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# STABLE AUTOMORPHIC FORMS FOR THE GENERAL LINEAR GROUP

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Abstract. In this paper, we introduce the notion of the stability of automorphic forms for the general linear group and relate the stability of automorphic forms to the moduli space of real tori and the Jacobian real locus.

## 1. Introduction

We let

$$\mathscr{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \right\}$$

be the open convex cone of positive definite symmetric real matrices of degree $n$ in the Euclidean space $\mathbb{R}^{n(n+1)/2}$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$ for two positive integers $k$ and $l$ and ${}^tM$ denotes the transpose of a matrix $M$. Then the general linear group $GL(n,\mathbb{R})$ acts on $\mathscr{P}_n$ transitively by

$$g \cdot Y = gY\,{}^tg, \quad g \in GL(n,\mathbb{R}),\ Y \in \mathscr{P}_n. \tag{1.1}$$

Therefore $\mathscr{P}_n$ is a symmetric space which is diffeomorphic to the quotient space $GL(n,\mathbb{R})/O(n,\mathbb{R})$, where $O(n,\mathbb{R})$ denotes the real orthogonal group of degree $n$. Atle Selberg [18] investigated differential operators on $\mathscr{P}_n$ invariant under the action (1.1) of $GL(n,\mathbb{R})$ (cf. [14, 15]). Using these invariant differential operators on $\mathscr{P}_n$, automorpic forms for $GL(n,\mathbb{R})$ were investigated thereafter (cf. [2,8,11,13,25]). The Siegel $\Phi$ operator plays an important role in the theory of Siegel modular forms (cf. [4, 6, 7, 15]). Douglas Grenier [11] constructed an analogue of the Siegel $\Phi$ operator called the *Grenier operator* for automorphic forms for $GL(n,\mathbb{R})$. The Grenier operator is applied to study the Maass-Selberg relation for $GL(n,\mathbb{R})$.

The goal of this article is to introduce the notion of stable automorphic forms for the general linear group using the Grenier operator and relate the stability of automorphic forms to the study of the moduli space of polarized real tori and

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the Jacobian real locus. This paper is organized as follows. In Section 2, we briefly review the geometry of the symmetric space $\mathscr{P}_n = GL(n,\mathbb{R})/O(n,\mathbb{R})$ and spherical functions on $\mathscr{P}_n$. In Section 3, we review some results on real polarized abelian varieties and then recall the notion of polarized real tori introduced by the author [26]. In Section 4, we roughly outline the moduli space of polarized real tori and the Jacobian real locus. In Section 5, we review the Fourier expansion of an automorphic form for $GL(n,\mathbb{R})$ and the Satake compactification of $GL(n,\mathbb{Z})\backslash\mathscr{P}_n$ obtained by Grenier [10–12]. In the final section, we introduce the notion of stable automorphic forms for the general linear group using the Grenier operator and relate the stability of automorphic forms for $GL(\infty)$ to the study of the moduli space of polarized real tori and the Jacobian real locus. We also give an example of a stable automorphic form for $GL(\infty)$. We prove that $(E_n(\alpha_n, Y)\,|\, n \geq 1)$ is a stable automorphic form for $\Gamma_\infty$, where $E_n(\alpha_n, Y)$ denotes the Selberg Eisenstein series defined by Formula (5.5). See Theorem 6.2 for the precise statement. This subject adds a new area to the theory of automorphic forms for the general linear group.

**Notations.** We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^\times$ (resp. $\mathbb{C}^\times$) denotes the group of nonzero real (resp. complex) numbers. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers $k$ and $l$, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $Tr(A)$ denotes the trace of $A$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of $M$. For a positive integer $n$, $I_n$ denotes the identity matrix of degree $n$. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$ (Siegel's notation). For a complex matrix $A$, $\overline{A}$ denotes the complex *conjugate* of $A$. $\mathrm{diag}(a_1, \ldots, a_n)$ denotes the $n \times n$ diagonal matrix with diagonal entries $a_1, \ldots, a_n$. For a smooth manifold, we denote by $C_c(X)$ (resp. $C_c^\infty(X)$) the algebra of all continuous (resp. infinitely differentiable) functions on $X$ with compact support.

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

denotes the symplectic matrix of degree $2g$.

$$\mathbb{H}_g = \{\, \Omega \in \mathbb{C}^{(g,g)} \,|\, \Omega = {}^t\Omega, \quad \mathrm{Im}\,\Omega > 0\}$$

denotes the Siegel upper half plane of degree $g$.

$$Sp(g,\mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} \,|\, {}^tMJ_gM = J_g\}$$

denotes the symplectic group of degree $g$ and

$$\Gamma_g^\flat = \{\gamma \in \mathbb{Z}^{(2g,2g)} \,|\, {}^t\gamma J_g \gamma = J_g\} \subset Sp(g,\mathbb{R})$$

denotes the Siegel modular group of degree $g$.

## 2. Review on the geometry of $GL(n,\mathbb{R})/O(n,\mathbb{R})$

For $Y = (y_{ij}) \in \mathscr{P}_n$, we put

$$dY = (dy_{ij}) \qquad \text{and} \qquad \frac{\partial}{\partial Y} = \left( \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).$$

For a fixed element $A \in GL(n,\mathbb{R})$, we put

$$Y_* = A \cdot Y = AY\,{}^tA, \quad Y \in \mathscr{P}_n.$$

Then

$$dY_* = A\,dY\,{}^tA \quad \text{and} \quad \frac{\partial}{\partial Y_*} = {}^tA^{-1} \frac{\partial}{\partial Y} A^{-1}. \tag{2.1}$$

We can see easily that

$$ds^2 = Tr((Y^{-1}dY)^2)$$

is a $GL(n,\mathbb{R})$-invariant Riemannian metric on $\mathscr{P}_n$ and its Laplacian is given by

$$\Delta = Tr\left( \left( Y \frac{\partial}{\partial Y} \right)^2 \right),$$

where $Tr(M)$ denotes the trace of a square matrix $M$. We also can see that

$$d\mu_n(Y) = (\det Y)^{-\frac{n+1}{2}} \prod_{i \le j} dy_{ij} \tag{2.2}$$

is a $GL(n,\mathbb{R})$-invariant volume element on $\mathscr{P}_n$.

**Theorem 2.1.** *A geodesic $\alpha(t)$ joining $I_n$ and $Y \in \mathscr{P}_n$ has the form*

$$\alpha(t) = \exp(tA[V]), \qquad t \in [0,1],$$

*where*

$$Y = (\exp A)[V] = \exp(A[V]) = \exp({}^tVAV)$$

*is the spectral decomposition of $Y$, where $V \in O(n,\mathbb{R})$, $A = \operatorname{diag}(a_1, \ldots, a_n)$ with all $a_j \in \mathbb{R}$. The distance of $\alpha(t)$ $(0 \le t \le 1)$ between $I_n$ and $Y$ is*

$$\left( \sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}}.$$

*Proof.* The proof can be found in [25, pp. 16–17]. □

We consider the following differential operators

$$D_k = Tr\left( \left( Y \frac{\partial}{\partial Y} \right)^k \right), \quad k = 1, 2, \ldots, n. \tag{2.3}$$

By Formula (2.1), we get

$$\left( Y_* \frac{\partial}{\partial Y_*} \right)^i = A \left( Y \frac{\partial}{\partial Y} \right)^i A^{-1}$$

for any $A \in GL(n,\mathbb{R})$. So each $D_i$ $(1 \le i \le n)$ is invariant under the action (1.1) of $GL(n,\mathbb{R})$.

Selberg [18] proved the following.

**Theorem 2.2.** *The algebra $\mathbb{D}(\mathscr{P}_n)$ of all $GL(n,\mathbb{R})$-invariant differential operators on $\mathscr{P}_n$ is generated by $D_1, D_2, \ldots, D_n$. Furthermore $D_1, D_2, \ldots, D_n$ are algebraically independent and $\mathbb{D}(\mathscr{P}_n)$ is isomorphic to the commutative ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with $n$ indeterminates $x_1, x_2, \ldots, x_n$.*

*Proof.* The proof can be found in [15, pp. 64–66]. □

For $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, Atle Selberg [18, pp. 57–58] introduced the power function $p_s : \mathscr{P}_n \longrightarrow \mathbb{C}$ defined by

$$p_s(Y) := \prod_{j=1}^{n} (\det Y_j)^{s_j}, \quad Y \in \mathscr{P}_n, \tag{2.4}$$

where $Y_j \in \mathscr{P}_j$ $(1 \le j \le n)$ is the $j \times j$ upper left corner of $Y$. Let

$$T_n := \left\{ t = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & t_{nn} \end{pmatrix} \in GL(n,\mathbb{R}) \,\middle|\, t_{jj} > 0,\ 1 \le j \le n \right\} \tag{2.5}$$

be the subgroup of $GL(n,\mathbb{R})$ consisting of upper triangular matrices. For $r = (r_1, \ldots, r_n) \in \mathbb{C}^n$, we define the group homomorphism $\tau_r : T_n \longrightarrow \mathbb{C}^\times$ by

$$\tau_r(t) := \prod_{j=1}^{n} t_{jj}^{r_j}, \quad t = (t_{ij}) \in T_n. \tag{2.6}$$

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define the function $\phi_z : T_n \longrightarrow \mathbb{C}^\times$ by

$$\phi_z(t) := \prod_{j=1}^{n} t_{jj}^{2z_j + j - \frac{n+1}{2}}, \quad t = (t_{ij}) \in T_n. \tag{2.7}$$

We note that $\phi_z(t) = p_s(I_n[t])$ for some $s \in \mathbb{C}^n$.

**Proposition 2.1.** (1) *For $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, we put $r_j = 2(s_j + \cdots + s_n)$, $j = 1, \ldots, n$. Then we have*

$$p_s(I_n[t]) = \tau_r(t), \quad t \in T_n.$$

(2) *$p_s(Y[t]) = p_s(I_n[t])\, p_s(Y)$ for any $Y \in \mathscr{P}_n$ and $t \in T_n$.*

(3) *For any $D \in \mathbb{D}(\mathscr{P}_n)$, we have $Dp_s = Dp_s(I_n)\, p_s$, that is, $p_s$ is a common eigenfunction of $\mathbb{D}(\mathscr{P}_n)$.*

*Proof.* The proof can be found in [25, pp. 39–40]. □

Hans Maass [15] proved the following theorem.

**Theorem 2.3.** (1) *Let $D_1, \ldots, D_n \in \mathbb{D}(\mathscr{P}_n)$ be algebraically independent invariant differential operators given by Formula* (2.3). *Then*

$$D_j \phi_z = \lambda_j(z)\phi_z, \quad 1 \le j \le n,$$

*where $\lambda_j(z)$ is a symmetric polynomial in $z_1, \ldots, z_n$ of degree $j$ and having the following form*:

$$\lambda_j(z_1, \ldots, z_n) = z_1^j + \cdots + z_n^j + \textit{terms of lower degree.}$$

(2) *The effect of $D \in \mathbb{D}(\mathscr{P}_n)$ on power functions $p_s(Y)$ determines $D$ uniquely.*

*Proof.* The proof can be found in [15, pp. 70–76] or [25, pp. 44–48]. □

A function $h : \mathscr{P}_n \longrightarrow \mathbb{C}$ is said to be *spherical* if $h$ satisfies the following properties (2.8)-(2.10):

$$h(Y[k]) = h({}^t kYk) = h(Y) \quad \text{for all } Y \in \mathscr{P}_n \text{ and } k \in O(n, \mathbb{R}). \tag{2.8}$$

$$h \text{ is a common eigenfunction of } \mathbb{D}(\mathscr{P}_n). \tag{2.9}$$

$$h(I_n) = 1. \tag{2.10}$$

For the present, we put $G = GL(n, \mathbb{R})$ and $K = O(n, \mathbb{R})$. For $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, we define the function

$$h_s(Y) := \int_K p_s(Y[k])\, dk, \quad Y \in \mathscr{P}_n, \tag{2.11}$$

where $dk$ is a normalized measure on $K$ so that $\int_K dk = 1$. It is easily seen that $h_s(Y)$ is a spherical function on $\mathscr{P}_n$. Selberg [18, pp. 53–57] proved that these $h_s(Y)$ are the only spherical functions on $\mathscr{P}_n$. If $f \in C_c(\mathscr{P}_n)$, the *Helgason-Fourier transform* of $f$ is defined to be the function $\mathscr{H} f : \mathbb{C}^n \times K \longrightarrow \mathbb{C}$:

$$\mathscr{H} f(s,k) := \int_{\mathscr{P}_n} f(Y)\, \overline{p_s(Y[k])}\, d\mu_n(Y), \quad (s,k) \in \mathbb{C}^n \times K, \tag{2.12}$$

where $p_s$ is the Selberg power function (see Formula (2.4)) and $d\mu_n(Y)$ is a $GL(n, \mathbb{R})$-invariant volume element on $\mathscr{P}_n$ (see Formula (2.2)).

**Proposition 2.2.** (1) *The spherical function $h$ on $\mathscr{P}_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with*

$$D_i h = \lambda_i h, \quad 1 \le i \le n$$

*is unique. Here $D_1, \ldots, D_n$ are invariant differential operators on $\mathscr{P}_n$ defined by Formula* (2.3).

(2) *Let $f \in C^c(\mathscr{P}_n)$ be a common eigenfunction of $\mathbb{D}(\mathscr{P}_n)$, that is, $Df = \lambda_D f$ ($\lambda_D \in \mathbb{C}$) for all $D \in \mathbb{D}(\mathscr{P}_n)$. Define $s \in \mathbb{C}^n$ by*

$$Dp_s = \lambda_D p_s, \quad D \in \mathbb{D}(\mathscr{P}_n).$$

*If $g \in C_c^\infty(K\backslash G/K)$ is a $K$-bi-invariant function on $G$ satisfying the condition $g(x) = g(x^{-1})$ for all $x \in G$, then*

$$f \star g = \hat{g}(\bar{s}) f,$$

*where $\star$ denotes the convolution operator and*

$$\hat{g}(\bar{s}) := \int_{\mathscr{P}_n} g(Y)\, \overline{p_{\bar{s}}(Y[k])}\, d\mu_n(Y).$$

*Conversely, suppose that $f \in C^\infty(\mathscr{P}_n)$ is a $K$-invariant eigenfunction of all convolution operators with $g \in C_c^\infty(\mathscr{P}_n)$. Then $f$ is a common eigenfunction of $\mathbb{D}(\mathscr{P}_n)$.*

*Proof.* The proof can be found in [18, pp. 53–56] or [25, pp. 67–69]. □

The fundamental domain $\mathfrak{R}_n$ for $GL(n,\mathbb{Z})\backslash\mathscr{P}_n$ which was found by H. Minkowski [16] is defined as a subset of $\mathscr{P}_n$ consisting of $Y = (y_{ij}) \in \mathscr{P}_n$ satisfying the following conditions (M.1)–(M.2) (cf. [15, p. 123]):

(M.1) $aY\,{}^t a \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \ldots, a_n$ are relatively prime for $k = 1, 2, \ldots, n$.
(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \ldots, n-1$.

We say that a point of $\mathfrak{R}_n$ is *Minkowski reduced* or simply *M-reduced*. $\mathfrak{R}_n$ has the following properties (R1)-(R4):

(R1) For any $Y \in \mathscr{P}_n$, there exist a matrix $A \in GL(n,\mathbb{Z})$ and $R \in \mathfrak{R}_n$ such that $Y = R[A]$ (cf. [15, p. 139]). That is,

$$GL(n,\mathbb{Z}) \circ \mathfrak{R}_n = \mathscr{P}_n.$$

(R2) $\mathfrak{R}_n$ is a convex cone through the origin bounded by a finite number of hyperplanes. $\mathfrak{R}_n$ is closed in $\mathscr{P}_n$ (cf. [15, p. 139]).
(R3) If $Y$ and $Y[A]$ lie in $\mathfrak{R}_n$ for $A \in GL(g,\mathbb{Z})$ with $A \neq \pm I_n$, then $Y$ lies on the boundary $\partial\mathfrak{R}_n$ of $\mathfrak{R}_n$. Moreover $\mathfrak{R}_n \cap (\mathfrak{R}_n[A]) \neq \emptyset$ for only finitely many $A \in GL(n,\mathbb{Z})$ (cf. [15, p. 139]).
(R4) If $Y = (y_{ij})$ is an element of $\mathfrak{R}_n$, then

$$y_{11} \leq y_{22} \leq \cdots \leq y_{nn} \quad \text{and} \quad |y_{ij}| < \frac{1}{2} y_{ii} \quad \text{for } 1 \leq i < j \leq n.$$

We refer to [15, pp. 123–124].

$\mathscr{P}_n$ parameterizes principally polarized real tori of dimension $n$ (see Section 3). The arithmetic quotient $GL(n,\mathbb{Z})\backslash\mathscr{P}_n$ is the moduli space of isomorphism classes of principally polarized real tori of dimension $n$. According to (R2) we see that $\mathfrak{R}_n$ is a semi-algebraic set with real analytic structure.

## 3. Polarized real tori

In this section, we recall the concept of polarized real tori (cf. [26]).

We review basic notions and some results on real principally polarized abelian varieties (cf. [9, 20, 22–24]).

**Definition 3.1.** A pair $(\mathfrak{A}, S)$ is said to be a *real abelian variety* if $\mathfrak{A}$ is a complex abelian variety and $S$ is an anti-holomorphic involution of $\mathfrak{A}$ leaving the origin of $\mathfrak{A}$ fixed. The set of all fixed points of $S$ is called the *real point* of $(\mathfrak{A}, S)$ and denoted by $(\mathfrak{A}, S)(\mathbb{R})$ or simply $\mathfrak{A}(\mathbb{R})$. We call $S$ a *real structure* on $\mathfrak{A}$.

**Definition 3.2.** (1) A *polarization* on a complex abelian variety $\mathfrak{A}$ is defined to be the Chern class $c_1(D) \in H^2(\mathfrak{A}, \mathbb{Z})$ of an ample divisor $D$ on $\mathfrak{A}$. We can identify $H^2(\mathfrak{A}, \mathbb{Z})$ with $\bigwedge^2 H^1(\mathfrak{A}, \mathbb{Z})$. We write $\mathfrak{A} = V/L$, where $V$ is a finite dimensional complex vector space and $L$ is a lattice in $V$. So a polarization on $\mathfrak{A}$ can be defined as an alternating form $E$ on $L \cong H_1(\mathfrak{A}, \mathbb{Z})$ satisfying the following conditions (E1) and (E2):

(E1) The Hermitian form $H : V \times V \longrightarrow \mathbb{C}$ defined by

$$H(u, v) = E(i\,u, v) + i\,E(u, v), \qquad u, v \in V \tag{3.1}$$

is positive definite. Here $E$ can be extended $\mathbb{R}$-linearly to an alternating form on $V$.

(E2) $E(L \times L) \subset \mathbb{Z}$, i.e., $E$ is integral valued on $L \times L$.

(2) Let $(\mathfrak{A}, S)$ be a real abelian variety with a polarization $E$ of dimension $g$. A polarization $E$ is said to be *real* or *S-real* if

$$E(S_*(a), S_*(b)) = -E(a, b), \qquad a, b \in H_1(\mathfrak{A}, \mathbb{Z}). \tag{3.2}$$

Here $S_* : H_1(\mathfrak{A}, \mathbb{Z}) \longrightarrow H_1(\mathfrak{A}, \mathbb{Z})$ is the map induced by a real structure $S$. If a polarization $E$ is real, the triple $(\mathfrak{A}, E, S)$ is called a *real polarized abelian variety*. A polarization $E$ on $\mathfrak{A}$ is said to be *principal* if for a suitable basis (that is, a symplectic basis) of $H_1(\mathfrak{A}, \mathbb{Z}) \cong L$, it is represented by the symplectic matrix $J_g$ (cf. see Notations in the introduction). A real abelian variety $(\mathfrak{A}, S)$ with a principal polarization $E$ is called a *real principally polarized abelian variety*.

(3) Let $(\mathfrak{A}, E)$ be a principally polarized abelian variety of dimension $g$ and let $\{\alpha_i \,|\, 1 \leq i \leq 2g\}$ be a symplectic basis of $H_1(\mathfrak{A}, \mathbb{Z})$. It is known that there is a basis $\{\omega_1, \ldots, \omega_g\}$ of the vector space $H^0(\mathfrak{A}, \Omega^1)$ of holomorphic 1-forms on $\mathfrak{A}$ such that

$$\left( \int_{\alpha_j} \omega_i \right) = (\Omega, I_g) \quad \text{for some } \Omega \in \mathbb{H}_g.$$

The $g \times 2g$ matrix $(\Omega, I_g)$ or simply $\Omega$ is called a *period matrix* for $(\mathfrak{A}, E)$.

The definition of a *real polarized abelian variety* is motivated by the following theorem.

**Theorem 3.1.** *Let* $(\mathfrak{A}, S)$ *be a real abelian variety and let* $E$ *be a polarization on* $\mathfrak{A}$*. Then there exists an ample* $S$*-invariant* (*or* $S$*-real*) *divisor with Chern class* $E$ *if and only if* $E$ *satisfies the condition* (3.2).

*Proof.* The proof can be found in [23, Theorem 3.4, pp. 81–84]. □

Now we consider a principally polarized abelian variety of dimension $g$ with a level structure. Let $N$ be a positive integer. Let $(\mathfrak{A} = \mathbb{C}^g/L, E)$ be a principally polarized abelian variety of dimension $g$. From now on we write $\mathfrak{A} = \mathbb{C}^g/L$, where $L$ is a lattice in $\mathbb{C}^g$. A *level* $N$ *structure* on $\mathfrak{A}$ is a choice of a basis $\{U_i, V_j\}\,(1 \leq i, j \leq g)$ for an $N$-torsion points of $\mathfrak{A}$ which is symplectic, in the sense that there exists a symplectic basis $\{u_i, v_j\}$ of $L$ such that

$$U_i \equiv \frac{u_i}{N} \pmod{L} \qquad \text{and} \qquad V_j \equiv \frac{v_j}{N} \pmod{L}, \qquad 1 \leq i, j \leq g.$$

For a given level $N$ structure, such a choice of a symplectic basis $\{u_i, v_j\}$ of $L$ determines a mapping

$$F : \mathbb{R}^g \oplus \mathbb{R}^g \longrightarrow \mathbb{C}^g$$

such that $F(\mathbb{Z}^g \oplus \mathbb{Z}^g) = L$ by $F(e_i) = u_i$ and $F(f_j) = v_j$, where $\{e_i, f_j\}\,(1 \leq i, j \leq g)$ is the standard basis of $\mathbb{R}^g \oplus \mathbb{R}^g$. The choice $\{u_i, v_j\}$ (or equivalently, the mapping $F$) will be referred to as a *lift* of the level $N$ structure. Such a mapping $F$ is well defined modulo the principal congruence subgroup $\Gamma_g(N)$, that is, if $F'$ is another lift of the level structure, then $F' \circ F^{-1} \in \Gamma_g(N)$. A level $N$ structure $\{U_i, V_j\}$ is said to be *compatible* with a real structure $S$ on $(\mathfrak{A}, E)$ if, for some (and hence for any) lift $\{u_i, v_j\}$ of the level structure,

$$S\left(\frac{u_i}{N}\right) \equiv -\frac{u_i}{N} \pmod{L} \qquad \text{and} \qquad S\left(\frac{v_j}{N}\right) \equiv \frac{v_j}{N} \pmod{L}, \qquad 1 \leq i, j \leq g.$$

**Definition 3.3.** A real principally polarized abelian variety of dimension $g$ with a level $N$ structure is a quadruple $\mathcal{A} = (\mathfrak{A}, E, S, \{U_i, V_j\})$ with $\mathfrak{A} = \mathbb{C}^g/L$, where $(\mathfrak{A}, E, S)$ is a real principally polarized abelian variety and $\{U_i, V_j\}$ is a level $N$ structure compatible with a real structure $S$. An isomorphism

$$\mathcal{A} = (\mathfrak{A}, E, S, \{U_i, V_j\}) \cong (\mathfrak{A}', E', S', \{U_i', V_j'\}) = \mathcal{A}'$$

is a complex linear mapping $\phi : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ such that

$$\phi(L) = L', \tag{3.3}$$

$$\phi_*(E) = E', \tag{3.4}$$

$$\phi_*(S) = S', \text{ that is, } \phi \circ S \circ \phi^{-1} = S', \tag{3.5}$$

$$\phi\left(\frac{u_i}{N}\right) \equiv \frac{u_i'}{N} \pmod{L'} \quad \text{and} \quad \phi\left(\frac{v_j}{N}\right) \equiv \frac{v_j'}{N} \pmod{L'}, \quad 1 \leq i, j \leq g \tag{3.6}$$

for some lift $\{u_i, v_j\}$ and $\{u_i', v_j'\}$ of the level structures.

Now we show that a given positive integer $N$ and a given $\Omega \in \mathbb{H}_g$ determine naturally a principally polarized abelian variety $(\mathfrak{A}_\Omega, E_\Omega)$ of dimension $g$ with a level $N$ structure. Let $E_0$ be the standard alternating form on $\mathbb{R}^g \oplus \mathbb{R}^g$ with the symplectic matrix $J_g$ with respect to the standard basis of $\mathbb{R}^g \oplus \mathbb{R}^g$. Let $F_\Omega : \mathbb{R}^g \oplus \mathbb{R}^g \longrightarrow \mathbb{C}^g$ be the real linear mapping with matrix $(\Omega, I_g)$, that is,

$$F_\Omega \begin{pmatrix} x \\ y \end{pmatrix} := \Omega\, x + y, \qquad x, y \in \mathbb{R}^g. \tag{3.7}$$

We define $E_\Omega := (F_\Omega)_*(E_0)$ and $L_\Omega := F_\Omega(\mathbb{Z}^g \oplus \mathbb{Z}^g)$. Then $(\mathfrak{A}_\Omega = \mathbb{C}^g/L_\Omega, E_\Omega)$ is a principally polarized abelian variety. The Hermitian form $H_\Omega$ on $\mathbb{C}^g$ corresponding to $E_\Omega$ is given by

$$H_\Omega(u, v) = {}^t u\, (\operatorname{Im} \Omega)^{-1}\, \overline{v}, \quad E_\Omega = \operatorname{Im} H_\Omega, \quad u, v \in \mathbb{C}^g. \tag{3.8}$$

If $z_1, \ldots, z_g$ are the standard coordinates on $\mathbb{C}^g$, then the holomorphic 1-forms $dz_1, \ldots, dz_g$ have the period matrix $(\Omega, I_g)$. If $\{e_i, f_j\}$ is the standard basis of $\mathbb{R}^g \oplus \mathbb{R}^g$, then $\{F_\Omega(e_i/N), F_\Omega(f_j/N)\}$ (mod $L_\Omega$) is a level $N$ structure on $(\mathfrak{A}_\Omega, E_\Omega)$, which we refer to as the *standard $N$ structure*. Assume that $\Omega_1$ and $\Omega_2$ are two elements of $\mathbb{H}_g$ such that

$$\psi : (\mathfrak{A}_{\Omega_1} = \mathbb{C}^g/L_{\Omega_1}, E_{\Omega_1}) \longrightarrow (\mathfrak{A}_{\Omega_2} = \mathbb{C}^g/L_{\Omega_2}, E_{\Omega_2})$$

is an isomorphism of the corresponding principally polarized abelian varieties, i.e., $\psi(L_{\Omega_1}) = L_{\Omega_2}$ and $\psi_*(E_{\Omega_1}) = E_{\Omega_2}$. We set

$$h = {}^t\big(F_{\Omega_2}^{-1} \circ \psi \circ F_{\Omega_1}\big) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then we see that $h \in \Gamma_g$. And we have

$$\Omega_1 = h \cdot \Omega_2 = (A\Omega_2 + B)(C\Omega_2 + D)^{-1} \tag{3.9}$$

and

$$\psi(Z) = {}^t(C\Omega_2 + D) Z, \qquad Z \in \mathbb{C}^g. \tag{3.10}$$

Let

$$I_* := \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

We define the involution $\tau : Sp(g, \mathbb{R}) \longrightarrow Sp(g, \mathbb{R})$ by

$$\tau(x) := I_* x I_*, \quad x \in Sp(g, \mathbb{R}).$$

Precisely $\tau$ is given by

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R}).$$

We note that $\tau : Sp(g, \mathbb{R}) \longrightarrow Sp(g, \mathbb{R})$ passes to an involution (which we denote by the same letter) $\tau : \mathbb{H}_g \longrightarrow \mathbb{H}_g$ such that

$$\tau(x \cdot \Omega) = \tau(x)\tau(\Omega) \quad \text{for all } x \in Sp(g, \mathbb{R}),\ \Omega \in \mathbb{H}_g.$$

In fact, we can see easily that the involution $\tau : \mathbb{H}_g \longrightarrow \mathbb{H}_g$ is the anti-holomorphic involution given by

$$\tau(\Omega) = -\overline{\Omega}, \quad \Omega \in \mathbb{H}_g.$$

Its fixed point set is the orbit

$$i\mathscr{P}_g = GL(g,\mathbb{R}) \cdot (iI_g) \subset \mathbb{C}^{(g,g)}.$$

We refer to [26, pp. 274–275, 278].

Let $\Omega \in \mathbb{H}_g$ such that $\gamma \cdot \Omega = \tau(\Omega) = -\overline{\Omega}$ for some $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g^\flat$. We define the mapping $S_{\gamma,\Omega} : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ by

$$S_{\gamma,\Omega}(Z) := {}^t(C\Omega + D)\,\overline{Z}, \qquad Z \in \mathbb{C}^g. \tag{3.11}$$

Then we can show that $S_{\gamma,\Omega}$ is a real structure on $(\mathfrak{A}_\Omega, E_\Omega)$ which is compatible with the polarization $E_\Omega$ (that is, $E_\Omega(S_{\gamma,\Omega}(u), S_{\gamma,\Omega}(v)) = -E_\Omega(u,v)$ for all $u, v \in \mathbb{C}^g$). Indeed according to Comessatti's Theorem (see Theorem 3.1), $S_{\gamma,\Omega}(Z) = \overline{Z}$, i.e., $S_{\gamma,\Omega}$ is a complex conjugation. Therefore we have

$$E_\Omega(S_{\gamma,\Omega}(u), S_{\gamma,\Omega}(v)) = E_\Omega(\overline{u}, \overline{v}) = -E_\Omega(u,v)$$

for all $u, v \in \mathbb{C}^g$. From now on we write simply $\sigma_\Omega = S_{\gamma,\Omega}$.

**Theorem 3.2.** *Let $(\mathfrak{A}, E, S)$ be a real principally polarized abelian variety of dimension $g$. Then there exists $\Omega = X + i\,Y \in \mathbb{H}_g$ such that $2X \in \mathbb{Z}^{(g,g)}$ and there exists an isomorphism of real principally polarized abelian varieties*

$$(\mathfrak{A}, E, S) \cong (\mathfrak{A}_\Omega, E_\Omega, \sigma_\Omega),$$

*where $\sigma_\Omega$ is a real structure on $\mathfrak{A}_\Omega$ induced by a complex conjugation $\sigma : \mathbb{C}^g \longrightarrow \mathbb{C}^g$.*

The above theorem is essentially due to Comessatti [5]. We refer to [22, 23] for the proof of Theorem 3.2.

Theorem 3.2 leads us to define the subset $\mathscr{H}_g$ of $\mathbb{H}_g$ by

$$\mathscr{H}_g := \left\{ \Omega \in \mathbb{H}_g \mid 2\,\mathrm{Re}\,\Omega \in \mathbb{Z}^{(g,g)} \right\}. \tag{3.12}$$

We note that $\mathscr{H}_g$ is a countable union of analytic subsets (see [24] or [26, p. 280]). Precisely

$$\mathscr{H}_g = \left\{ \Omega = x + i\,y = {}^t\Omega \mid x, y \in \mathbb{R}^{(g,g)},\ y > 0,\ 2x_{ij} \in \mathbb{Z},\ y_{ij} \in \mathbb{R},\ 1 \le i \le j \le g \right\},$$

where $x = (x_{ij}) = {}^t x \in \mathbb{R}^{(g,g)}$ and $y = (y_{ij}) = {}^t y \in \mathbb{R}^{(g,g)}$ with $y > 0$, or

$$\mathscr{H}_g = \bigcup_{2x \in S_g(\mathbb{Z})} (x + i\,\mathscr{P}_g),$$

where

$$S_g(\mathbb{Z}) := \left\{ x \in \mathbb{R}^{(g,g)} \mid x \text{ is integeral},\ x = {}^t x \right\}$$

and

$$\mathscr{P}_g := \left\{ y \in \mathbb{R}^{(g,g)} \,|\, x = {}^t x > 0, \text{ positive definite} \right\}.$$

$\mathscr{H}_g$ may be considered as a countable union of semialgebraic subsets.

Assume $\Omega = X + i\,Y \in \mathscr{H}_g$. Then according to Theorem 3.2, $(\mathfrak{A}_\Omega, E_\Omega, \sigma_\Omega)$ is a real principally polarized abelian variety of dimension $g$. The matrix $M_\sigma$ for the action of a complex conjugation $\sigma$ on the lattice $L_\Omega = \Omega\mathbb{Z}^g + \mathbb{Z}^g$ with respect to the basis given by the columns of $(\Omega, I_g)$ is given by

$$M_\sigma = \begin{pmatrix} -I_g & 0 \\ 2\,X & I_g \end{pmatrix}. \tag{3.13}$$

Since

$${}^t M_\sigma \, J_g \, M_\sigma = \begin{pmatrix} -I_g & 2\,X \\ 0 & I_g \end{pmatrix} J_g \begin{pmatrix} -I_g & 0 \\ 2\,X & I_g \end{pmatrix} = -J_g,$$

the canonical polarization $J_g$ is $\sigma$-real.

**Theorem 3.3.** *Let $\Omega$ and $\Omega_*$ be two elements in $\mathscr{H}_g$. Then $\Omega$ and $\Omega_*$ represent (real) isomorphic triples $(\mathfrak{A}, E, \sigma)$ and $(\mathfrak{A}_*, E_*, \sigma_*)$ if and only if there exists an element $A \in GL(g, \mathbb{Z})$ such that*

$$2\,\mathrm{Re}\,\Omega_* = 2\,A\,(\mathrm{Re}\,\Omega)\,{}^t A \pmod 2 \tag{3.14}$$

*and*

$$\mathrm{Im}\,\Omega_* = A\,(\mathrm{Im}\,\Omega)\,{}^t A. \tag{3.15}$$

*Proof.* Suppose $(\mathfrak{A}, E, \sigma)$ and $(\mathfrak{A}_*, E_*, \sigma_*)$ are real isomorphic. Then we can find an element $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ such that

$$\Omega_* = (A\Omega + B)(C\Omega + D)^{-1}.$$

The map

$$\varphi : \mathbb{C}^g / L_{\Omega_*} = \mathfrak{A}_{\Omega_*} \longrightarrow \mathfrak{A}_\Omega = \mathbb{C}^g / L_\Omega$$

induced by the map

$$\widetilde{\varphi} : \mathbb{C}^g \longrightarrow \mathbb{C}^g, \qquad Z \longmapsto {}^t(C\Omega + D)\,Z$$

is a real isomorphism. Since $\widetilde{\varphi} \circ \sigma_* = \sigma \circ \widetilde{\varphi}$, i.e., $\widetilde{\varphi}$ commutes with complex conjugation on $\mathbb{C}^g$, we have $C = 0$. Therefore

$$\Omega_* = (A\Omega + B)\,{}^t A = (AX\,{}^t A + B\,{}^t A) + i\,AY\,{}^t A,$$

where $\Omega = X + i\,Y$. Hence we obtain the desired results (3.14) and (3.15).

Conversely, we assume that there exists $A \in GL(g, \mathbb{Z})$ satisfying the conditions (3.14) and (3.15). Then

$$\Omega_* = \gamma \cdot \Omega = (A\Omega + B)\,{}^t A$$

for some $\gamma = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_g$ with $B \in \mathbb{Z}^{(g,g)}$ with $B\,{}^tA = A\,{}^tB$. The map $\psi : \mathfrak{A}_\Omega \longrightarrow \mathfrak{A}_{\Omega_*}$ induced by the map

$$\widetilde{\psi} : \mathbb{C}^g \longrightarrow \mathbb{C}^g, \qquad Z \longmapsto A^{-1}Z$$

is a complex isomorphism commuting complex conjugation $\sigma$. Therefore $\psi$ is a real isomorphism of $(\mathfrak{A}, E, \sigma)$ onto $(\mathfrak{A}_*, E_*, \sigma_*)$. □

According to Theorem 3.3, we are led to define the subgroup $\Gamma_g^\star$ of $\Gamma_g$ by

$$\Gamma_g^\star := \left\{ \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_g \;\middle|\; B \in \mathbb{Z}^{(g,g)}, \quad A\,{}^tB = B\,{}^tA \right\}. \tag{3.16}$$

It is easily seen that $\Gamma_g^\star$ acts on $\mathscr{H}_g$ properly discontinuously by

$$\gamma \cdot \Omega = A\Omega\,{}^tA + B\,{}^tA, \tag{3.17}$$

where $\gamma = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_g^\star$ and $\Omega \in \mathscr{H}_g$.

Now we define the notion of polarized real tori.

**Definition 3.4.** A real torus $T = \mathbb{R}^n/\Lambda$ with a lattice $\Lambda$ in $\mathbb{R}^n$ is said to be *polarized* if the associated complex torus $\mathfrak{A} = \mathbb{C}^n/L$ is a polarized real abelian variety, where $L = \mathbb{Z}^n + i\,\Lambda$ is a lattice in $\mathbb{C}^n$. Moreover if $\mathfrak{A}$ is a principally polarized real abelian variety, $T$ is said to be *principally polarized.* Let $\Phi : T \longrightarrow \mathfrak{A}$ be the smooth embedding of $T$ into $\mathfrak{A}$ defined by

$$\Phi(v + \Lambda) := i\,v + L, \qquad v \in \mathbb{R}^n. \tag{3.18}$$

Let $\mathfrak{L}$ be a polarization of $\mathfrak{A}$, that is, an ample line bundle over $\mathfrak{A}$. The pullback $\Phi^*\mathfrak{L}$ is called a *polarization* of $T$. We say that a pair $(T, \Phi^*\mathfrak{L})$ is a *polarized real torus.*

**Example 3.1.** Let $Y \in \mathscr{P}_n$ be an $n \times n$ positive definite symmetric real matrix. Then $\Lambda_Y = Y\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$. Then the $n$-dimensional torus $T_Y = \mathbb{R}^n/\Lambda_Y$ is a principally polarized real torus. Indeed,

$$\mathfrak{A}_Y = \mathbb{C}^n/L_Y, \qquad L_Y = \mathbb{Z}^n + i\,\Lambda_Y$$

is a princially polarized real abelian variety. Its corresponding hermitian form $H_Y$ is given by

$$H_Y(x,y) = E_Y(i\,x,y) + i\,E_Y(x,y) = {}^tx\,Y^{-1}\,\overline{y}, \qquad x, y \in \mathbb{C}^n,$$

where $E_Y$ denotes the imaginary part of $H_Y$. It is easily checked that $H_Y$ is positive definite and $E_Y(L_Y \times L_Y) \subset \mathbb{Z}$ (cf. [17, pp. 29–30]). The real structure $\sigma_Y$ on $\mathfrak{A}_Y$ is a complex conjugation. In addition, if $\det Y = 1$, the real torus $T_Y$ is said to be *special.*

**Example 3.2.** Let $Q = \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & -\sqrt{5} \end{pmatrix}$ be a $2 \times 2$ symmetric real matrix of signature $(1,1)$. Then $\Lambda_Q = Q\mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$. Then the real torus $T_Q =$

$\mathbb{R}^2/\Lambda_Q$ is not polarized because the associated complex torus $\mathfrak{A}_Q = \mathbb{C}^2/L_Q$ is not an abelian variety, where $L_Q = \mathbb{Z}^2 + i\,\Lambda_Q$ is a lattice in $\mathbb{C}^2$.

**Definition 3.5.** Two polarized tori $T_1 = \mathbb{R}^n/\Lambda_1$ and $T_2 = \mathbb{R}^n/\Lambda_2$ are said to be isomorphic if the associated polarized real abelian varieties $\mathfrak{A}_1 = \mathbb{C}^n/L_1$ and $\mathfrak{A}_2 = \mathbb{C}^n/L_2$ are isomorphic, where $L_i = \mathbb{Z}^n + i\,\Lambda_i$ $(i = 1, 2)$, more precisely, if there exists a linear isomorphism $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that

$$\varphi(L_1) = L_2, \tag{3.19}$$

$$\varphi_*(E_1) = E_2, \tag{3.20}$$

$$\varphi_*(\sigma_1) = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_2, \tag{3.21}$$

where $E_1$ and $E_2$ are polarizations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, and $\sigma_1$ and $\sigma_2$ denotes the real structures (in fact complex conjugations) on $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively.

**Example 3.3.** Let $Y_1$ and $Y_2$ be two $n \times n$ positive definite symmetric real matrices. Then $\Lambda_i := Y_i\,\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ $(i = 1, 2)$. We let

$$T_i := \mathbb{R}^n/\Lambda_i, \qquad i = 1, 2$$

be real tori of dimension $n$. Then according to Example 3.1, $T_1$ and $T_2$ are principally polarized real tori. We see that $T_1$ is isomorphic to $T_2$ as polarized real tori if and only if there is an element $A \in GL(n, \mathbb{Z})$ such that $Y_2 = A\,Y_1\,{}^tA$.

## 4. The moduli space of polarized real tori

For a given fixed positive integer $n$, we let

$$\mathbb{H}_n = \{\,\Omega \in \mathbb{C}^{(n,n)} \,|\, \Omega = {}^t\Omega, \ \ \operatorname{Im}\Omega > 0\}$$

be the Siegel upper half plane of degree $n$ and let

$$Sp(n, \mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n\}$$

be the symplectic group of degree $n$, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $Sp(n, \mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{4.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$. Let

$$\Gamma_n^\flat := Sp(n, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \,\Big|\, A, B, C, D \text{ integral} \right\}$$

be the Siegel modular group of degree $n$. This group acts on $\mathbb{H}_n$ properly discontinuously.

Let $\mathcal{A}_n := \Gamma_n^\flat \backslash \mathbb{H}_n$ be the Siegel modular variety of degree $n$, that is, the moduli space of $n$-dimensional principally polarized abelian varieties, and let

$\mathcal{M}_n$ be the the moduli space of projective curves of genus $n$. Then according to Torelli's theorem, the Jacobi mapping

$$T_n : \mathcal{M}_n \longrightarrow \mathcal{A}_n \tag{4.2}$$

defined by

$$C \longmapsto J(C) := \text{the Jacobian of } C$$

is injective. The Jacobian locus $J_n := T_n(\mathcal{M}_n)$ is a $(3n-3)$-dimensional subvariety of $\mathcal{A}_n$ if $n \geq 2$. We denote by $Hyp_n$ the hyperelliptic locus in $\mathcal{A}_n$.

If $Y \in \mathscr{P}_n$, according to Example 3.1, $T_Y = \mathbb{R}^n/\Lambda_Y$ is a principally polarized real torus of dimension $n$ and $\mathfrak{A}_Y = \mathbb{C}^n/L_Y$ is a principally polarized abelian variety of dimension $n$. Here $\Lambda_Y = Y\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ and $L_Y = \mathbb{Z}^n + i\Lambda_Y$ is a lattice in $\mathbb{C}^n$. We denote by $[\mathfrak{A}_Y]$ the isomorphism class of $\mathfrak{A}_Y$.

The arithmetic quotient

$$\mathfrak{Y}_n := \Gamma_n \backslash GL(n,\mathbb{R})/O(n,\mathbb{R}), \qquad \Gamma_n := GL(n,\mathbb{Z})/\{\pm I_n\}$$

is the moduli space of principally polarized real tori of dimension $n$.

We define

$$\mathfrak{J}_{n,J} := \{Y \in \mathscr{P}_n \mid \mathfrak{A}_Y \text{ is the Jacobian of a curve of genus } n,\ i.e.,\ [\mathfrak{A}_Y] \in J_n\}$$

and

$$\mathfrak{J}_{n,H} := \{Y \in \mathscr{P}_n \mid \mathfrak{A}_Y \text{ is the Jacobian of a hyperelliptic curve of genus } n\}.$$

We see that $\Gamma_n$ acts on both $\mathfrak{J}_{n,J}$ and $\mathfrak{J}_{n,H}$ properly discontinously. So we may define

$$\mathfrak{Y}_{n,J} := \Gamma_n \backslash \mathfrak{J}_{n,J} \qquad \text{and} \qquad \mathfrak{Y}_{n,H} := \Gamma_n \backslash \mathfrak{J}_{n,H}.$$

$\mathfrak{Y}_{n,J}$ and $\mathfrak{Y}_{n,H}$ are called the *Jacobian real locus* and the *hyperelliptic real locus*, respectively.

The following natural problem may be regarded as the real version of the Schottky problem.

**Problem.** Characterize the Jacobian real locus $\mathfrak{Y}_{n,J}$.

For any positive integer $n \in \mathbb{Z}^+$, let

$$G_n = GL(n,\mathbb{R}), \quad K_n = O(n,\mathbb{R}) \quad \text{and} \quad \Gamma_n = GL(n,\mathbb{Z})/\{\pm I_n\}.$$

For any $m, n \in \mathbb{Z}^+$ with $m < n$, we define

$$\xi_{m,n} : G_m \longrightarrow G_n$$

by

$$\xi_{m,n}(A) := \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix}, \qquad A \in G_m. \tag{4.3}$$

We let

$$G_\infty := \varinjlim_n G_n, \qquad K_\infty := \varinjlim_n K_n \quad \text{and} \quad \Gamma_\infty := \varinjlim_n \Gamma_n$$

be the inductive limits of the directed systems $(G_n, \xi_{m,n})$, $(K_n, \xi_{m,n})$ and $(\Gamma_n, \xi_{m,n})$, respectively.

For any two positive integers $m, n \in \mathbb{Z}^+$ with $m < n$, we embed $\mathscr{P}_m$ into $\mathscr{P}_n$ as follows:

$$\psi_{m,n} : \mathscr{P}_m \longrightarrow \mathscr{P}_n, \qquad Y \mapsto \begin{pmatrix} Y & 0 \\ 0 & I_{n-m} \end{pmatrix}, \qquad Y \in \mathscr{P}_m.$$

We let

$$\mathscr{P}_\infty = \varinjlim_n \mathscr{P}_n$$

be the inductive limit of the directed system $(\mathscr{P}_n, \psi_{m,n})$. We can show that

$$\mathscr{P}_\infty = G_\infty / K_\infty.$$

Let $\mathfrak{Y}_n^S$ be the Satake compactification of $\mathfrak{Y}_n$ (cf. Theorem 3 in [12, pp. 62–65] or Theorem 5.2 in this article). We denote by $\mathfrak{Y}_{n,J}^S$ (resp. $\mathfrak{Y}_{n,H}^S$) the Satake compactification of $\mathfrak{Y}_{n,J}$ (resp. $\mathfrak{Y}_{n,H}$). We can show that $\mathfrak{Y}_{n,J}^S$ (resp. $\mathfrak{Y}_{n,H}^S$) is the closure of $\mathfrak{Y}_{n,J}$ (resp. $\mathfrak{Y}_{n,H}$) inside $\mathfrak{Y}_n^S$. We have the following sequences

$$\mathfrak{Y}_1^S \longrightarrow \mathfrak{Y}_2^S \longrightarrow \mathfrak{Y}_3^S \longrightarrow \cdots,$$

$$\mathfrak{Y}_{1,J}^S \longrightarrow \mathfrak{Y}_{2,J}^S \longrightarrow \mathfrak{Y}_{3,J}^S \longrightarrow \cdots$$

and

$$\mathfrak{Y}_{1,H}^S \longrightarrow \mathfrak{Y}_{2,H}^S \longrightarrow \mathfrak{Y}_{3,H}^S \longrightarrow \cdots.$$

We put

$$\mathfrak{Y}_\infty^S := \varinjlim_n \mathfrak{Y}_n^S, \qquad \mathfrak{Y}_{\infty,J}^S := \varinjlim_n \mathfrak{Y}_{n,J}^S \quad \text{and} \quad \mathfrak{Y}_{\infty,H}^S := \varinjlim_n \mathfrak{Y}_{n,H}^S.$$

## 5. Automorphic forms for $GL(n, \mathbb{R})$

Let

$$\mathfrak{P}_n := \left\{ Y \in \mathbb{R}^{(n,n)} \,|\, Y = {}^tY > 0, \ \det(Y) = 1 \right\}$$

be a symmetric space associated to $SL(n, \mathbb{R})$. Indeed, $SL(n, \mathbb{R})$ acts on $\mathfrak{P}_n$ transitively by

$$g \cdot Y = gY\, {}^tg, \qquad g \in SL(n, \mathbb{R}), \ Y \in \mathfrak{P}_n. \tag{5.1}$$

Thus $\mathfrak{P}_n$ is a smooth manifold diffeomorphic to the symmetric space $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ through the bijective map

$$SL(n, \mathbb{R})/SO(n, \mathbb{R}) \longrightarrow \mathfrak{P}_n, \qquad g\, SO(n, \mathbb{R}) \mapsto g\, {}^tg, \quad g \in SL(n, \mathbb{R}).$$

For $Y \in \mathfrak{P}_n$, we have a partial Iwasawa decomposition

$$Y = \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)} W \end{pmatrix} \left[ \begin{pmatrix} 1 & {}^tx \\ 0 & I_{n-1} \end{pmatrix} \right] = \begin{pmatrix} v^{-1} & v^{-1}\, {}^tx \\ v^{-1} x & v^{-1} x\, {}^tx + v^{1/(n-1)} W \end{pmatrix}, \tag{5.2}$$

where $v > 0,\ x \in \mathbb{R}^{(n-1,1)}$ and $W \in \mathfrak{P}_{n-1}$. From now on, for brevity, we write $Y = [v, x, W]$ instead of the decomposition (5.2). In these coordinates $Y = [v, x, W]$,

$$ds_Y^2 = \frac{n}{n-1}\, v^{-2} dv^2 + 2\, v^{-n/(n-1)} W^{-1}[dx] + ds_W^2$$

is an $SL(n, \mathbb{R})$-invariant metric on $\mathfrak{P}_n$, where $dx = {}^t(dx_1, \ldots, dx_{n-1})$ and $ds_W^2$ is an $SL(n-1, \mathbb{R})$-invariant metric on $\mathfrak{P}_{n-1}$. The Laplace operator $\Delta_n$ of $(\mathfrak{P}_n, ds_Y^2)$ is given by

$$\Delta_n = \frac{n-1}{n}\, v^2 \frac{\partial^2}{\partial v^2} - \frac{1}{n}\frac{\partial}{\partial v} + v^{n/(n-1)}\, W \left[\frac{\partial}{\partial x}\right] + \Delta_{n-1}$$

inductively, where if $x = {}^t(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{(n-1,1)}$,

$$\frac{\partial}{\partial x} = {}^t\!\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$$

and $\Delta_{n-1}$ is the Laplace operator of $(\mathfrak{P}_{n-1}, ds_W^2)$.

$$d\mu_n = v^{-(n+2)/2}\, dv\, dx\, d\mu_{n-1}$$

is an $SL(n, \mathbb{R})$-invariant volume element on $\mathfrak{P}_n$, where $dx = dx_1 \cdots dx_{n-1}$ and $d\mu_{n-1}$ is an $SL(n-1, \mathbb{R})$-invariant volume element on $\mathfrak{P}_{n-1}$.

Following earlier work of Minkowski, Siegel [21] showed that the volume of the fundamental domain $SL(n, \mathbb{Z})\backslash\mathfrak{P}_n$ is given as follows:

$$\mathrm{Vol}(SL(n, \mathbb{Z})\backslash\mathfrak{P}_n) = \int_{SL(n,\mathbb{Z})\backslash\mathfrak{P}_n} d\mu_n = n\, 2^{n-1} \prod_{k=2}^{n} \frac{\zeta(k)}{\mathrm{Vol}(S^{k-1})}, \tag{5.3}$$

where

$$\mathrm{Vol}(S^{k-1}) = \frac{2\,(\sqrt{\pi})^k}{\Gamma(k/2)}$$

denotes the volume of the $(k-1)$-dimensional sphere $S^{k-1}$, $\Gamma(x)$ denotes the usual Gamma function and $\zeta(k) = \sum_{m=1}^{\infty} m^{-k}$ denotes the Riemann zeta function. The proof of (5.3) can be found in [8, pp. 27–37] and [21].

If we repeat this partial decomposition process for $W$, we get the Iwasawa decomposition

(5.4)

$$Y = y^{-1}\mathrm{diag}\left(1, y_1^2, (y_1 y_2)^2, \ldots, (y_1 y_2 \cdots y_{n-1})^2\right) \left[\begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \cdots & x_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}\right],$$

where $y > 0,\ y_j \in \mathbb{R}\,(1 \leq j \leq n-1)$ and $x_{ij} \in \mathbb{R}(1 \leq i < j \leq n)$. Here $y = y_1^{2(n-1)} \cdots y_{n-1}^2$ and $\mathrm{diag}(a_1, \ldots, a_n)$ denotes the $n \times n$ diagonal matrix with diagonal entries $a_1, \ldots, a_n$. In this case we denote $Y = (y_1, \ldots, y_{n-1}, x_{12}, \ldots, x_{n-1,n})$.

Define $\Gamma_n = GL(n,\mathbb{Z})/\{\pm I_n\}$. We observe that $\Gamma_n = SL(n,\mathbb{Z})/\{\pm I_n\}$ if $n$ is even, and $\Gamma_n = SL(n,\mathbb{Z})$ if $n$ is odd. An *automorphic form* for $\Gamma_n$ is defined to be a real analytic function $f : \mathfrak{P}_n \longrightarrow \mathbb{C}$ satisfying the following conditions (AF1)–(AF3):

(AF1) $f$ is an eigenfunction for all $GL(n,\mathbb{R})$-invariant differential operators on $\mathfrak{P}_n$.

(AF2) $f(\gamma Y\, {}^t\gamma) = f(Y)$ for all $\gamma \in SL(n,\mathbb{R})$ and $Y \in \mathfrak{P}_n$.

(AF3) There exist a constant $C > 0$ and $s \in \mathbb{C}^{n-1}$ with $s = (s_1, \dots, s_{n-1})$ such that $|f(Y)| \le C\,|p_{-s}(Y)|$ as the upper left determinants $\det Y_j \longrightarrow \infty$, $j = 1, 2, \dots, n$, where

$$p_{-s}(Y) := \prod_{j=1}^{n-1} (\det Y_j)^{-s_j}$$

is the Selberg's power function (cf. [18, 25]).

We denote by $A(\Gamma_n)$ the space of all automorphic forms for $\Gamma_n$. A *cusp form* $f \in A(\Gamma_n)$ is an automorphic form for $\Gamma_n$ satisfying the following conditions:

$$\int_{X \in (\mathbb{R}/\mathbb{Z})^{(j,n-j)}} f\left( Y \left[ \begin{pmatrix} I_j & X \\ 0 & I_{n-j} \end{pmatrix} \right] \right) dX = 0, \quad 1 \le j \le n-1.$$

We denote by $A_0(\Gamma_n)$ the space of all cusp forms for $\Gamma_n$.

For $s = (s_1, \dots, s_{n-1}) \in \mathbb{C}^{n-1}$, we now consider the following Eisenstein series

$$E_n(s, Y) := \sum_{\gamma \in \Gamma_n/\Gamma_\star} p_{-s}(Y[\gamma]), \qquad Y \in \mathfrak{P}_n, \tag{5.5}$$

where $\Gamma_\star$ is a subgroup of $\Gamma_n$ consisting of all upper triangular matrices. $E_n(s,Y)$ is a special type of the more general Eisenstein series introduced by Atle Selberg [19]. It is known that the series converges for $\mathrm{Re}(s_j) > 1$, $j = 1, 2, \dots, n-1$, and has analytic continuation for each of the variables $s_1, \dots, s_{n-1}$ (cf. [19, 25]). It is seen that $E_n(s,Y)$ is a common eigenfunction of all invariant differential operators in $\mathbb{D}(\mathscr{P}_n)$. Its corresponding eigenvalue of the Laplace operator $\Delta_n$ is given by

$$\begin{aligned}\lambda = {} & \frac{n-1}{n}(s_1+\xi_1)\left(s_1 - 1 + \xi_1 - \frac{1}{n-1}\right) \\ & + \frac{n-2}{n-1}(s_2+\xi_2)\left(s_2 - 1 + \xi_2 - \frac{1}{n-2}\right) + \cdots + \frac{1}{2}s_{n-1}(s_{n-1}-2),\end{aligned}$$

where $\xi_{n-1} = 0$ and

$$\xi_j = \frac{1}{n-j}\sum_{k=j+1}^{n-1} (n-k)s_k, \qquad j = 1, 2, \dots, n-2. \tag{5.6}$$

Let $f \in A(\Gamma_n)$ be an automorphic form. Since $f(Y)$ is invariant under the action of the subgroup $\left\{ \begin{pmatrix} 1 & a \\ 0 & I_{n-1} \end{pmatrix} \,\Big|\, a \in \mathbb{Z}^{(1,n-1)} \right\}$ of $\Gamma_n$, we have the Fourier expansion

$$f(Y) = \sum_{N \in \mathbb{Z}^{(n-1,1)}} a_N(v,W) e^{2\pi i\, {}^t x N}, \tag{5.7}$$

where $Y = [v, x, W] \in \mathfrak{P}_n$ and

$$a_N(v,W) = \int_0^1 \int_0^1 \cdots \int_0^1 f([v,x,W])\, e^{-2\pi i\, {}^t x N} dx.$$

For $s \in \mathbb{C}^{n-1}$ and $A, B \in \mathfrak{P}_n$, we define the $K$-Bessel function

$$K_n(s \,|\, A, B) := \int_{\mathscr{P}_n} p_s(Y)\, e^{Tr(AY + BY^{-1})} d\mu_n, \tag{5.8}$$

where $d\mu_n$ is the $GL(n,\mathbb{R})$-invariant volume element on $\mathscr{P}_n$ (see (2.2)).

Let $f \in A(\Gamma_n)$ be an automorphic form for $\Gamma_n$. Thus we have $n$ differential equations $D_j f = \lambda_j f\, (1 \le j \le n)$. Here $D_1, \ldots, D_n$ are $GL(n,\mathbb{R})$-invariant differential operators defined by (2.3). We can find $s = (s_j) \in \mathbb{C}^{n-1}$ satisfying the various relations determined by the $\lambda_j\, (1 \le j \le n)$. D. Grenier [11, Theorem 1, pp. 469–471] proved that $f$ has the following Fourier expansion

$$\begin{aligned} f(Y) &= f([v,x,W]) \\ &= a_0(v,W) + \sum_{0 \ne m \in \mathbb{Z}^{n-1}} \sum_{\gamma \in \Gamma_{n-1}/P} a_m(v,W) v^{(n-1)/2} \\ &\quad \times K_{n-1}(\hat{s} \,|\, v^{1/(n-1)}\, {}^t\gamma W \gamma, \pi^2 v\, m\, {}^t m)\, e^{2\pi i\, {}^t x \gamma m}, \end{aligned} \tag{5.9}$$

where $\hat{s} = \left(s_1 - \frac{1}{2}, s_2, \ldots, s_{n-1}\right)$ and $P$ denotes the parabolic subgroup of $\Gamma_{n-1}$ consisting of the form $\begin{pmatrix} \pm 1 & b \\ 0 & d \end{pmatrix}$ with $d \in \Gamma_{n-2}$.

D. Grenier [10, 12] found a fundamental domain $\mathfrak{F}_n$ for $\Gamma_n$ in $\mathfrak{P}_n$. The fundamental domain $\mathfrak{F}_n$ is precisely the set of all $Y = [v,x,W] \in \mathfrak{P}_n$ satisfying the following conditions (F1)-(F3):

- (F1) $(a + {}^t x c)^2 + v^{n/(n-1)} W[c] \ge 1$ for all $\begin{pmatrix} a & {}^t b \\ c & d \end{pmatrix} \in \Gamma_n$ with $a \in \mathbb{Z},\ b, c \in \mathbb{Z}^{(n-1,1)}$ and $d \in \mathbb{Z}^{(n-1,n-1)}$.
- (F2) $W \in \mathfrak{F}_{n-1}$.
- (F3) $0 \le x_1 \le \frac{1}{2}$, $|x_j| \le 2$ for $2 \le j \le n-2$. Here $x = {}^t(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{(n-1,1)}$.

For a positive real number $t > 0$, we define the *Siegel set* $\mathscr{S}_{t,1/2}$ by

$$\mathscr{S}_{t,1/2} := \left\{ Y \in \mathfrak{P}_n \,|\, y_i \ge t^{-1/2}\, (1 \le i \le n-1),\ |x_{ij}| \le \frac{1}{2}\, (1 \le i < j \le n) \right\}.$$

Here we used the coordinates on $\mathfrak{P}_n$ given in Formula (5.4).

Grenier [12] proved the following theorems:

**Theorem 5.1.** *Let*

$$\mathfrak{F}_n^\sharp := \bigcup_{\gamma\in D_n} \mathfrak{F}_n[\gamma] \subset \mathfrak{P}_n,$$

*where $D_n$ is the subgroup of $\Gamma_n$ consisting of diagonal matrices* $\mathrm{diag}(\pm 1, \ldots, \pm 1)$. *Then*

$$\mathscr{S}_{1,1/2} \subset \mathfrak{F}_n^\sharp \subset \mathscr{S}_{4/3,1/2}.$$

*Proof.* See Theorem 1 in [12, pp. 58–59]. □

**Theorem 5.2.** *Let*

$$\mathfrak{F}_n^* := \mathfrak{F}_n \cup \mathfrak{F}_{n-1} \cup \cdots \cup \mathfrak{F}_2 \cup \mathfrak{F}_1$$

*and*

$$V_n^* := V_n \cup V_{n-1} \cup \cdots \cup V_1 \cup V_0, \qquad V_n := \Gamma_n \backslash \mathfrak{P}_n.$$

*Then $\mathfrak{F}_n^*$ is a compact Hausdorff space whose topology is induced by the closure of $\mathfrak{F}_n$ in $\mathfrak{P}_n$, and $V_n^*$ is a compact Hausdorff space called the* **Satake compactification** *of $V_n$.*

*Proof.* See Theorem 3 in [12, pp. 62–65]. □

## 6. Stable automorphic forms for the general linear group

In this section, we introduce the concept of the stability of automorphic forms for the general linear group using the Grenier operator, and relate the stability of automorphic forms to the moduli space of principally polarized real tori and the Jacobian real locus.

**Definition 6.1.** Let $f \in A(\Gamma_n)$ be an automorphic form for $\Gamma_n$ with eigenvalues determined by $s = (s_1, \ldots, s_{n-1}) \in \mathbb{C}^{(n-1)}$. We set

$$\xi_1 = \frac{1}{n-1}\sum_{n=2}^{n-1}(n-k)s_k \qquad \text{(cf. Formula (5.6))}.$$

We define formally, for any $f \in A(\Gamma_n)$,

$$\mathfrak{L}_n f(W) := \lim_{v\longrightarrow\infty} v^{-s_1-\xi_1} f(Y), \quad v > 0,\ W \in \mathfrak{P}_{n-1},\ Y \in \mathfrak{P}_n, \tag{6.1}$$

where $Y$, $v$, $W$ are determined by the unique partial Iwasawa decomposition of $Y$ given by

$$Y = \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix} \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{\frac{1}{n-1}} W \end{pmatrix} \begin{pmatrix} 1 & {}^t x \\ 0 & I_{n-1} \end{pmatrix} \quad \text{(see (5.2))}.$$

D. Grenier [11] defined the formula (6.1) and proved the following result.

**Theorem 6.1.** *If $f \in A(\Gamma_n)$, then $\mathfrak{L}_n f \in A(\Gamma_{n-1})$. Thus $\mathfrak{L}_n$ is a linear mapping of $A(\Gamma_n)$ into $A(\Gamma_{n-1})$. Moreover if $f \in A_0(\Gamma_n)$ is a cusp form, then $\mathfrak{L}_n f = 0$. In general,* $\ker \mathfrak{L}_n \neq A_0(\Gamma_n)$.

*Proof.* The detailed proof can be found in [11, Theorem 2, pp. 472–473]. For the convenience of the reader, we sketch Grenier's proof. We consider the Fourier expansion (5.9) of $f$. Using the properties of the $K$-Bessel functions (cf. Formula (5.8)) and the Selberg power functions, Grenier showed that as $v \to \infty$,

$$f(Y) = f([v, x, W]) \sim a_0(v, W).$$

Therefore it suffices to show that

$$\mathfrak{L}_n f(W) := \lim_{v \to \infty} v^{-s_1 - \xi_1} a_0(v, W)$$

satisfies the properties (AF1), (AF2) and (AF3).

(1) Since $a_0(v, W)$ contains no $x = {}^t(x_1, \ldots, x_{n-1})$ terms, each $D_j$ reduces to $D_j = D_{j,v} + D_{j,W}$ $(1 \le j \le n)$, where $D_{j,v}$ operates on $v$ alone and $D_{j,W}$ is the corresponding $GL(n-1, \mathbb{R})$-invariant differential operator on $\mathfrak{P}_{n-1}$. Thus $\mathfrak{L}_n f(W)$ is a joint eigenfunction of $\mathbb{D}(\mathfrak{P}_{n-1})$.

(2) It is easily seen that $a_N(v, W[\gamma]) = a_{\gamma N}(v, W)$ for all $\gamma \in \Gamma_{n-1}$ (see Formula (5.7)). Taking $N = 0$, we get $a_0(v, W[\gamma]) = a_0(v, W)$ for all $\gamma \in \Gamma_{n-1}$. Thus $\mathfrak{L}_n f(W[\gamma]) = \mathfrak{L}_n f(W)$ for all $\gamma \in \Gamma_{n-1}$.

(3) There exist a constant $C > 0$ and $s \in \mathbb{C}^{n-1}$ with $s = (s_1, \ldots, c_{n-1})$ such that $|f(Y)| \le C |p_{-s}(Y)|$ as $\det Y_j \to \infty$ $(1 \le j \le n)$. Since

$$a_0(v, W) = \int_0^1 \int_0^1 \cdots \int_0^1 f([v, x, W]) dx,$$

we get

$$\begin{aligned}
|a_0(v, W)| &\le \int_0^1 \int_0^1 \cdots \int_0^1 |f([v, x, W])| \, dx \\
&\le C \int_0^1 \int_0^1 \cdots \int_0^1 |p_{-s}([v, x, W])| \, dx \\
&= C \int_0^1 \int_0^1 \cdots \int_0^1 |v^{s_1 + \xi_1}| \, |p_{-s'}(W)| \, dx \\
&= C \, |v^{s_1 + \xi_1}| \, |p_{-s'}(W)|,
\end{aligned}$$

where $s' = (s_2, \ldots, s_{n-1})$. Hence $|\mathfrak{L}_n f(W)| \le C \, |p_{-s'}(W)|$. □

*Remark* 6.1. The reason that $\ker \mathfrak{L}_n \neq A_0(\Gamma_n)$ in general is that $\mathfrak{L}_n$ is only one of several such operators associated with the various maximal parabolic subgroups.

For any $m, n \in \mathbb{Z}^+$ with $m < n$, we define

$$\xi_{m,n} : \Gamma_m \longrightarrow \Gamma_n$$

by

$$\xi_{m,n}(\gamma) := \begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m} \end{pmatrix}, \qquad \gamma \in \Gamma_m.$$

We let

$$\Gamma_\infty := \varinjlim_n \Gamma_n$$

be the inductive limit of the directed system $(\Gamma_n, \xi_{m,n})$.

**Definition 6.2.** A collection $(f_n)_{n\geq 1}$ is said to be a *stable automorphic form* for $\Gamma_\infty$ if it satisfies the following conditions (6.2) and (6.3):

$$f_n \in A(\Gamma_n), \quad n \geq 1 \tag{6.2}$$

and

$$\mathfrak{L}_{n+1} f_{n+1} = f_n, \quad n \geq 1. \tag{6.3}$$

Let

$$\mathbb{A}_\infty = A(\Gamma_\infty) := \varprojlim_n A(\Gamma_n)$$

be the inverse limit of the directed system $(A(\Gamma_n), \mathfrak{L}_n)$, that is, the space of all stable automorphic forms for $\Gamma_\infty$.

We propose the following problems.

**Problem 6.1.** Discuss the injectivity, the surjectivity and the bijectivity of $\mathfrak{L}_n$.

**Problem 6.2.** Give examples of stable automorphic forms for $\Gamma_\infty$.

**Problem 6.3.** Investigate the structure of $\mathbb{A}_\infty$.

*Remark* 6.2. In the classical case of Siegel modular forms, Freitag [6] showed that the ring structure of stable Siegel modular forms corresponding similarly to $\mathbb{A}_\infty$ is the polynomial ring in the theta series associated to irreducible, positive definite, unimodular even quadratic forms.

We give an example of stable automorphic forms for $\Gamma_\infty$.

**Theorem 6.2.** *Let $\{\alpha_n\}$ be the sequence such that if $\alpha_n = (s_1, s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$, then $\alpha_{n-1} = (s_2, s_3, \ldots, s_{n-1}) \in \mathbb{C}^{n-2}$ for any $n \in \mathbb{Z}^+$. Then $(E_n(\alpha_n, Y))$ is a stable automorphic form for $\Gamma_\infty$. Here $E_n(\alpha_n, Y)$ is the Selberg Eisenstein series defined by Formula* (5.5).

*Proof.* We prove the above theorem using the Grenier's result [11, p. 472]. Let

$$Y = [v, x, W] = \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)} W \end{pmatrix} \left[ \begin{pmatrix} 1 & {}^t x \\ 0 & I_{n-1} \end{pmatrix} \right].$$

Let $s = (s_1, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$. For any $\gamma = \begin{pmatrix} a & {}^t b \\ c & D \end{pmatrix} \in \Gamma_n$ with $a \in \mathbb{Z},\ b, c \in \mathbb{Z}^{(n-1,1)}$ and $D \in \mathbb{Z}^{(n-1,n-1)}$, we have

$$Y[\gamma] = {}^t\gamma Y \gamma = \begin{pmatrix} \alpha & q \\ {}^t q & R \end{pmatrix},$$

where

$$\alpha = v^{-1}(a + {}^t cx)^2 + v^{1/(n-1)} W[c],$$
$$q = v^{-1}(a + {}^t cx)({}^t b + {}^t xD) + v^{1/(n-1)}\, {}^t cWD,$$
$$R = v^{-1}(b + {}^t Dx)({}^t b + {}^t xD) + v^{1/(n-1)} W[D].$$

We observe that

$$Y[\gamma] = \begin{pmatrix} \alpha & 0 \\ 0 & R - \alpha^{-1}\, {}^t qq \end{pmatrix} \left[ \begin{pmatrix} 1 & \alpha^{-1} q \\ 0 & I_{n-1} \end{pmatrix} \right].$$

Let $\alpha_n = (s_1, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$. Set $\xi_1 = \frac{1}{n-1} \sum_{k=2}^{n-1} (n-k) s_k$. By Proposition 2.1, we see that

$$p_{-\alpha_n}(Y[\gamma]) = \alpha^{-(s_1+\xi_1)} p_{-\alpha_{n-1}}((v\alpha)^{\frac{2-n}{n-1}}\, Y_*),$$

where $\alpha_{n-1} = (s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-2}$ and

$$Y_* = W[(a + {}^t xc)D - c({}^t b + {}^t xD)].$$

It is easily seen that

$$p_{-\alpha_n}(Y[\gamma]) = v^{s_1+\xi_1} (v\alpha)^{-s_1+s_3+2s_4+\cdots+(n-3)s_{n-1}} p_{-\alpha_{n-1}}(Y_*).$$

We note that

$$v\alpha = (a + {}^t xc)^2 + v^{n/(n-1)} W[c].$$

Thus we have

$$\begin{aligned}
& E_n(\alpha_n, Y) \\
= & \sum_{\gamma \in \Gamma_n/\Gamma_\infty} p_{-\alpha_n}(Y[\gamma]) \\
= & \, v^{s_1+\xi_1} \sum_{\gamma \in \Gamma_n/\Gamma_\star} (v\alpha)^{-s_1+s_3+2s_4+\cdots+(n-3)s_{n-1}} p_{-\alpha_{n-1}}(Y_*) \\
= & \, v^{s_1+\xi_1} \sum_{\gamma \in \Gamma_n/\Gamma_\star} \left\{ (a + {}^t cx)^2 + v^{n/(n-1)} W[c] \right\}^{-s_1+s_3+2s_4+\cdots+(n-3)s_{n-1}} \\
& \times\, p_{-\alpha_{n-1}}(W[(a + {}^t xc)D - c({}^t b + {}^t xD)]),
\end{aligned}$$

where $\Gamma_\star$ is a subgroup of $\Gamma_n$ consisting of all upper triangular matrices. As $v \longrightarrow \infty$, if $s$ is chosen to make the exponent negative, all the terms with $c \neq 0$ approach to zero. If $c = 0$, then $\gamma = \begin{pmatrix} a & {}^t b \\ 0 & D \end{pmatrix} \in \Gamma_n$ and so $a = \pm 1$. Thus as $v \longrightarrow \infty$, we have

$$E_n(\alpha_n, Y) \sim v^{s_1+\xi_1} \sum_{\gamma \in \Gamma_{n-1}/\Gamma_\diamond} p_{-\alpha_{n-1}}(W[\gamma]) = v^{s_1+\xi_1} E_{n-1}(\alpha_{n-1}, W),$$

where $\Gamma_\diamond$ is a subgroup of $\Gamma_{n-1}$ consisting of all upper triangular matrices. Therefore

$$\mathfrak{L}_n E_n(\alpha_n, Y) = \lim_{v \to \infty} v^{-(s_1+\xi_1)} E_n(\alpha_n, Y) = E_{n-1}(\alpha_{n-1}, W).$$

Hence $(E_n(\alpha_n, Y))$ is a stable automorphic form for $\Gamma_\infty$. $\square$

Let

$$G_n = SL(n,\mathbb{R}), \quad K_n = SO(n,\mathbb{R}) \quad \text{and} \quad \Gamma_n = GL(n,\mathbb{Z})/\{\pm I_n\}.$$

We observe that $\Gamma_n = SL(n,\mathbb{Z})/\{\pm I_n\}$ if $n$ is even, and $\Gamma_n = SL(n,\mathbb{Z})$ if $n$ is odd.

Let

$$\mathfrak{X}_n := \Gamma_n\backslash\mathfrak{P}_n = \Gamma_n\backslash G_n/K_n \tag{6.4}$$

be the moduli space of *special* principally polarized real tori of dimension $n$.

For any $m, n \in \mathbb{Z}^+$, we define

$$\xi_{m,n} : G_m \longrightarrow G_n$$

by

$$\xi_{m,n}(A) := \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix}, \qquad A \in G_m. \tag{6.5}$$

We let

$$G_\infty := \varinjlim_n G_n, \qquad K_\infty := \varinjlim_n K_n \quad \text{and} \quad \Gamma_\infty := \varinjlim_n \Gamma_n$$

be the inductive limits of the directed systems $(G_n, \xi_{m,n})$, $(K_n, \xi_{m,n})$ and $(\Gamma_n, \xi_{m,n})$, respectively.

We recall the Jacobian locus $J_n$ (resp. the hyperelliptic locus $\mathrm{Hyp}_n$) in the Siegel modular variety $\mathcal{A}_n$ (see Section 4). We define

$$\mathscr{J}_{n,J} := \{Y \in \mathfrak{P}_n \,|\, \mathfrak{A}_Y \text{ is the Jacobian of a curve of genus } n, \ i.e., \ [\mathfrak{A}_Y] \in J_n\}$$

and

$$\mathscr{J}_{n,H} := \{Y \in \mathfrak{P}_n \,|\, \mathfrak{A}_Y \text{ is the Jacobian of a hyperelliptic curve of genus } n\}.$$

See Example 3.1 for the definition of $\mathfrak{A}_Y$. We see that $\Gamma_n$ acts on both $\mathscr{J}_{n,J}$ and $\mathscr{J}_{n,H}$ properly discontinuously. So we may define

$$\mathfrak{X}_{n,J} := \Gamma_n\backslash\mathscr{J}_{n,J} \qquad \text{and} \qquad \mathfrak{X}_{n,H} := \Gamma_n\backslash\mathscr{J}_{n,H}.$$

$\mathfrak{X}_{n,J}$ and $\mathfrak{X}_{n,H}$ are defined over the real numbers. $\mathfrak{X}_{n,J}$ and $\mathfrak{X}_{n,H}$ are called the *Jacobian real locus* and the *hyperelliptic real locus*, respectively.

**Problem 6.4.** Characterize the Jacobian real locus $\mathfrak{X}_{n,J}$. This problem may be the real version of the Schottky problem.

Let $\mathfrak{X}_n^S$ be the Satake compactification of $\mathfrak{X}_n$. We denote by $\mathfrak{X}_{n,J}^S$ (resp. $\mathfrak{X}_{n,H}^S$) the Satake compactification of $\mathfrak{X}_{n,J}$ (resp. $\mathfrak{X}_{n,H}$). We can show that $\mathfrak{X}_{n,J}^S$ (resp. $\mathfrak{X}_{n,H}^S$) is the closure of $\mathfrak{X}_{n,J}$ (resp. $\mathfrak{X}_{n,H}$) inside $\mathfrak{X}_n^S$. We have the following sequences

$$\mathfrak{X}_1^S \longrightarrow \mathfrak{X}_2^S \longrightarrow \mathfrak{X}_3^S \longrightarrow \cdots,$$

$$\mathfrak{X}_{1,J}^S \longrightarrow \mathfrak{X}_{2,J}^S \longrightarrow \mathfrak{X}_{3,J}^S \longrightarrow \cdots$$

and

$$\mathfrak{X}^S_{1,H} \longrightarrow \mathfrak{X}^S_{2,H} \longrightarrow \mathfrak{X}^S_{3,H} \longrightarrow \cdots .$$

As far as the author knows, nobody proved or disproved so far that the Satake compactifications $\mathfrak{X}^S_j \, (j \geq 1)$ are projective and also normal. We propose the following problems.

**Problem 6.5.** Are $\mathfrak{X}^S_j \, (j = 1, 2, 3, \ldots)$ projective and normal?

**Problem 6.6.** Discuss the injectivity and the surjectivity of the maps $\mathfrak{X}^S_j \longrightarrow \mathfrak{X}^S_{j+1} \, (j = 1, 2, 3, \ldots)$.

*Remark* 6.3. For the classical Satake compactification $\mathcal{A}^S_n := \overline{\Gamma_n \backslash \mathbb{H}_n} \, (n \geq 1)$, Baily [1] proved that each $\mathcal{A}^S_n \, (n \geq 1)$ is a projective and normal subvariety. For the more detailed discussion of this subject, we refer to Freitag's book [7, pp. 111–124].

We put

$$\mathfrak{X}^S_\infty := \varinjlim_n \mathfrak{X}^S_n, \quad \mathfrak{X}^S_{\infty,J} := \varinjlim_n \mathfrak{X}^S_{n,J} \quad \text{and} \quad \mathfrak{X}^S_{\infty,H} := \varinjlim_n \mathfrak{X}^S_{n,H}.$$

For any two positive integers $m, n \in \mathbb{Z}^+$ with $m < n$, we embed $\mathfrak{P}_m$ into $\mathfrak{P}_n$ as follows:

$$\psi_{m,n} : \mathfrak{P}_m \longrightarrow \mathfrak{P}_n, \qquad Y \mapsto \begin{pmatrix} Y & 0 \\ 0 & I_{n-m} \end{pmatrix}, \qquad Y \in X_m.$$

We let

$$\mathfrak{P}_\infty = \varinjlim_n \mathfrak{P}_n$$

be the inductive limit of the directed system $(\mathfrak{P}_n, \psi_{m,n})$. We can show that

$$\mathfrak{P}_\infty = G_\infty / K_\infty.$$

Now we have the Grenier operator

$$\mathfrak{L}_n : A(\Gamma_n) \longrightarrow A(\Gamma_{n-1})$$

defined by the formula (6.1).

**Definition 6.3.** An automorphic form $f \in A(\Gamma_n)$ is said to be a *Grenier-Schottky automorphic form* for the Jacobian real locus (resp. the hyperelliptic real locus) if it vanishes along $\mathfrak{X}_{n,J}$ (resp. $\mathfrak{X}_{n,H}$). A collection $(f_n)_{n \geq 1}$ is called a *stable Grenier-Schottky automorphic form* for the Jacobian real locus (resp. the hyperelliptic real locus) if it satisfies the following conditions (SGS1) and (SGS2):

(SGS1) $f_n$ is a Grenier-Schottky automorphic form for the Jacobian real locus (resp. the hyperelliptic real locus) for each $n \geq 1$.

(SGS2) $\mathfrak{L}_n f_n = f_{n-1}$ for all $n > 1$.

The following natural question arises:

**Question 6.1.** Are there stable Grenier-Schottky automorphic forms for the Jacobian real locus (resp. the hyperelliptic real locus)?

*Remark* 6.4. In the classical case for the Jacobian locus, Codogni and Shepherd-Barron [4] showed that there do not exist stable Schottky-Siegel modular forms. In the classical case for the hyperelliptic locus, Codogni [3] showed that there exist nontrivial stable Schottky-Siegel modular forms.

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