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AVERAGE ENTROPY AND ASYMPTOTICS

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Abstract. We determine the $N \to \infty$ asymptotics of the expected value of entanglement entropy for pure states in $H_{1,N} \otimes H_{2,N}$, where $H_{1,N}$ and $H_{2,N}$ are the spaces of holomorphic sections of the N-th tensor powers of hermitian ample line bundles on compact complex manifolds.

1. Introduction

There are various mathematical notions of entropy. It quantifies chaos, mixing, disorder or complexity. The Shannon entropy used in information theory has a probabilistic interpretation or it can be viewed as a way to quantify information. The Shannon entropy of a probability measure P_X on a finite set $X = \{x_1, \dots, x_r\}$ with masses $\{p_1, \dots, p_r\}$ ($p_j = P_X(x_j)$, $1 \leq j \leq r$) equals $-\sum_{j=1}^{r} p_j \ln p_j$, with the convention $p_j \ln p_j = 0$ when $p_j = 0$. Let H_1 and H_2 be finite-dimensional Hilbert spaces. The partial trace tr_2 is the linear map $\mathrm{tr}_2 : \mathrm{End}(H_1 \otimes H_2) \to \mathrm{End}(H_1)$ given by $\mathrm{tr}_2(A \otimes B) = \mathrm{tr}(B)A$ and extended by linearity. The entanglement entropy $E(v)$ of a vector $v \in H_1 \otimes H_2$ is $E(v) = -\sum_{j=1}^{m} \lambda_j \ln \lambda_j$, where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of $\mathrm{tr}_2(P_v)$, the linear map P_v is the orthogonal projection from $H_1 \otimes H_2$ onto the one-dimensional linear subspace spanned by v, and as before we use the convention $0 \ln 0 = 0$. Note: $P_v = vv^*$. The vector v is decomposable if and only if $E(v) = 0$. Calculations of entropy on the Hilbert spaces of geometric quantization or Toeplitz quantization lead to interesting insights [1, 2]. In [9], the main result is the $k \to \infty$ asymptotics of the Shannon entropies of μ_z^k, where $k \in \mathbb{N}$, $z \in M$, M is a toric Kähler manifold with an ample toric hermitian line bundle, and μ_z^k are the Bergman measures that were introduced by Zelditch in [8] to define generalized Bernstein polynomials and were subsequently used in [7, 10]. In a series of papers on random sections of line bundles, starting with [6], Shiffmann

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and Zelditch worked with the probability space

$$\prod_{k=1}^{\infty} SH^0(M, L^k),$$

where $L \to M$ is an ample holomorphic hermitian line bundle on a compact complex manifold M and $SH^0(M, L^k)$ is the unit sphere in the finite-dimensional Hilbert space $H^0(M, L^k)$. In this paper, we consider instead the probability space

$$\Omega = \prod_{k=1}^{\infty} S(H^0(M, L^k) \otimes H^0(M, L^k)) \tag{1}$$

and a sequence of random variables ($\mathbb{R}$-valued functions on Ω) $E_k \circ p_k$, where p_k is the projection to the k-th component in the product $\prod_{k=1}^{\infty}$ above in (1), and E_k is the entanglement entropy. We find the $k \to \infty$ asymptotics of the sequence of expected values of these random variables. In fact, we prove a more general result.

Theorem 1.1. *Let $L_1 \to M_1$ and $L_2 \to M_2$ be positive holomorphic hermitian line bundles on compact complex manifolds M_1 and M_2 of complex dimensions d_1 and d_2, respectively. Assume without loss of generality $d_1 \le d_2$. Let $d\mu_N$, for each $N \in \mathbb{N}$, be the measure on the unit sphere $S_N = S(H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N))$ induced by the hermitian metrics. There are the following $N \to \infty$ asymptotics for the average entanglement entropy*

$$\langle E_N \rangle = \frac{\int_{S_N} E_N(v) d\mu_N(v)}{\int_{S_N} d\mu_N(v)}.$$

Let

$$\beta_j = \int_{M_j} \frac{c_1(L_j)^{d_j}}{d_j!},$$

$$\gamma_j = \int_{M_j} \Big(c_1(L_j) + \frac{1}{2} c_1(TM_j)\Big) \frac{c_1(L_j)^{d_j-1}}{(d_j-1)!}$$

for $j \in \{1, 2\}$. As $N \to \infty$,

$$\langle E_N \rangle \sim \begin{cases} \ln \beta_1 + d_1 \ln N - \frac{\beta_1}{2\beta_2} + \Big(\frac{\gamma_1}{\beta_1} - \frac{\beta_1}{2\beta_2}(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2})\Big)\frac{1}{N} + O(\frac{1}{N^2}), & \text{if } d_1 = d_2; \\ \ln \beta_1 + d_1 \ln N + \Big(\frac{\gamma_1}{\beta_1} - \frac{\beta_1}{2\beta_2}\Big)\frac{1}{N} + O(\frac{1}{N^2}), & \text{if } d_1 = d_2 - 1; \\ \ln \beta_1 + d_1 \ln N + \frac{\gamma_1}{\beta_1}\frac{1}{N} + O(\frac{1}{N^2}), & \text{if } d_1 - d_2 \le -2. \end{cases}$$

Remark 1.2. We observe that a statement analogous to Theorem 1.1 holds for semipositive line bundles L_j on Moishezon manifolds M_j, $j \in \{1, 2\}$. The following is true. Let $L_1 \to M_1$ and $L_2 \to M_2$ be holomorphic hermitian line bundles on compact connected complex manifolds M_1 and M_2 of complex dimensions d_1 and d_2, respectively. Assume without loss of generality $d_1 \le d_2$. Assume M_1 and M_2 are Moishezon and L_1, L_2 are semipositive. Let $d\mu_N$, for each $N \in \mathbb{N}$, be the measure on the unit sphere $S_N = S(H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N))$ induced by the hermitian metrics. There are the following $N \to \infty$ asymptotics for the average entanglement entropy on the Hilbert spaces $H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N)$: as $N \to \infty$

$$\langle E_N \rangle \sim \begin{cases} \ln \beta_1 + d_1 \ln N - \frac{\beta_1}{2\beta_2} + o(1), & \text{if } d_1 = d_2; \\ \ln \beta_1 + d_1 \ln N + o(1), & \text{if } d_1 < d_2, \end{cases}$$

where, as before, $\beta_j = \int_{M_j} \frac{c_1(L_j)^{d_j}}{d_j!}$ for $j = 1, 2$. The proof is similar to the proof of Theorem 1.1 in Section 2.2 below, with (17), (18) replaced by (from Th. 1.7.1 [3])

$$m = m(N) = \dim H^0(M_1, L_1^N) = N^{d_1} \int_{M_1} \frac{c_1(L_1)^{d_1}}{d_1!} + o(N^{d_1}),$$

$$n = n(N) = \dim H^0(M_2, L_2^N) = N^{d_2} \int_{M_2} \frac{c_1(L_2)^{d_2}}{d_2!} + o(N^{d_2}).$$

2. Asymptotics

In this section, we establish the background and write the proofs needed for Theorem 1.1. An expression for the average entanglement entropy for the tensor product of two finite-dimensional Hilbert spaces is the statement of Page conjecture [4]. There were several derivations of this formula in physics literature, including [5]. They assume the equality (2) (see below) as a starting point. Our Theorem 2.2 below is a proof of (2). Then, our proof of Theorem 2.3 follows the idea of Sen [5]. We rely on the semiclassical methods, together with the statement of Theorem 2.3, to prove our main result, Theorem 1.1 above.

2.1. Preliminaries

Let H_1 and H_2 be two complex Hilbert spaces of complex-dimension m and n, respectively, with $m \leq n$. We note that $H_1 \otimes H_2 \cong \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{R}^{2mn}$. Let $\mathbb{S}^{2mn-1} = \{v \in H_1 \otimes H_2 : \|v\| = 1\} \subset H_1 \otimes H_2$ be the unit sphere in $H_1 \otimes H_2$ and $d\mu$ be the standard spherical measure on $\mathbb{S}^{2mn-1}$, normalized so that $\int_{\mathbb{S}^{2mn-1}} d\mu = 1$. We fix an orthonormal basis $\{e_1, e_2, \dots, e_m\}$ for H_1 and $\{f_1, f_2, \dots, f_n\}$ for H_2. Then we have a canonical isomorphism $A : H_1 \otimes H_2 \to M_{n\times m}(\mathbb{C})$ given by $v = \sum_{j=1}^n \sum_{k=1}^m a_{jk}(v) e_k \otimes f_j \mapsto A(v) = (a_{jk}(v))_{n\times m}$ (this is also a diffeomorphism). Using $M_{n\times m}(\mathbb{C}) \cong \mathbb{R}^{2mn}$, we identify an $n \times m$ matrix with a point in $\mathbb{R}^{2mn}$. For $v \in H_1 \otimes H_2$, we have the Singular Value Decomposition (SVD) of $A(v) = U(v) \left(\begin{smallmatrix} \Sigma(v) \\ 0 \end{smallmatrix}\right) V(v)^*$, where $U(v) \in U(n)$ and $V(v) \in U(m)$ and $\Sigma(v) = \operatorname{diag}\{\sigma_1(v), \sigma_2(v), \dots, \sigma_m(v)\}$ is a diagonal matrix with $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$, called the singular values of the matrix $A(v)$. Note that in the SVD of a matrix, the diagonal matrix $\Sigma(v)$ is unique (up to permutation), however the choice of $U(v)$ and $V(v)$ is not unique.

The Schmidt coefficients of v are the same as the singular values of $A(v)$. If $\sigma_1(v), \sigma_2(v), \ldots, \sigma_m(v)$ are Schmidt coefficients of $v \in \mathbb{B}^{2mn}$ (where $\mathbb{B}^{2mn}$ is the closed unit ball in $\mathbb{R}^{2mn}$), then $\sum_{j=1}^m \sigma_j(v)^2 = \|v\|^2 \leq 1$, i.e., for every $v \in \mathbb{B}^{2mn}$, there exists a triple $(U(v), \Sigma(v), V(v))$ such that $A(v) = U(v) \left(\begin{smallmatrix} \Sigma(v) \\ 0 \end{smallmatrix}\right) V(v)^*$, where $U(v) \in U(n)$ and $V(v) \in U(m)$ are unitary matrices and $\Sigma(v) = \operatorname{diag}\{\sigma_1(v), \sigma_2(v), \ldots, \sigma_m(v)\}$ a diagonal matrix with $0 \leq \sigma_1(v) \leq \sigma_2(v) \leq \cdots \leq \sigma_m(v)$ and $\sum_{j=1}^m \sigma_j(v)^2 \leq 1$. Conversely, if $A(v) \in M_{n\times m}$ has a SVD $A(v) = U(v) \left(\begin{smallmatrix} \Sigma(v) \\ 0 \end{smallmatrix}\right) V(v)^*$ such that $\sum_{j=1}^m \sigma_j^2 \leq 1$, then $v \in \mathbb{B}^{2mn}$.

Lemma 2.1. *Singular value decomposition $A = U \left(\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix}\right) V^*$ of a matrix $A \in M_{n\times m}(\mathbb{C})$ is not unique. However, if we put the condition that diagonal entries of Σ are in the ascending order, the diagonal entries of the unitary matrix V are non-negative and consider U as an element of the complex Stiefel manifold $V_m^n(\mathbb{C})(= U(n)/U(n-m))$, then this "Modified SVD" is unique.*

Proof. Let $A = U \left(\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix}\right) V^*$ be an SVD of A. Let $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$, where U_1 is the matrix whose columns are the first m columns of U and U_2 is the matrix whose columns are the last $n-m$ columns of U. Then we see that $A = U_1 \Sigma V^*$. To see that SVD is not unique, let $\tau = \operatorname{diag}\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_m}\}$ for $-\pi < \theta_1, \theta_2, \ldots, \theta_m \leq \pi$. Let $V' = V\tau$ and $U' = U \left(\begin{smallmatrix} \tau & 0 \\ 0 & I \end{smallmatrix}\right)$ be unitary matrices respective size and $U'\Sigma V'^* = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \left(\begin{smallmatrix} \tau & 0 \\ 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix}\right) (V\tau)^* = U_1 \tau \Sigma \tau^* V^* = U_1 \Sigma \tau \tau^* V^* = U_1 \Sigma V^* = A$ is also an SVD of A.

Let $V = [v_{jk}] \in U(m)$. For each $j \in \{1, 2, \ldots, m\}$, we can write $v_{jj} = |v_{jj}| e^{i\theta_j}$, where $\theta_j \in (-\pi, \pi]$. Then

$$V = \begin{pmatrix} |v_{11}| & v_{12}e^{-i\theta_2} & \ldots & v_{1m}e^{-i\theta_m} \\ v_{21}e^{-i\theta_1} & |v_{22}| & \ldots & v_{2m}e^{-i\theta_m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}e^{-i\theta_1} & v_{m2}e^{-i\theta_2} & \ldots & |v_{mm}| \end{pmatrix} \begin{pmatrix} e^{i\theta_1} & 0 & \ldots & 0 \\ 0 & e^{i\theta_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{i\theta_m} \end{pmatrix}$$
$$= \tilde{V}\tau \text{ (say)}.$$

Therefore, we get that any unitary matrix V can be factored into a product of unitary matrix $\tilde{V}$ with diagonal entries being non-negative and a unitary diagonal matrix τ. The factorization is unique and this gives us the one-to-one correspondence $U(m) \cong (U(m)/(U(1))^m) \times (U(1))^m$.

If $A = U \left(\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix}\right) V^*$ is an SVD of A and $V = \tilde{V}\tau$ is the factorization of V as above and $\tilde{U} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \left(\begin{smallmatrix} \tau^* & 0 \\ 0 & I \end{smallmatrix}\right) = \begin{pmatrix} U_1\tau^* & U_2 \end{pmatrix} = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}$ (say), then let us call $A = \tilde{U} \left(\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix}\right) \tilde{V}^*$ the "modified SVD" of A. Because of the uniqueness of the factors $\tilde{V}$ and τ, the modified SVD gives the unique triple $(\tilde{U}, \Sigma, \tilde{V})$ with $\tilde{U} \in V_m^n(\mathbb{C})$, $\Sigma \in \mathbb{R}^m_{\geq}$ (with ascending diagonal) and $\tilde{V} \in U(m)/(U(1))^m$.

This modified SVD gives one-to-one correspondence (up to permutation of the diagonal entries of the diagonal matrix) between $M_{n\times m}(\mathbb{C})$ and $V_m^n(\mathbb{C}) \times \mathbb{R}^m_{\geq} \times (U(m)/(U(1))^m)$ via the map $V_m^n(\mathbb{C}) \times \mathbb{R}^m_{\geq} \times (U(m)/(U(1))^m) \to M_{n\times m}(\mathbb{C})$ by $(\tilde{U}, \Sigma, \tilde{V}) = \tilde{U} \left(\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix}\right) \tilde{V}^*$. □

Theorem 2.2. *Let f be a continuous function on $\mathbb{S}^{2mn-1}$ that depends only on the squares of the Schmidt coefficients. For $v \in \mathbb{B}^{2mn}$, let $\sigma_1(v), \dots, \sigma_m(v)$ be the Schmidt coefficients of v and $p_1(v), \dots, p_m(v)$ be the eigenvalues of $\operatorname{tr}_2(vv^*)$ (i.e., $p_j(v) = \sigma^2_{\tau(j)}(v)$ for some $\tau \in S_m$). Then*

$$\int_{\mathbb{S}^{2mn-1}} f(u)d\mu(u)$$

$$= \frac{\int_{T_m} f\left(\frac{p_1(v)}{\sum_{k=1}^m p_k(v)}, \dots, \frac{p_m(v)}{\sum_{k=1}^m p_k(v)}\right) \prod_{1\le j<k\le m}^{m} (p_k(v)-p_j(v))^2 \prod_{j=1}^{m} p_j(v)^{n-m} dp_j(v)}{\int_{T_m} \prod_{1\le j<k\le m}^{m} (p_k(v)-p_j(v))^2 \prod_{j=1}^{m} p_j(v)^{n-m} dp_j(v)}, \tag{2}$$

where $T_m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1, \dots, x_m \ge 0 \text{ and } x_1 + \dots + x_m \le 1\}$.

Proof. We will use the Lebesgue measure ν of the ambient Euclidean space $\mathbb{R}^{2mn}$ and then for $X \subset \mathbb{S}^{2mn-1}$, $\mu(X) = \nu(\{tx \,|\, x \in X \text{ and } t \in [0,1]\})$. Then for an integrable function defined on $\mathbb{S}^{2mn-1}$, we have

$$\int_{\mathbb{S}^{2mn-1}} f(u)d\mu(u) = \int_{\mathbb{B}^{2mn}} f\left(\frac{v}{\|v\|}\right) d\nu(v).$$

For $v \in \mathbb{B}^{2mn}$, let $A(v)$ be the matrix of v as above. We will change variables using "modified SVD" $A(v) = \tilde{U}(v)\left(\begin{smallmatrix}\Sigma(v)\\ 0\end{smallmatrix}\right)\tilde{V}(v)^*$, where $\tilde{U} = \begin{pmatrix}\tilde{U}_1 & \tilde{U}_2\end{pmatrix}$ with $\tilde{U}_1$ and $\tilde{U}_2$ the matrices with columns the first m columns of $\tilde{U}$ and last $n-m$ columns of $\tilde{U}$, respectively. Let $\tilde{u}_j(v)$ be the j-th column of $\tilde{U}(v)$ and $\tilde{v}_k(v)$ be the k-th column of $\tilde{V}(v)$. Now,

$$A(v) = \tilde{U}(v)\begin{pmatrix}\Sigma(v)\\ 0\end{pmatrix}\tilde{V}^*(v)$$

hence

$$dA(v) = d\tilde{U}(v)\begin{pmatrix}\Sigma(v)\\ 0\end{pmatrix}\tilde{V}^*(v) + \tilde{U}(v)\begin{pmatrix}d\Sigma(v)\\ 0\end{pmatrix}\tilde{V}^*(v) + \tilde{U}(v)\begin{pmatrix}\Sigma(v)\\ 0\end{pmatrix}d\tilde{V}^*(v).$$

The volume measure at $v_0 \in \mathbb{B}^{2mn}$ can be written as $\tilde{U}^*(v_0)dA(v)\tilde{V}(v_0)|_{v=v_0}$ (as $\tilde{U}^*(v_0)$ and $\tilde{V}(v_0)$ being unitary matrices do not affect the volume). Let

$$dB(v_0) = \tilde{U}^*(v_0)dA(v)\tilde{V}(v_0)|_{v=v_0}.$$

Then

$$\begin{aligned} dB(v_0) &= \tilde{U}^*(v_0)d\tilde{U}(v_0)\begin{pmatrix}\Sigma(v_0)\\ 0\end{pmatrix} + \begin{pmatrix}d\Sigma(v_0)\\ 0\end{pmatrix} + \begin{pmatrix}\Sigma(v_0)\\ 0\end{pmatrix}d\tilde{V}^*(v_0)\tilde{V}(v_0) \\ &= \begin{pmatrix}\tilde{U}_1^*(v_0)d\tilde{U}_1(v_0)\Sigma(v_0) + d\Sigma(v_0) + \Sigma(v_0)d\tilde{V}^*(v_0)\tilde{V}(v_0)\\ \tilde{U}_2^*(v_0)d\tilde{U}_1(v_0)\Sigma(v_0)\end{pmatrix}. \end{aligned}$$

As v_0 was an arbitrary point in $\mathbb{B}^{2mn}$, so we have

$$dB(v) = \begin{pmatrix}\tilde{U}_1^*(v)d\tilde{U}_1(v)\Sigma(v) + d\Sigma(v) + \Sigma(v)d\tilde{V}^*(v)\tilde{V}(v)\\ \tilde{U}_2^*(v)d\tilde{U}_1(v)\Sigma(v)\end{pmatrix}.$$

For notational convenience, we drop the "(v)" to write

$$dB = \begin{pmatrix} \tilde{U}_1^* d\tilde{U}_1 \Sigma + d\Sigma + \Sigma d\tilde{V}^* \tilde{V} \\ \tilde{U}_2^* d\tilde{U}_1 \Sigma \end{pmatrix}.$$

Since $\tilde{U}_1^* \tilde{U}_1 = I$, it follows that $d\tilde{U}_1^* \tilde{U}_1 + \tilde{U}_1^* d\tilde{U}_1 = 0$, then

$$\tilde{U}_1^* d\tilde{U}_1 = -d\tilde{U}_1^* \tilde{U}_1 = -(\tilde{U}_1^* d\tilde{U}_1)^*,$$

so $\tilde{U}_1^* d\tilde{U}_1$ is an anti-hermitian matrix. Similarly, $\tilde{V}^* d\tilde{V}$ is also anti-hermitian. Denote $E = \tilde{U}_1^* d\tilde{U}_1$ and $F = \tilde{V}^* d\tilde{V}$, then for $j, k \in \{1, 2, \ldots, m\}$ we have $E_{jk} = \tilde{u}_j^* d\tilde{u}_k$ and $F_{jk} = \tilde{v}_j^* d\tilde{v}_k$. Now,

$$dB = \begin{pmatrix} E\Sigma + d\Sigma - \Sigma F \\ \tilde{U}_2^* d\tilde{U}_1 \Sigma \end{pmatrix}.$$

As E and F are anti-hermitian, the diagonal elements of E and F are imaginary. It follows that the real parts of diagonal elements of dB are exactly the diagonal elements of $d\Sigma$ and imaginary parts of the diagonal elements of dB come from the matrix $E\Sigma - \Sigma F$, i.e.,

$$\Re(dB_{jj}) = d\Sigma_{jj} = d\sigma_j,$$

$$\Im(dB_{jj}) = \sigma_j(\Im(\tilde{u}_j^* d\tilde{u}_j) - \Im(\tilde{v}_j^* d\tilde{v}_j))$$

for $j \in \{1, 2, \ldots, m\}$. Let $j \in \{1, 2, \ldots, m\}$ with $j > k$, then

$$dB_{jk} = \sigma_k E_{jk} - \sigma_j F_{jk}, \quad dB_{kj} = \sigma_j E_{kj} - \sigma_k F_{kj} = -\sigma_j \overline{E_{jk}} + \sigma_k \overline{F_{jk}}.$$

We get

$$\begin{aligned} \Re(dB_{jk}) &= \sigma_k \Re(E_{jk}) - \sigma_j \Re(F_{jk}), \\ \Im(dB_{jk}) &= \sigma_k \Im(E_{jk}) - \sigma_j \Im(F_{jk}), \\ \Re(dB_{kj}) &= -\sigma_j \Re(E_{jk)} + \sigma_k \Re(F_{jk}), \\ \Im(dB_{kj}) &= \sigma_j \Im(E_{jk}) - \sigma_k \Im F_{jk}. \end{aligned}$$

Computing the wedge of real parts,

$$\Re(dB_{jk}) \Re(dB_{kj}) = (\sigma_k^2 - \sigma_j^2) \Re(E_{jk}) \Re(F_{jk})$$

and the imaginary parts,

$$\Im(dB_{jk}) \Im(dB_{kj}) = (\sigma_k^2 - \sigma_j^2) \Im(F_{jk}) \Im(E_{jk})$$

we get that

$$\begin{aligned} &\Re(dB_{jk}) \Im(dB_{jk}) \Re(dB_{kj}) \Im(dB_{kj}) \\ = &-(\sigma_k^2 - \sigma_j^2)^2 \Re(\tilde{u}_j^* d\tilde{u}_k) \Im(\tilde{u}_j^* d\tilde{u}_k) \Re(\tilde{v}_j^* d\tilde{v}_k) \Im(\tilde{v}_j^* d\tilde{v}_k). \end{aligned}$$

Combining all these and ignoring the sign, we get the form

$$\bigwedge_{j=1}^{m} \Re(dB_{jj}) \bigwedge_{j,k=1, j \neq k}^{m} \Re(dB_{jk}) \Im(dB_{jk})$$

$$= \bigwedge_{j=1}^{m} d\sigma_j \prod_{1\le j<k\le m}^{m} (\sigma_k^2 - \sigma_j^2)^2 \eta \bigwedge_{1\le j<k\le m}^{m} \Re(\tilde{u}_j^* d\tilde{u}_k)\Im(\tilde{u}_j^* d\tilde{u}_k),$$

where

$$\eta = \bigwedge_{1\le j<k\le m}^{m} \Re(\tilde{v}_j^* d\tilde{v}_k)\Im(\tilde{v}_j^* d\tilde{v}_k).$$

We note that η is a volume form of $U(m)/(U(1)^m)$. For the imaginary part of the diagonal entries, for $j,k \in \{1,\dots,m\}$ with $j \neq k$, let's look at the following

$$\begin{aligned}
&\Im dB_{jj} \wedge \Im dB_{kk} \\
&= \sigma_j(\Im(\tilde{u}_j^* d\tilde{u}_j) - \Im(\tilde{v}_j^* d\tilde{v}_j)) \wedge \sigma_k(\Im(\tilde{u}_k^* d\tilde{u}_k) - \Im(\tilde{v}_k^* d\tilde{v}_k)) \\
&= \sigma_j\sigma_k \Im(\tilde{u}_j^* d\tilde{u}_j)\Im(\tilde{u}_k^* d\tilde{u}_k) + \kappa,
\end{aligned}$$

where $\kappa = \sigma_j\sigma_k[\Im(\tilde{v}_j^* d\tilde{v}_j)\Im(\tilde{v}_k^* d\tilde{v}_k) - \Im(\tilde{u}_j^* d\tilde{u}_j)\Im(\tilde{v}_k^* d\tilde{v}_k) - \Im(\tilde{u}_k^* d\tilde{u}_k)\Im(\tilde{v}_j^* d\tilde{v}_j)]$. We note that η is the volume form of the compact group $U(m)/U(1)^m$, so $\kappa \wedge \eta = 0$ (as η already contains all the independent variables coming from V). Therefore,

$$\begin{aligned}
&\bigwedge_{j,k=1}^{m} \Re(dB_{jk})\Im(dB_{jk}) \\
&= \prod_{1\le j<k\le m}^{m} (\sigma_k^2 - \sigma_j^2)^2 \eta \bigwedge_{1\le j<k\le m}^{m} \Re(\tilde{u}_j^* d\tilde{u}_k)\Im(\tilde{u}_j^* d\tilde{u}_k) \bigwedge_{j=1}^{m} \Im(\tilde{u}_j^* d\tilde{u}_j) \bigwedge_{j=1}^{m} d\sigma_j.
\end{aligned}$$

Now, for $k \in \{1,2,\dots,m\}$ and $j \in \{m+1, m+2, \dots, n\}$, we have

$$dB_{jk} = (\tilde{U}_2 d\tilde{U}_1 \Sigma)_{j-m,k} = \sigma_k \tilde{u}_j^* d\tilde{u}_k.$$

Therefore,

$$\begin{aligned}
\bigwedge_{j=m+1}^{n} \bigwedge_{k=1}^{m} \Re(dB_{jk})\Im(dB_{jk}) &= \bigwedge_{j=m+1}^{n} \bigwedge_{k=1}^{m} \sigma_k^2 \Re(\tilde{u}_j^* d\tilde{u}_k)\Im(\tilde{u}_j^* d\tilde{u}_k) \\
&= \prod_{j=1}^{m} \sigma_k^{2(n-m)} \bigwedge_{j=m+1}^{n} \bigwedge_{k=1}^{m} \Re(\tilde{u}_j^* d\tilde{u}_k)\Im(\tilde{u}_j^* d\tilde{u}_k).
\end{aligned}$$

Gathering all these and ignoring sign, we get that

$$\begin{aligned}
\rho &= \bigwedge_{j=1}^{n} \bigwedge_{k=1}^{m} \Re(dB_{jk})\Im(dB_{jk}) \\
&= \prod_{j=1}^{m} \sigma_j \prod_{1\le j<k\le m}^{m} (\sigma_k^2 - \sigma_j^2)^2 \prod_{j=1}^{m} \sigma_k^{2(n-m)} \eta\omega \bigwedge_{j=1}^{m} d\sigma_j,
\end{aligned}$$

where

$$\omega = \bigwedge_{1\le j<k\le m}^{m} \Re(\tilde{u}_j^* d\tilde{u}_k)\Im(\tilde{u}_j^* d\tilde{u}_k) \bigwedge_{j=1}^{m} \Im(\tilde{u}_j^* d\tilde{u}_j) \bigwedge_{j=m+1}^{n} \bigwedge_{k=1}^{m} \Re(\tilde{u}_j^* d\tilde{u}_k)\Im(\tilde{u}_j^* d\tilde{u}_k)$$

is a form independent of σ_j's. We have

$$\begin{aligned}
&\int_{\mathbb{S}^{2mn-1}} d\mu \\
&= \int_{\mathbb{B}^{2mn}} d\nu \\
&= c(m,n)\int_{(U(m)/(U(1))^m)\times\mathbb{B}^m_{\geq 0}\times V^n_m(\mathbb{C})} \rho \\
&= c(m,n)\int_{U(m)/(U(1))^m} \eta \int_{V^n_m(\mathbb{C})} \omega \int_{\mathbb{B}^m_{\geq 0}} \prod_{j=1}^m \sigma_j \prod_{1\leq j<k\leq m}^m (\sigma_k^2-\sigma_j^2)^2 \prod_{j=1}^m \sigma_j^{2(n-m)} d\sigma_j,
\end{aligned}$$

where $c(m,n)$ is a constant due to the requirement that $\int_{\mathbb{S}^{2mn-1}} d\mu = 1$ and $\mathbb{B}^m_{\geq 0} = \{(x_1,\ldots,x_m)\in\mathbb{R}^m : x_1,\ldots,x_m \geq 0 \text{ and } x_1^2+\cdots+x_m^2 \leq 1\}$. For the integral $\int_{\mathbb{B}^m_{\geq 0}} \prod_{j=1}^m \sigma_j \prod_{1\leq j<k\leq m}^m (\sigma_k^2-\sigma_j^2)^2 \prod_{j=1}^m \sigma_j^{2(n-m)} d\sigma_j$, we change variables to $p_j=\sigma_j^2$ to get

$$\begin{aligned}
&\int_{\mathbb{B}^m_{\geq 0}} \prod_{j=1}^m \sigma_j \prod_{1\leq j<k\leq m}^m (\sigma_k^2-\sigma_j^2)^2 \prod_{j=1}^m \sigma_j^{2(n-m)} d\sigma_j \\
&= \frac{1}{2^m}\int_{T_m} \prod_{1\leq j<k\leq m}^m (p_k-p_j)^2 \prod_{j=1}^m p_j^{n-m} dp_j.
\end{aligned}$$

We note that for $v\in\mathbb{B}^{2mn}$, if $p_1,\ldots,p_m$ are the eigenvalues of $\mathrm{tr}_2(vv^*)$, then $\frac{p_1}{\sum_{j=1}^m p_j},\ldots,\frac{p_m}{\sum_{j=1}^m p_j}$ are the eigenvalues of $\mathrm{tr}_2(\frac{vv^*}{\|v\|^2})$. Therefore, for a function f on $\mathbb{S}^{2mn-1}$ that depends only on the squares of the Schmidt coefficients $\sigma_j(u)$ of $u\in\mathbb{S}^{2mn-1}$ (in other words, depends only on $p_j(u)$'s), we get

$$\begin{aligned}
&\int_{\mathbb{S}^{2mn-1}} f(u)d\mu(u) \\
&= \int_{\mathbb{B}^{2mn}} f\left(\frac{v}{\|v\|}\right) d\nu(v) \\
&= c(m,n)\int_{(U(m)/(U(1))^m)\times\mathbb{B}^m_{\geq 0}\times V^n_m(\mathbb{C})} f\left(\frac{v}{\|v\|}\right)\rho \\
&= \frac{c(m,n)\int_{(U(m)/(U(1))^m)\times\mathbb{B}^m_{\geq 0}\times V^n_m(\mathbb{C})} f\left(\frac{v}{\|v\|}\right)\rho}{c(m,n)\int_{(U(m)/(U(1))^m)\times\mathbb{B}^m_{\geq 0}\times V^n_m(\mathbb{C})} \rho} \\
&\quad \left(\text{as the denominator is } \int_S d\mu = 1\right) \\
&= \frac{\int_{T_m} f\left(\frac{p_1(v)}{\sum_{k=1}^m p_k(v)},\ldots,\frac{p_m(v)}{\sum_{k=1}^m p_k(v)}\right) \prod_{1\leq j<k\leq m}^m (p_k(v)-p_j(v))^2 \prod_{j=1}^m p_j(v)^{n-m} dp_j(v)}{\int_{T_m} \prod_{1\leq j<k\leq m}^m (p_k(v)-p_j(v))^2 \prod_{j=1}^m p_j(v)^{n-m} dp_j(v)}.
\end{aligned}$$

□

Theorem 2.3. *The expected value of the entropy of entanglement of all the pure states in $H_1 \otimes H_2$ is*

$$\langle E_{(m,n)} \rangle = \sum_{k=n+1}^{mn} \frac{1}{k} + \frac{m-1}{2n}.$$

Proof. We base our proof on the general idea of the proof in [5]. The expected value of the entropy of entanglement is given by

$$\begin{aligned} &\int_{\mathbb{S}^{2mn-1}} E_{(m,n)}(u)d\mu(u) \\ &= \int_{\mathbb{B}^{2mn}} E_{(m,n)}\left(\frac{v}{\|v\|}\right) d\nu(v) \\ &= \frac{\int_{T_m} \left(-\sum_{j=1}^{m} \frac{p_j}{\sum_{k=1}^m p_k} \ln \frac{p_j}{\sum_{k=1}^m p_k}\right) \prod_{1\le j<k\le m}^{m} (p_k-p_j)^2 \prod_{j=1}^{m} p_j^{n-m} dp_j}{\int_{T_m} \prod_{1\le j<k\le m}^{m} (p_k-p_j)^2 \prod_{j=1}^{m} p_j^{n-m} dp_j}. \end{aligned} \tag{3}$$

We change variables to $(q_1, \dots, q_{m-1}, r)$ such that $q_1 + \dots + q_{m-1} + q_m = 1$, and $q_k = rp_k$ for $k = 1, 2, \dots, m$ (i.e., $r = \frac{1}{\sum_{j=1}^m p_j}$), so $r \in [1, \infty)$. Then for $j \in \{1, \dots, m\}$ we have,

$$\begin{aligned} \frac{\partial q_k}{\partial p_j} &= \delta_{jk} r \text{ for } k \in \{1, \dots, m-1\}, \\ \frac{\partial r}{\partial p_j} &= \frac{\partial}{\partial p_j}\left(\frac{1}{\sum_k p_k}\right) = \frac{-1}{(\sum_k p_k)^2} = -r^2. \end{aligned}$$

We denote $F_m = \{(x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_1, x_2, \dots, x_m \ge 0 \text{ and } x_1 + x_2 + \dots + x_m = 1\}$. So, the integral becomes

$$\begin{aligned} &\frac{\int_1^\infty \int_{F_m} \left(-\sum_{j=1}^{m} q_j \ln q_j\right) \prod_{1\le j<k\le m}^{m} (\frac{q_k}{r} - \frac{q_j}{r})^2 \prod_{j=1}^{m} \left(\frac{q_j}{r}\right)^{n-m} \frac{1}{r^{m+1}} dr \prod_{j=1}^{m-1} dq_j}{\int_1^\infty \int_{F_m} \prod_{1\le j<k\le m}^{m} (\frac{q_k}{r} - \frac{q_j}{r})^2 \prod_{j=1}^{m} \left(\frac{q_j}{r}\right)^{n-m} \frac{1}{r^{m+1}} dr \prod_{j=1}^{m-1} dq_j} \\ &= \frac{\int_1^\infty \frac{dr}{r^{mn+1}} \int_{F_m} \left(-\sum_{j=1}^{m} q_j \ln q_j\right) \prod_{1\le j<k\le m}^{m} (q_k-q_j)^2 \prod_{j=1}^{m} q_j^{n-m} \prod_{j=1}^{m-1} dq_j}{\int_1^\infty \frac{dr}{r^{mn+1}} \int_{F_m} \prod_{1\le j<k\le m}^{m} (q_k-q_j)^2 \prod_{j=1}^{m} q_j^{n-m} \prod_{j=1}^{m-1} dq_j} \end{aligned}$$

$$= \frac{\int_{F_m} \left(-\sum_{j=1}^{m} q_j \ln q_j\right) \prod_{1\le j<k\le m}^{m} (q_k-q_j)^2 \prod_{j=1}^{m} q_j^{n-m} \prod_{j=1}^{m-1} dq_j}{\int_{F_m} \prod_{1\le j<k\le m}^{m} (q_k-q_j)^2 \prod_{j=1}^{m} q_j^{n-m} \prod_{j=1}^{m-1} dq_j}. \tag{4}$$

Let $x_k = tq_k$ for $k = 1, 2, \dots, m$ with $t \in [0, \infty)$, so that $x_1 + x_2 + \dots + x_m = t$ and $x_k \in [0, \infty)$. Then, for $j \in \{1, 2, \dots, m\}$, we have

$$\frac{\partial q_k}{\partial x_j} = \delta_{jk} t \text{ for } k \in \{1, 2, \dots, m-1\},$$

$$\frac{\partial t}{\partial x_j} = 1.$$

Using the determinant of the Jacobian, we get

$$\begin{aligned} dq_1 \wedge \cdots \wedge dq_{m-1} \wedge dt &= \frac{1}{t^{m-1}} dx_1 \wedge \cdots \wedge dx_{m-1} \wedge dx_m \\ &= \frac{1}{t^{m-1}} dx_1 \wedge \cdots \wedge dx_{m-1} \wedge d(x_1 + \cdots + x_{m-1} + x_m) \\ &= \frac{1}{t^{m-1}} dx_1 \wedge \cdots \wedge dx_{m-1} \wedge dt. \end{aligned} \tag{5}$$

Now, we use the gamma function $\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy$ and the derivative of the gamma function $\Gamma'(z) = \int_0^\infty y^{z-1}e^{-y}\ln y dy$. In particular,

$$\begin{aligned} \int_0^\infty t^{mn}e^{-t}dt &= \Gamma(mn+1) = (mn)!, \\ \int_0^\infty t^{mn}e^{-t}\ln t dt &= \Gamma'(mn+1) = (mn)!\psi(mn+1), \end{aligned}$$

where $\psi(N+1) = -\gamma + \sum_{k=1}^N \frac{1}{k}$ and γ is the Euler constant. We have,

$$\begin{aligned} &\frac{\int_{F_m}\left(-\sum\limits_{j=1}^m q_j \ln q_j\right)\prod\limits_{1\le j<k\le m}(q_k-q_j)^2\prod\limits_{j=1}^m q_j^{n-m}\prod\limits_{j=1}^{m-1}dq_j}{\int_{F_m}\prod\limits_{1\le j<k\le m}(q_k-q_j)^2\prod\limits_{j=1}^m q_j^{n-m}\prod\limits_{j=1}^{m-1}dq_j} \\ &= \frac{(mn-1)!}{(mn)!}\frac{\int_0^\infty t^{mn}e^{-t}dt\int_{F_m}\left(-\sum\limits_{j=1}^m q_j\ln q_j\right)\prod\limits_{1\le j<k\le m}(q_k-q_j)^2\prod\limits_{j=1}^m q_j^{n-m}\prod\limits_{j=1}^{m-1}dq_j}{\int_0^\infty t^{mn-1}e^{-t}dt\int_{F_m}\prod\limits_{1\le j<k\le m}(q_k-q_j)^2\prod\limits_{j=1}^m q_j^{n-m}\prod\limits_{j=1}^{m-1}dq_j} \\ &= \frac{1}{mn}\frac{\int_{[0,\infty)^m}\left(-\sum\limits_{j=1}^m\left(\frac{x_j}{t}\right)\ln\left(\frac{x_j}{t}\right)\right)\prod\limits_{1\le j<k\le m}\left(\frac{x_k}{t}-\frac{x_j}{t}\right)^2\prod\limits_{j=1}^m\left(\frac{x_j}{t}\right)^{n-m}e^{-t}t^{mn}\frac{dt}{t^{m-1}}\prod\limits_{j=1}^{m-1}dx_j}{\int_{[0,\infty)^m}\prod\limits_{1\le j<k\le m}\left(\frac{x_k}{t}-\frac{x_j}{t}\right)^2\prod\limits_{j=1}^m\left(\frac{x_j}{t}\right)^{n-m}e^{-t}t^{mn-1}\frac{dt}{t^{m-1}}\prod\limits_{j=1}^{m-1}dx_j} \\ &= \frac{1}{mn}\frac{\int_{[0,\infty)^m}\left(-\sum\limits_{j=1}^m x_j\ln\left(\frac{x_j}{t}\right)\right)\prod\limits_{1\le j<k\le m}(x_k-x_j)^2\prod\limits_{j=1}^m x_j^{n-m}e^{-t}dt\prod\limits_{j=1}^{m-1}dx_j}{\int_{[0,\infty)^m}\prod\limits_{1\le j<k\le m}(x_k-x_j)^2\prod\limits_{j=1}^m x_j^{n-m}e^{-t}dt\prod\limits_{j=1}^{m-1}dx_j} \\ &= \frac{\int_{[0,\infty)^m}t\ln t\prod\limits_{1\le j<k\le m}(x_k-x_j)^2\prod\limits_{j=1}^m x_j^{n-m}e^{-t}dt\prod\limits_{j=1}^{m-1}dx_j}{mn\int_{[0,\infty)^m}\prod\limits_{1\le j<k\le m}(x_k-x_j)^2\prod\limits_{j=1}^m x_j^{n-m}e^{-t}dt\prod\limits_{j=1}^{m-1}dx_j} \\ &\quad - \frac{\int_{[0,\infty)^m}\left(\sum\limits_{j=1}^m x_j\ln x_j\right)\prod\limits_{1\le j<k\le m}(x_k-x_j)^2\prod\limits_{j=1}^m x_j^{n-m}e^{-t}dt\prod\limits_{j=1}^{m-1}dx_j}{mn\int_{[0,\infty)^m}\prod\limits_{1\le j<k\le m}(x_k-x_j)^2\prod\limits_{j=1}^m x_j^{n-m}e^{-t}dt\prod\limits_{j=1}^{m-1}dx_j}. \end{aligned} \tag{6}$$

Let the first and the second integral in equation (6) denoted by I_1 and I_2, respectively. We have

$$I_1 = \frac{\int_{[0,\infty)^m} t\ln t \prod_{1\le j<k\le m}^{m} (x_k - x_j)^2 \prod_{j=1}^{m} x_j^{n-m} e^{-t} dt \prod_{j=1}^{m-1} dx_j}{mn \int_{[0,\infty)^m} \prod_{1\le j<k\le m}^{m} (x_k - x_j)^2 \prod_{j=1}^{m} x_j^{n-m} e^{-t} dt \prod_{j=1}^{m-1} dx_j}$$

$$= \frac{\int_0^\infty \int_{F_m} t\ln t \prod_{1\le j<k\le m}^{m} (tq_k - tq_j)^2 \prod_{j=1}^{m} (tq_j)^{n-m} e^{-t} t^{m-1} dt \prod_{j=1}^{m-1} dq_j}{mn \int_0^\infty \int_{F_m} \prod_{1\le j<k\le m}^{m} (tq_k - tq_j)^2 \prod_{j=1}^{m} (tq_j)^{n-m} e^{-t} t^{m-1} dt \prod_{j=1}^{m-1} dq_j}$$

$$= \frac{\int_0^\infty t^{mn} \ln t e^{-t} dt \int_{F_m} \prod_{1\le j<k\le m}^{m} (q_k - q_j)^2 \prod_{j=1}^{m} q_j^{n-m} \prod_{j=1}^{m-1} dq_j}{mn \int_0^\infty t^{mn-1} e^{-t} dt \int_{F_m} \prod_{1\le j<k\le m}^{m} (q_k - q_j)^2 \prod_{j=1}^{m} q_j^{n-m} \prod_{j=1}^{m-1} dq_j}$$

$$= \frac{\Gamma'(mn+1)}{mn\Gamma(mn)}$$

$$= \psi(mn+1). \tag{7}$$

Using equation (5), the second integral in equation (6) becomes

$$I_2 = \frac{\int_{[0,\infty)^m} \left(\sum_{j=1}^{m} x_j \ln x_j\right) \prod_{1\le j<k\le m}^{m} (x_k - x_j)^2 \prod_{j=1}^{m} x_j^{n-m} e^{-t} dt \prod_{j=1}^{m-1} dx_j}{mn \int_{[0,\infty)^m} \prod_{1\le j<k\le m}^{m} (x_k - x_j)^2 \prod_{j=1}^{m} x_j^{n-m} e^{-t} dt \prod_{j=1}^{m-1} dx_j}$$

$$= \frac{\sum_{l=1}^{m} \int_{[0,\infty)^m} x_l \ln x_l \prod_{1\le j<k\le m}^{m} (x_k - x_j)^2 \prod_{j=1}^{m} x_j^{n-m} e^{-(x_1+\cdots+x_m)} \prod_{j=1}^{m} dx_j}{mn \int_{[0,\infty)^m} \prod_{1\le j<k\le m}^{m} (x_k - x_j)^2 \prod_{j=1}^{m} x_j^{n-m} e^{-(x_1+\cdots+x_m)} \prod_{j=1}^{m} dx_j}. \tag{8}$$

We observe that the van der Monde determinant

$$\Delta(x_1, \ldots, x_m) = \prod_{1\le j<k\le m}^{m} (x_k - x_j) = \det \begin{pmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_m \\ \vdots & \ddots & \vdots \\ x_1^{m-1} & \ldots & x_m^{m-1} \end{pmatrix}.$$

As determinant remains unchanged after applying elementary row operations, we see that

$$\prod_{1\le j<k\le m}^{m} (x_k - x_j) = \det \begin{pmatrix} f_0(x_1) & \ldots & f_0(x_m) \\ f_1(x_1) & \ldots & f_1(x_m) \\ \vdots & \ddots & \vdots \\ f_{m-1}(x_1) & \ldots & f_{m-1}(x_m) \end{pmatrix} = \det_{m\times m} (f_{j-1}(x_k))$$

for any set of polynomials $\{f_0, f_1, \ldots, f_{m-1}\}$ with $f_0 = 1$ and f_k's are monic with $\deg(f_k) = k$ for $k = 1, \ldots, m-1$. We choose these polynomials cleverly for the $\Delta(x_1, \ldots, x_m)^2$ appearing in the numerator and the denominator in equation (8). We use a class of orthogonal polynomials, generalized Laguerre polynomials, given by

$$L_k^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{k!}\frac{d^k}{dx^k}(e^{-x}x^{k+\alpha}),$$

where $\alpha \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$. These polynomials satisfy the orthogonality relation,

$$\int_0^\infty x^\alpha e^{-x} L_k^{(\alpha)}(x) L_j^{(\alpha)}(x)dx = \frac{\Gamma(k+\alpha+1)}{k!}\delta_{jk}. \tag{9}$$

Using these polynomials,

$$\Delta(x_1,\ldots,x_m) = \det_{m\times m}\left(L_{j-1}^{(\alpha)}(x_k)\right) = \sum_{\sigma\in S_m} \operatorname{sgn}(\sigma)\prod_{k=1}^m L_{\sigma(k-1)}^{(n-m)}(x_k).$$

The integral in equation (8) becomes

$$\begin{aligned}
I_2 &= \frac{\sum_{l=1}^m \int_{[0,\infty)^m} x_l \ln x_l \prod_{1\le j<k\le m}(x_k-x_j)^2 \prod_{j=1}^m x_j^{n-m} e^{-(x_1+\cdots+x_m)} \prod_{j=1}^m dx_j}{mn\int_{[0,\infty)^m} \prod_{1\le j<k\le m}(x_k-x_j)^2 \prod_{j=1}^m x_j^{n-m} e^{-(x_1+\cdots+x_m)} \prod_{j=1}^m dx_j} \\
&= \frac{\sum_{l=1}^m \int_{[0,\infty)^m} x_l \ln x_l \Delta(x_1,\ldots,x_m)\Delta(x_1,\ldots,x_m) \prod_{j=1}^m x_j^{n-m} e^{-x_j}\, dx_j}{mn\int_{[0,\infty)^m} \Delta(x_1,\ldots,x_m)\Delta(x_1,\ldots,x_m) \prod_{j=1}^m x_j^{n-m} e^{-x_j}\, dx_j} \\
&= \frac{\sum_{l=1}^m \int_{[0,\infty)^m} x_l \ln x_l \sum_{\sigma,\tau\in S_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{k=1}^m L_{\sigma(k-1)}^{(n-m)}(x_k) L_{\tau(k-1)}^{(n-m)}(x_k) \prod_{j=1}^m x_j^{n-m} e^{-x_j}\, dx_j}{mn\int_{[0,\infty)^m} \sum_{\sigma,\tau\in S_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{k=1}^m L_{\sigma(k-1)}^{(n-m)}(x_k) L_{\tau(k-1)}^{(n-m)}(x_k) \prod_{j=1}^m x_j^{n-m} e^{-x_j}\, dx_j} \\
&= \frac{\sum_{l=1}^m \sum_{\sigma,\tau\in S_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \int_{[0,\infty)^m} x_l \ln x_l \prod_{k=1}^m L_{\sigma(k-1)}^{(n-m)}(x_k) L_{\tau(k-1)}^{(n-m)}(x_k) x_k^{n-m} e^{-x_k}\, dx_k}{mn \sum_{\sigma,\tau\in S_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \int_{[0,\infty)^m} \prod_{k=1}^m L_{\sigma(k-1)}^{(n-m)}(x_k) L_{\tau(k-1)}^{(n-m)}(x_k) x_k^{n-m} e^{-x_k}\, dx_k} \\
&= \frac{\sum_{l=1}^m \sum_{\sigma,\tau\in S_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{k=1}^m \int_0^\infty (x_k \ln x_k)^{\delta_{lk}} L_{\sigma(k-1)}^{(n-m)}(x_k) L_{\tau(k-1)}^{(n-m)}(x_k) x_k^{n-m} e^{-x_k}\, dx_k}{mn \sum_{\sigma,\tau\in S_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{k=1}^m \int_0^\infty L_{\sigma(k-1)}^{(n-m)}(x_k) L_{\tau(k-1)}^{(n-m)}(x_k) x_k^{n-m} e^{-x_k}\, dx_k} \\
&= \frac{\sum_{l=1}^m \sum_{\sigma\in S_m} \prod_{k=1}^m \int_0^\infty (x_k \ln x_k)^{\delta_{lk}} [L_{\sigma(k-1)}^{(n-m)}(x_k)]^2 x_k^{n-m} e^{-x_k}\, dx_k}{mn \sum_{\sigma\in S_m} \prod_{k=1}^m \int_0^\infty [L_{\sigma(k-1)}^{(n-m)}(x_k)]^2 x_k^{n-m} e^{-x_k}\, dx_k} \\
&\quad \text{(using the orthogonal property)} \\
&= \frac{\sum_{l=1}^m \sum_{\sigma\in S_m} \int_0^\infty x_l^{n-m+1} \ln x_l [L_{\sigma(l-1)}^{(n-m)}(x_l)]^2 e^{-x_l}\, dx_l \prod_{k=1,k\neq l}^m \int_0^\infty [L_{\sigma(k-1)}^{(n-m)}(x_k)]^2 x_k^{n-m} e^{-x_k}\, dx_k}{mn \sum_{\sigma\in S_m} \prod_{k=1}^m \int_0^\infty [L_{\sigma(k-1)}^{(n-m)}(x_k)]^2 x_k^{n-m} e^{-x_k}\, dx_k} \\
&= \frac{\sum_{l=1}^m \sum_{\sigma\in S_m} \int_0^\infty x^{n-m+1} \ln x [L_{\sigma(l-1)}^{(n-m)}(x)]^2 e^{-x}\, dx \prod_{k=1,k\neq l}^m \int_0^\infty [L_{\sigma(k-1)}^{(n-m)}(x)]^2 x^{n-m} e^{-x}\, dx}{mn \sum_{\sigma\in S_m} \prod_{k=1}^m \int_0^\infty [L_{\sigma(k-1)}^{(n-m)}(x)]^2 x^{n-m} e^{-x}\, dx}
\end{aligned}$$

$$= \frac{\sum_{l=1}^{m}(m-1)!\sum_{k=1}^{m}\int_0^\infty x^{n-m+1}\ln x[L_{l-1}^{(n-m)}(x)]^2e^{-x}dx\prod_{k=1,k\neq l}^{m}\int_0^\infty [L_{k-1}^{(n-m)}(x)]^2x^{n-m}e^{-x}dx}{mn(m)!\prod_{j=1}^{m}\int_0^\infty [L_{j-1}^{(n-m)}(x)]^2x^{n-m}e^{-x}dx}$$

$$= \sum_{l=1}^{m}\sum_{k=1}^{m}\frac{\int_0^\infty x^{n-m+1}\ln x[L_{l-1}^{(n-m)}(x)]^2e^{-x}dx}{m^2n\int_0^\infty [L_{l-1}^{(n-m)}(x)]^2x^{n-m}e^{-x}dx}$$

$$= m\sum_{l=1}^{m}\frac{\int_0^\infty x^{n-m+1}\ln x[L_{l-1}^{(n-m)}(x)]^2e^{-x}dx}{m^2n\int_0^\infty [L_{l-1}^{(n-m)}(x)]^2x^{n-m}e^{-x}dx}$$

$$= \frac{1}{mn}\sum_{k=0}^{m-1}\frac{\int_0^\infty x^{n-m+1}\ln x[L_{k}^{(n-m)}(x)]^2e^{-x}dx}{\int_0^\infty [L_{k}^{(n-m)}(x)]^2x^{n-m}e^{-x}dx}. \tag{10}$$

Let $I_k^{(\alpha)} = \int_0^\infty x^{\alpha+1}\ln x[L_k^{(\alpha)}(x)]^2e^{-x}dx$ and $J_k(\alpha) = \int_0^\infty x^{\alpha+1}[L_k^{(\alpha)}(x)]^2e^{-x}dx$. By properties of Laguerre polynomials, we have

$$J_k(\alpha) = \frac{(2k+\alpha+1)\Gamma(k+\alpha+1)}{k!}. \tag{11}$$

Now,

$$\frac{d}{d\alpha}J_k(\alpha) = \int_0^\infty x^{\alpha+1}\ln x[L_k^{(\alpha)}(x)]^2e^{-x}dx + 2\int_0^\infty x^{\alpha+1}L_k^{(\alpha)}(x)\frac{dL_k^{(\alpha)}(x)}{d\alpha}e^{-x}dx$$

$$\implies I_k^{(n-m)} = \left[\frac{d}{d\alpha}J_k(\alpha) - 2\int_0^\infty x^{\alpha+1}L_k^{(\alpha)}(x)\frac{dL_k^{(\alpha)}(x)}{d\alpha}e^{-x}dx\right]_{\alpha=n-m}. \tag{12}$$

Using equation (11), we get

$$\begin{aligned}\frac{d}{d\alpha}J_k(\alpha) &= \frac{d}{d\alpha}\left(\frac{(2k+\alpha+1)\Gamma(k+\alpha+1)}{k!}\right)\\ &= \frac{\Gamma(k+\alpha+1)}{k!} + \frac{2k+\alpha+1}{k!}\frac{d\Gamma(k+\alpha+1)}{d\alpha}\\ &= \frac{\Gamma(k+\alpha+1)}{k!} + \frac{2k+\alpha+1}{k!}\Gamma(k+\alpha+1)\psi(k+\alpha+1)\\ &= \frac{\Gamma(k+\alpha+1)}{k!}[1+(2k+\alpha+1)\psi(k+\alpha+1)]. \end{aligned}\tag{13}$$

We use the property $L_k^{(\alpha)}(x) = L_k^{(\alpha+1)}(x) - L_{k-1}^{(\alpha+1)}(x)$ and $\frac{dL_k^{(\alpha)}(x)}{d\alpha} = \sum_{j=0}^{k-1}\frac{L_j^\alpha(x)}{k-j}$ to compute

$$\begin{aligned}&\int_0^\infty x^{\alpha+1}L_k^{(\alpha)}(x)\frac{dL_k^{(\alpha)}(x)}{d\alpha}e^{-x}dx\\ &= \int_0^\infty x^{\alpha+1}L_k^{(\alpha)}(x)\sum_{j=0}^{k-1}\frac{L_j^\alpha(x)}{k-j}e^{-x}dx\\ &= \sum_{j=0}^{k-1}\frac{1}{k-j}\int_0^\infty x^{\alpha+1}\left(L_k^{(\alpha+1)}(x) - L_{k-1}^{(\alpha+1)}(x)\right)\left(L_j^{(\alpha+1)}(x) - L_{j-1}^{(\alpha+1)}(x)\right)e^{-x}dx\end{aligned}$$

$$= -\int_0^\infty x^{\alpha+1}[L_{k-1}^{(\alpha+1)}(x)]^2 e^{-x}dx$$

$$(14) \quad = -\frac{\Gamma(k+\alpha+1)}{(k-1)!}.$$

Using (13) and (14) in (12), we get

$$I_k^{(n-m)} = \left[\tfrac{\Gamma(k+\alpha+1)}{k!}[1+(2k+\alpha+1)\psi(k+\alpha+1)] + 2\tfrac{\Gamma(k+\alpha+1)}{(k-1)!}\right]_{\alpha=n-m}$$

$$= \left[\tfrac{\Gamma(k+\alpha+1)}{k!}(1+2k+(2k+\alpha+1)\psi(k+\alpha+1))\right]_{\alpha=n-m}$$

$$(15) \quad = \tfrac{\Gamma(k+n-m+1)}{k!}[1+2k+(2k+n-m+1)\psi(k+n-m+1)].$$

Using equation (9) and (15) in equation (10), we get

$$I_2 = \frac{1}{mn}\sum_{k=0}^{m-1}[1+2k+(2k+n-m+1)\psi(k+n-m+1)]$$

$$= \frac{1}{mn}\sum_{0}^{m-1}(1+2k) + \frac{1}{mn}\sum_{k=0}^{m-1}\left[(2k+n-m+1)\left(-\gamma+\sum_{r=1}^{n-m+k}\frac{1}{r}\right)\right]$$

$$= \frac{m+m(m-1)}{mn} - \gamma\frac{1}{mn}\sum_{k=0}^{m-1}(2k+n-m+1) + \frac{1}{mn}\sum_{k=0}^{m-1}\sum_{r=1}^{n-m+k}\frac{2k+n-m+1}{r}$$

$$= \frac{m}{n} - \gamma + \left[mn + \frac{mn}{2} + \cdots + \frac{mn}{n-m} + \frac{mn-(n-m+1)}{n-m+1} + \cdots \right.$$
$$\left. + \frac{mn-(n-m+1)-(n-m+1+2)-\cdots-((n-m-1)+2(m-1)))}{n-m+m-1}\right] \times \frac{1}{mn}$$

$$= \frac{m}{n} - \gamma + \sum_{k=1}^{n-1}\frac{1}{k} - \frac{1}{mn}\left[\frac{n-m+1}{n-m+1} + \cdots + \frac{(m-1)(n-m)+(m-1)^2}{n-m+m-1}\right]$$

$$= \frac{m}{n} - \gamma + \sum_{k=1}^{n-1}\frac{1}{k} - \frac{1}{mn}[1+\cdots+(m-1)]$$

$$= -\gamma + \sum_{k=1}^{mn}\frac{1}{k} - \sum_{k=n+1}^{mn}\frac{1}{k} + \frac{m-1}{2n}$$

$$(16) \quad = \psi(mn+1) - \sum_{k=n+1}^{mn}\frac{1}{k} + \frac{m-1}{2n}.$$

Using equation (7), (16), (6) and (4), we get the expected value of Entropy of Entanglement over the pure states is

$$\sum_{k=n+1}^{mn}\frac{1}{k} - \frac{m-1}{2n}.$$

□

2.2. Proof of Theorem 1.1

For $j \in \{1,2\}$, let (L_j, h_j) be a holomorphic hermitian line bundle on a compact Kähler manifold (M_j, ω_j) of complex dimension $d_j \geq 1$ such that the curvature of the Chern connection on L_j is $-i\omega_j$, and $d_1 \leq d_2$. For $N \in \mathbb{N}$, the Hilbert spaces H_1 (of dimension $m = m(N)$) and H_2 (of dimension $n = n(N)$) will be $H^0(M_1, L_1^N)$ and $H^0(M_2, L_2^N)$. Let $N \to \infty$. We have [3, Sec. 4.1.1]:

$$m = m(N) = \beta_1 N^{d_1} + \gamma_1 N^{d_1-1} + O(N^{d_1-2}), \tag{17}$$

$$n = n(N) = \beta_2 N^{d_2} + \gamma_2 N^{d_2-1} + O(N^{d_2-2}), \tag{18}$$

where

$$\beta_j = \int_{M_j} \frac{c_1(L_j)^{d_j}}{d_j!},$$

$$\gamma_j = \int_{M_j} \Big(c_1(L_j) + \frac{1}{2}c_1(TM_j)\Big)\frac{c_1(L_j)^{d_j-1}}{(d_j-1)!}$$

for $j \in \{1,2\}$.

We notice that $m \leq n$ for large N. By Theorem 2.3, the average entanglement entropy $\langle E_N \rangle$ over all the pure states in $H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N)$ equals

$$\Big(\sum_{k=n+1}^{mn} \frac{1}{k}\Big) - \frac{m-1}{2n}. \tag{19}$$

To figure out the asymptotics of (19), we apply the Euler-Maclaurin formula to $f(x) = \frac{1}{x}$, to conclude that

$$\sum_{k=n+1}^{mn} \frac{1}{k} = \int_n^{mn} \frac{1}{x}dx + \frac{f(mn)-f(n)}{2} + \sum_{k=1}^{[\frac{p}{2}]} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(mn) - f^{(2k-1)}(n)) + R_p,$$

where B_{2k} are the Bernoulli numbers, in particular $B_2 = \frac{1}{6}$, and for the remainder we have the estimate

$$|R_p| \leq \frac{2\zeta(p)}{(2\pi)^p} \int_n^{mn} |f^{(p)}(x)|dx.$$

Therefore, $\langle E_N \rangle$ becomes

$$\ln m + \frac{1}{2mn} - \frac{m}{2n} + \sum_{k=1}^{[\frac{p}{2}]} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(mn) - f^{(2k-1)}(n)) + R_p. \tag{20}$$

In (20), let us set $p = 2$ in the part

$$\frac{1}{2mn} + \sum_{k=1}^{[\frac{p}{2}]} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(mn) - f^{(2k-1)}(n)) + R_p$$

and we can now conclude that this part is $O(N^{-2d_2})$, because

$$|R_2| \leq \frac{\zeta(2)}{2\pi^2} \int_n^{mn} |f''(x)| dx = \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{m^2 n^2} \right),$$

$$f'(mn) - f'(n) = -\frac{1}{m^2 n^2} + \frac{1}{n^2}$$

and by (17), (18).

It remains to consider the term $\ln m - \frac{m}{2n}$ in (20). By (17)

$$\ln m = \ln(\beta_1 N^{d_1} (1 + \frac{\gamma_1}{\beta_1} \frac{1}{N} + O(\frac{1}{N^2}))) \sim \ln \beta_1 + d_1 \ln N + \frac{\gamma_1}{\beta_1} \frac{1}{N} + O(\frac{1}{N^2}).$$

If $d_1 = d_2$, then by (17), (18), we get

$$\frac{m}{2n} = \frac{\beta_1 (1 + \frac{\gamma_1}{\beta_1} \frac{1}{N} + O(\frac{1}{N^2}))}{2\beta_2 (1 + \frac{\gamma_2}{\beta_2} \frac{1}{N} + O(\frac{1}{N^2}))} \sim \frac{\beta_1}{2\beta_2} \Big(1 + (\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}) \frac{1}{N} \Big) + O(\frac{1}{N^2}).$$

Similarly, if $d_1 = d_2 - 1$, then

$$\frac{m}{2n} = \frac{\beta_1 (1 + \frac{\gamma_1}{\beta_1} \frac{1}{N} + O(\frac{1}{N^2}))}{2\beta_2 N (1 + \frac{\gamma_2}{\beta_2} \frac{1}{N} + O(\frac{1}{N^2}))} \sim \frac{\beta_1}{2\beta_2} \frac{1}{N} + O(\frac{1}{N^2}),$$

and if $d_1 - d_2 \leq -2$, then

$$\frac{m}{2n} \sim O(\frac{1}{N^2}).$$

The statement of the theorem follows.

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