\right)-I_\lambda\left(v_\lambda\right) &= \frac{\lambda}{6}\int_V\left[\left(e^{u_0+v_\lambda-c}-1\right)^6-\left(e^{u_0+v_\lambda-1}\right)^6\right]d\mu-4\pi Nc\\ &< \frac{\lambda}{6}|V|C_{13}-4\pi Nc\to-\infty \text{ as } c\to+\infty.\end{aligned}\tag{36}$$

There are two possibilities: (I) v_λ is not a strict local minimum for I_λ, (II) v_λ is a strict local minimum for I_λ. If case (I) happens, then we deduce that

$$\inf_{\|v-v_\lambda\|_{H^1(V)}=\rho} I_\lambda = I_\lambda(v_\lambda) =: \alpha_\lambda$$

for all $0 < \rho < \rho_0$. It follows that there exists a local minimum $v_\rho \in H^1(V)$ such that

$$||v_\rho - v_\lambda|| = \rho,$$

and

$$I_\lambda(v_\rho) = \alpha_\lambda$$

for all $\rho \in (0, \rho_0)$. Therefore, in this situation, we get a one-parameter family of solutions of (7). If case (II) happens, we can find $\rho_1 \in (0, \rho_0)$ such that

$$\inf_{\|v-v_\lambda\|_{H^1(V)}=\rho_1} I_\lambda(v) > I_\lambda(v_\lambda) = \alpha_\lambda. \tag{37}$$

By (36), we deduce that

$$I_\lambda(u_\lambda - c_0) \le I_\lambda(u_\lambda) - 1 < I_\lambda(v_\lambda)$$

for some $c_0 > |V|^{-\frac{1}{2}}\rho_1$.

We now define

$$\mathcal{P} = \left\{\gamma : [0,1] \to H^1(V) \,|\, \gamma \text{ is continuous and satisfies } \gamma(0) = v_\lambda, \gamma(1) = v_\lambda - c_0\right\}$$

and

$$\alpha = \inf_{\gamma \in \mathcal{P}} \sup_{t \in [0,1]} I_\lambda(\gamma(t)).$$

From (37), we conclude that

$$\alpha > I_\lambda(v_\lambda) \ge \max\{I_\lambda(\gamma(0)), I_\lambda(\gamma(1))\} \;\; \forall \gamma \in \mathcal{P}.$$

Thus, by Lemma 3.6 , I_λ satisfies the hypothesis of the mountain-pass theorem. Thus α is a critical point of I_λ. By virtue of $\alpha > I_\lambda(v_\lambda)$, we get a second solution of (7).

We now complete the proof of Theorem 2.5.

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