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RESEARCH ARTICLE

A method to overcome the double discontinuity of pre-service mathematics teachers: Focusing on didactic transpositions

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Abstract

Pre-service mathematics teachers who entered the College of Education experienced a disconnection between higher mathematics and school mathematics. Moreover, in-service mathematics teachers also experienced the disconnections in teaching school mathematics after graduating the university. These disconnections have been recognized as an issue for a very long time since Klein called it as double discontinuity in 1908. This study attempted a pedagogical approach to overcome the double discontinuity problem of pre-service mathematics teachers. According to the results of this study, pre-service mathematics teachers basically tried didactic transpositions based on textbooks in the pre-assignment. After the project, pre-service mathematics teachers attempted to make additional didactic transpositions based on the differences they recognized through project activities comparing the content between university mathematics and school mathematics. In addition, through teachers' impressions, it was confirmed that this project helped resolve the double disconnection that teachers were experiencing. Implications based on the findings were discussed.

Keywords: algebra, quadratic equation, collective case study, college mathematics

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I. INTRODUCTION

Teacher education is one of the most important methods to improve the quality of education for students (Ambussaidi & Yang, 2019; Kennedy et al., 2008; Wang et al., 2011). For better teacher education, numerous studies have examined what knowledge mathematics teachers need to have and how to develop them. Nevertheless, according to some recent research results (Liang et al., 2023), many teachers acknowledged that they do not effectively utilize the knowledge they have acquired at teacher training institutions and are conducting classes based on what they learned and how they were learned in secondary school. Even though they received education at teacher training institutes for 3-5 years, mathematics teachers had difficulties in using the knowledge they have learned from teacher training institutes.

In the Korean context, as noted in studies by Kang and Jun (2006) and Kwon (2019), there are more courses focusing on mathematics content knowledge than on pedagogical knowledge throughout the curriculum for pre-service mathematics teachers. Moreover, in the mathematics content courses, it was found that the higher mathematics was treated academically, but how it was converted into school mathematics was not well dealt with. For that reason, in-service teachers pointed out that the mathematics content courses had very low connection with school mathematics in evaluating the curriculum they experienced in the department of mathematics education at the college of education. According to a study by Hwang (2007), more than 70% of the in-service teachers evaluated that courses such as algebra and analysis that were taken at university were not related to the contents of school mathematics. In other words, in-service teachers perceived that the mathematics content knowledge they had learned at university did not directly affect their teaching activities.

The disconnect between university mathematics and school mathematics that mathematics teachers experience begins when they enter university. In other words, pre-service mathematics teachers who entered the college of education experienced a disconnection between higher mathematics and school mathematics. Moreover, in-service mathematics teachers also experienced the disconnections in teaching school mathematics after graduating the university. These disconnections have been recognized as an issue for a very long time since Klein called it as "double discontinuity" in 1908. The important reason the double discontinuity is considered problematic is that mathematics teachers cannot use the academic mathematics learned in the college of education in their teaching (Gueudet et al., 2016). To help mathematics teachers to more actively apply the mathematics contents they have learned during four years of college to school mathematics, it is necessary to overcome the problem of double discontinuity by revising and improving the curriculum and instructions of teacher training institutions. We attempted to provide a pedagogical approach to help pre-service mathematics teachers recognize and address the double discontinuity they might experience between university mathematics and school mathematics. While the double discontinuity was a multidimensional issue that cannot be fully 'overcome' in a simple or absolute sense, the goal of this study was to help pre-service teachers begin to bridge the gap by reflecting on the differences between academic and school mathematics and adapting their knowledge for teaching practice.

Nevertheless, methods for overcoming the double discontinuity have been studied very limitedly. Among domestic studies in Korea, Park (2009), Yang and Lee (2015), and Lee and Choi (2011) focused on introducing the content of the double discontinuity of pre-service mathematics teachers. Only the study by Lee and Kim (2016) focused on methods to overcome double discontinuity by comparing the school mathematics and university mathematics textbooks and examining the textbook factors that can cause double discontinuity. In the same manner, according to several international studies (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013; Isaev & Eichler, 2017; Kaiser & Buchholtz, 2014; Krauss et al. 2013; Winsløw & Grønbæk, 2014), most of them only present the problem of double discontinuity and suggest solutions. Only Winsløw and Grønbæk (2014) conducted an experimental study that implemented the program to overcome double discontinuity for pre-service teachers. As such, studies on the approaches for overcoming the double discontinuity of pre-service mathematics teachers have been very limited around the world.

Although many years have passed since Klein (1908) presented the problem of double discontinuity, pre-service mathematics teachers still have the same problem, and there is not enough educational approach to help them. Therefore, this study sought to explore ways to overcome the double discontinuity of pre-service mathematics teachers. In particular, among the double discontinuity defined by Klein (1908), this study focused on the discontinuity that pre-service mathematics teachers experience when they graduate from college of education and enter the school field. The reason why mathematics teachers experience discontinuity in the school field is because they lack the experience to understand the connection between university mathematics and school mathematics, and further transform university mathematics into school mathematics (Gueudet et al., 2016; Kaiser & Buchholtz, 2014). Both in-service and pre-service mathematics teachers have limited opportunities to engage in active didactical transposition while preparing for their lessons. Only a small number of teachers participate in curriculum development or textbook writing, while most teachers tend to conduct lessons based on pre-developed curricula and textbooks. This tendency suggests that mathematics teachers are more inclined to adopt the transformations made by others rather than actively engaging in their own transformative processes.

In this vein, this study provided pre-service mathematics teachers with opportunities to compare the contents of university mathematics and school mathematics and directly to convert university mathematics to school mathematics. The purpose of this research was to explore the specific types of transformations that students attempted through these experiences of didactical transposition. The findings from this exploration are expected to offer valuable insights for designing future curricula in teacher education programs. Specifically, this study provided an opportunity to pedagogically transform the content of 'relationship between roots and coefficients, for short, RBRC' in an algebra course for first-year university students. The 'relationship between roots and coefficients' is covered in both university and school mathematics, but there are differences in the background and context in which it is introduced between university and school mathematics. In this sense, it was judged to be suitable as a topic on which first-year university students can attempt didactic transpositions. Therefore, regarding this topic, the research attempted a pedagogical approach to overcome the double discontinuity problem of pre-service mathematics teachers, and intended to suggest implications by describing this case.

The research questions are as follows.

1. What kind of didactic transpositions (based on the textbook) did the pre-service mathematics teacher make in designing the lesson on 'relationship between roots and coefficients'?
2. What differences did pre-service mathematics teachers recognize between the content of university mathematics and school mathematics regarding the relationship between roots and coefficients?
3. After the project utilized in this study, what changes have been seen in the didactic transpositions of pre-service mathematics teachers?
4. What changes was shown in the pre-service teachers' beliefs regarding the double discontinuity through the project to solve double discontinuity?

II. LITERATURE REVIEW

Double Discontinuity

The pre-service teachers' double discontinuity between university mathematics and secondary school mathematics has been indicated by Klein (1908). As can be seen from the meaning of *double*, Klein (1908) described pre-service mathematics teachers experience discontinuity twice, one when entering university from secondary school, and the other when going back from university to secondary school (Liang et al., 2023). The first discontinuity that pre-service teachers face comes from entering college of education and learning higher mathematics. That is, pre-service mathematics teachers were interested in the mathematics they learned in middle and high school, and that motivated them to enter the university of education. However, when they entered to university from secondary school, they are struggle to learn the increased precision of mathematical language or abstract concepts or proofs as challenges for students in the transition from school to university (Eichler & Isaev, 2023). Right at this point, pre-service mathematics teachers perceive that the mathematics they learned at university was far from relevant to the mathematics they learned in middle and high school. In the other word, when entering college of education and learning higher mathematics, pre-service teachers have the belief that there is no relevance between school mathematics and university mathematics. This pre-service teachers' belief was referred as first discontinuity of double discontinuity (Kilpatrick, 2019).

After that, a second discontinuity occurs again when pre-service teachers graduate from college and teach students as secondary school teachers. They learned higher mathematics in college, but as they struggle to convert the mathematics they learned at that university into school mathematics, they again recognize no relevance between the mathematics they learned in university and school mathematics. In other words, the belief that university mathematics is no relevance to teachers' professional practice in the school and the university mathematics is not useful to teach the school mathematics was referred as the second discontinuity. These two discontinuities are what Klein (1908) called the double discontinuity.

Several previous studies (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013) have pointed out the double discontinuity of teachers as a problem. The important reason the double discontinuity is considered problematic is that mathematics teachers cannot use the academic mathematics learned in the college of education in their teaching. Considering that pre-service mathematics teachers spend a lot of time learning academic mathematics in the college of education, the fact that they do not recognize the relationship between academic mathematics and school mathematics and do not use it well in their lessons indicates that teacher education is very inefficient. In addition, double discontinuity makes mathematics teachers lose a deep understanding of academic mathematics in teaching school mathematics, and teach school mathematics based on the way how they learn in their own schooldays (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013).

Early studies (Isaev & Eichler, 2017; Lee & Kim, 2016) regarding double discontinuity focused on examining whether mathematics teachers had problems with double discontinuity. Whether mathematics teachers faced the double discontinuity problem was explored with various approaches. For example, Isaev and Eichler (2017) regarded the pre-service teachers' perception on the relationships between school mathematics and university mathematics as part of their beliefs. They developed a scale measuring pre-service teachers' beliefs on the connections between university mathematics and school mathematics, the relevance of university mathematics for the later profession as a school teacher, and the usefulness of university mathematics as a higher standpoint for elementary mathematics. Rather than directly investigating double discontinuity in mathematics teachers, there was also an effort (Lee & Kim, 2016) to find specific contents that could cause double discontinuity by examining the contents of textbooks. Lee and Kim (2016) tried to examine the discontinuity between university and secondary textbooks, especially on the topic of functions, which may cause the double discontinuity of pre-service mathematics teachers. In examining the double discontinuity problem of mathematics teachers, there have been studies (Liang et al., 2023) focusing on other perspectives other than mathematics content. According to Winsløw and Grønbæk (2014), double discontinuity includes three dimensions such as the institutional context (university vs. high school), the subject's role within the institution (secondary students vs. university student vs. secondary mathematics teacher), and the difference in mathematical content (higher vs. elementary). Among these dimensions, Liang et al. (2023) focused on the institutional context and the subject's role within the institution rather than the content dimension in examining the double discontinuity of pre-service mathematics teachers. As they indicated in their study, the studies exploring the two dimensions rather than the mathematical content are very limited with the exception of Liang et al. (2023).

On the other hand, research on how to overcome the double discontinuity faced by mathematics teachers was very limited. As the concept of double discontinuity implies, mathematics teachers experience two discontinuities: when they go from high school to college, and when they go back to school as teachers from college. At this time, considering that the key to determining the quality of mathematics education in school is the mathematics teacher, the discontinuity of the latter should be resolved more urgently. As one of the limited studies that explored how to overcome the double discontinuity problem of mathematics teachers, Lee and Kim (2016) analyzed the discontinuity between school mathematics and university mathematics textbooks, and suggested how to improve the textbooks for university mathematics. However, this study was not a study targeting teachers, and practical approaches to overcome double discontinuity were not addressed. Therefore, now is the time to go beyond exploring whether mathematics teachers face the double discontinuity problem and to study how to overcome it.

Didactic Transposition

Chevallard and Bosch (2020) conceptualized the process of didactic transposition as “the transformations an object or a body of knowledge undergoes from the moment it is produced, put into use, selected, and designed to be taught until it is actually taught in a given educational institution” (p. 170). Chevallard (1985) classified the didactic transposition into two major types such as external and internal didactic transposition. The process of transforming academic knowledge into knowledge to be taught was called external didactic transposition, and the process of transforming knowledge to be taught into knowledge taught in the actual teaching field was called internal didactic transposition. The didactic transposition model proposed by Chevallard (1985) showed a sequential transformation process in which academic knowledge is transformed into knowledge to be taught, and it is transformed into taught knowledge.

In the process of didactic transposition, teachers are key actors in particular, teachers have a direct influence on transforming the knowledge to be taught from textbooks or curriculum into taught knowledge. In the process of transforming mathematical knowledge presented in curriculum and textbooks into taught knowledge in class, teachers must make pedagogical decisions, such as selecting the content to be taught, how to teach it, and to what level the content will be covered (Østergaard, 2013). However, Chevallard and Bosch (2020) emphasized the transformation from academic knowledge to the knowledge to be taught, and enforced that the taught knowledge should be based on and resemble academic knowledge. In other words, although the taught knowledge is a transposition from knowledge to be taught, knowledge to be taught is from academic knowledge, and so that, taught knowledge should also not be disconnected from academic knowledge. In addition, it should be noted that the transformation between the knowledge to be taught and the taught knowledge occurs mainly in the classroom setting, and its subjects are teachers.

Therefore, teachers should be able to understand not only internal didactic transpositions but also external didactic transpositions. In other words, the mathematics teacher should understand mathematics as academic knowledge using various materials including mathematics learned at the university, and understand how it is reflected in the school mathematics curriculum and textbooks. Although the transformation from academic knowledge to knowledge to be taught is important, teachers' participation in curriculum and textbook development is very limited. This means that teachers have limited opportunities to engage in external didactic transpositions. Moreover, sometimes it is also necessary for teachers to transform the academic knowledge they have learned at university into the knowledge to be taught and then taught knowledge. For example, although the textbook is already the result of external didactic transposition, teachers should understand what academic mathematics each content of the textbook has been converted based on when selecting and modifying the contents of the textbook. In addition, in their classes, teachers should independently determine which the new contents will be added based on academic mathematics and convert it into taught knowledge. Nevertheless, in reality, teachers tend to focus only on developing classes based on knowledge to be taught, which is already transposed from academic knowledge by others such as curriculum and textbook developers, not based on relevance with academic knowledge.

Korean mathematics teachers also exhibit this tendency (Lee & Bae, 2013). Research on didactic transposition among Korean mathematics teachers primarily focuses on how teachers convert academic mathematical knowledge into a teachable form to effectively convey it to students. In general, mathematics teachers tended to teach students after making didactic changes based on the curriculum or textbook contents (Lee & Bae, 2013). Additionally, it has been found that mathematics teachers' didactic transposition is mainly influenced by factors related to mathematical concepts, textbooks, students, and external factors (Lee et al., 2017). To support teachers' understanding and self-reflection on their didactic transposition processes, Lee et al. (2017) proposed using reflective journals. Based on these previous studies, it seems necessary to explore opportunities for mathematics teachers to enhance their didactic transposition competencies by providing them with experiences that allow self-directed engagement with various factors affecting their didactic transposition.

According to the existing model developed by Chevallard and Bosch (2020), it is rare for a teacher to apply academic mathematics directly to the classroom and to convert it into the taught knowledge. The problem of mathematics teachers' double discontinuity discussed above is that they do not recognize the conversion between scholarly knowledge, the knowledge to be taught, and the taught knowledge in that it means the disconnection between academic mathematics and school mathematics (Chevallard & Bosch, 2020). Therefore, this study provides an opportunity to participate in the didactic transposition mentioned by Chevallard and Bosch (2020) as an attempt to prevent the double discontinuity problem of pre-service mathematics teachers, and explored how it affects their perception of the relationship between academic mathematics and school mathematics.

III. METHODS

Participants

The participants were 29 students enrolled in a College of Education at a university in Seoul, Korea. They were all pre-service mathematics teachers and were taking an algebra course in 2021. All 27 pre-service mathematics teachers except for two were in the first year, and the other two were in the third year. The first-year students had taken courses in algebra, set theory, and other general education courses, as well as introductory education courses. However, the majority of the participants had not yet taken courses specifically related to mathematics education."

Overview of the Study

This study designed a project with the main content of ‘didactic transposition' as an approach to overcome the double discontinuity of pre-service mathematics teachers. This study aimed to find out what kind of changes in didactic transpositions the pre-service mathematics teachers showed in designing their lessons before and after this project. The overall study process for this is shown in Figure 1.

Figure 1. The process of the study

First, before starting the project in earnest, pre-service mathematics teachers were asked to submit pre-assignments to write dialogues between teachers and students about ‘the relationship between roots and coefficients’ in the high school mathematics classrooms. Considering that the participants were in the first grade, rather than a formal lesson plans, dialogue that can be written easily while imagining the actual classroom was adopted as an appropriate task format. To perform a pre-assignment, the pre-service mathematics teachers were provided with some pages of a high school first year mathematics textbook (see Figure 2).

Next, in the algebra class conducted after the pre-assignment, the pre-service mathematics teachers learned the relationship between roots and coefficients. The contents of university mathematics on the relationship between roots and coefficients are as follows. First of all, the relationship between roots and coefficients in quadratic equations was introduced. Likewise, the relationship between roots and coefficients in a cubic equation was introduced, followed by the relationship between roots and coefficients in an *n*-th degree equation. In addition, the concepts of elementary symmetric expression, symmetric expression, alternating expression, and difference product were introduced, and the theorems on the relationship between these concepts were introduced and proved.

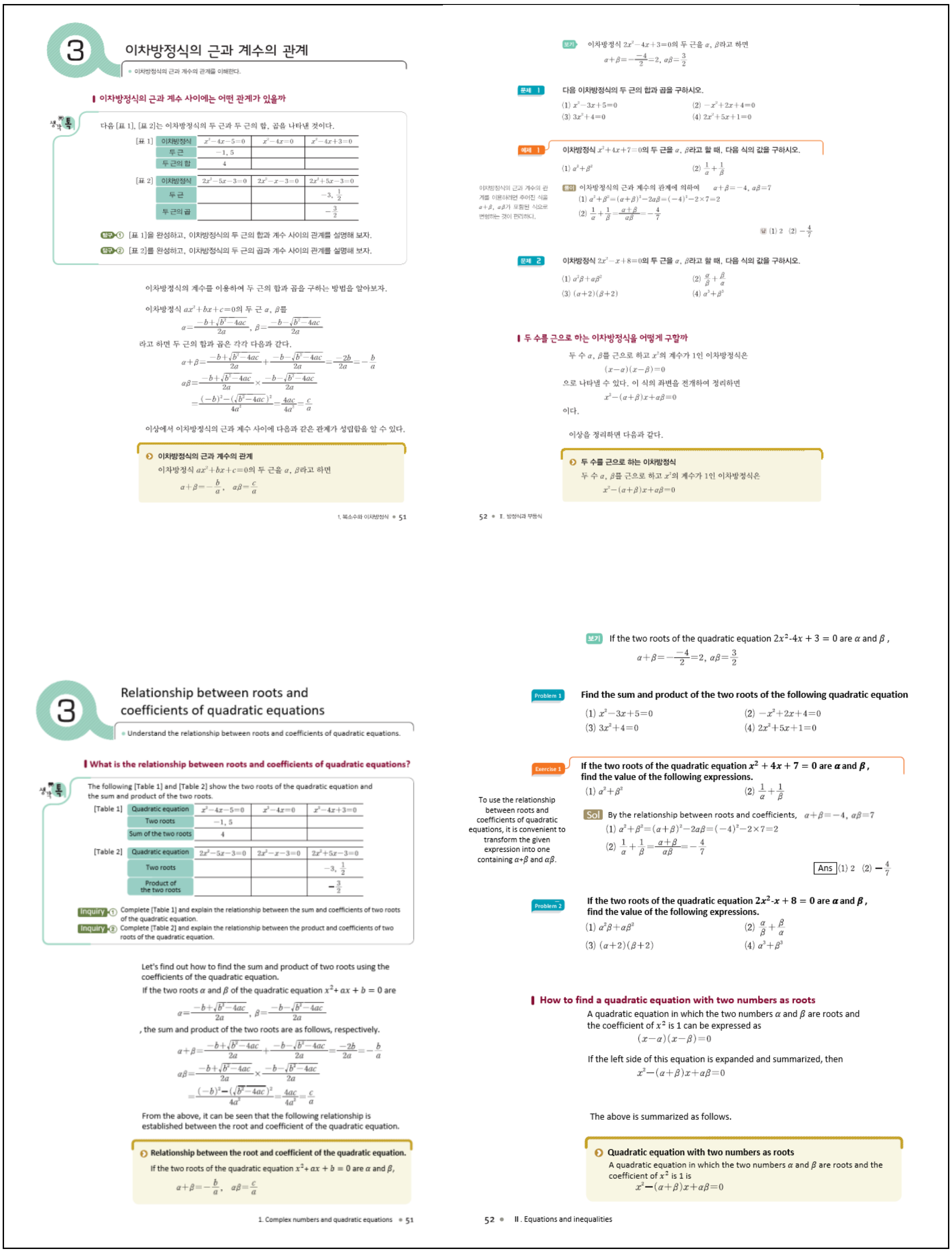

Figure 2. Textbook contents (top: original Korean version, bottom: translated English version)

After the lecture on algebra on the relationship between roots and coefficients, a project on didactic transposition was conducted as follows. First, studies on the disconnection (e.g., Hwang, 2007) between university and school mathematics that mathematics teachers were experiencing were introduced. Second, the concept of didactic transposition was explained with specific examples. Third, pre-service mathematics teachers had time to organize the contents of university mathematics that they learned about the relationship between roots and coefficients. Fourth, teachers had time to look at the school mathematics curriculum corresponding to the relationship between the root coefficient. After that, teachers also looked for contents on the relationship between the root coefficient in middle school textbooks. Finally, teachers have had time to check how university mathematics has been converted into school mathematics for each content based on the university mathematics contents previously organized by teachers. In other word, teachers have the opportunity to compare the contents dealt with in university mathematics and school mathematics with filling in a comparison table prepared by the instructor. Finally, the instructor has the pre-service mathematics teachers decide whether there were contents of university mathematics that were not covered in school mathematics that they wanted to transform for school mathematics classes, and if there were such contents, think about how to attempt a didactic transposition.

Based on these activities, the pre-service mathematics teachers were asked to submit the post-assignment was to modify, delete, and supplement the dialogues written in the pre-assignment and explain the reason. Finally, pre-service mathematics teachers wrote their impressions of the overall project in the final report.

Data Collection

The main data of the study were pre-service mathematics teachers' responses to the pre-assignment before the project, and the post-assignment and final report. In the pre-assignment, pre-service mathematics teachers imagined a situation in which the relationship between roots and coefficients was taught in high schools, and tried to describe it in a dialogue between the teacher and the student. Through this assignment, the researchers tried to examine what kind of didactic transpositions the pre-service mathematics teachers made by focusing on which learning elements. In the post-assignment, pre-service mathematics teachers were asked to provide a table comparing the contents of "the relationship between roots and coefficients" from the school mathematics and higher mathematics. Moreover, they were asked to revise the dialogues that they wrote for the pre-assignment and to explain why they did. In the final report, researchers let pre-service mathematics teachers fill out one open question. The open question was "Please write your overall impression of this project in 1000 words or less (including spaces). (Write whatever you want, such as what you learned from the project, what was helpful to you, what was disappointing about the project contents or the process, etc.) "

Data Analysis

The data analyses were implemented as follow. For the first research question, the dialogues that the pre-service mathematics teachers wrote in the pre-assignment were analyzed. The procedure of the data analysis followed the 5-step process suggested by Creswell & Poth (2018). First, most of the pre-service mathematics teachers submitted their assignments as word or PDF files, which were prepared as data for analysis. Second, the researchers quickly read all the pre-service mathematics teachers ' assignments to see what general ideas the dialogues submitted by all pre-service mathematics teachers represented. Third, coding was started to confirm what kind of didactic transpositions were shown in the pre-assignments submitted by pre-service mathematics teachers. Coding was carried out using the Nvivo 12 program. For coding, the class scene was used as an analysis unit, and each class scene was coded as a topic that encapsulated what kind of didactic transposition each class scene represents. Subsequently, the coded topics were listed and similar topics were combined into one topic. As a result, as shown in Table 1, a total of 10 topics appeared. After that, each class scenes were coded once again using 10 topics. Finally, the decision on the abbreviation for each topic was made (Tesch, 2013). By each topic that appeared frequently among each coded topic, representative cases were observed to confirm the common characteristics of the teacher's didactic transposition.

For the second research question, the researchers analyzed the comparison table between the contents of university mathematics and school mathematics written by the teachers and implemented coding in the same way as research question 1. After that, the frequency of the number of cases for each coding was measured, and representative characteristics were observed by classifying them together by similar content.

Table 1. Coded topics regarding the dialogue of pre- and post-assignments

	Coded Topics
Class scene of pre-assignments' dialogue	Proof of relationship between roots and coefficients
	Suggest learning objectives.
	Application of the relationship between roots and coefficients
	The usefulness of the relationship between roots and coefficients
	Simple calculation on a textbook problem using relationship between roots and coefficients
	Another specific example of the relationship formula between roots and coefficients
	Encourage students to predict the relationship between roots and coefficients on their own.
	Review of coefficients
	Review of a root or quadratic formula
	Review of a discriminant
Added or modified class scene of post-assignments' dialogue	Another way to introduce the relationship between roots and coefficients
	Expansion to equations of degree 3 and higher
	Introduction to Symmetric Expressions

For the third research question, pre-service mathematics teachers' responses to the post-assignment were analyzed. In the post-assignment, students were asked to add, remove, or revise the dialogues and explained why they did those changes. First of all, the changes made to dialogues were classified according to whether the changes were additions, deletions, or modifications to class scenes. Then, the newly added or modified class scenes that correspond to didactic transpositions were coded in the same way as in Research Question 1, and as shown in Table 1, a total of 3 coded topics were added newly. After that, a second classification was made according to which didactic transpositions the contents of the changes applied to the dialogues corresponded to.

For the fourth research question, pre-service mathematics teachers' responses to open-question of the final report were analyzed similar to Creswell & Poth (2018)'s analysis method used for the first research question. First, the researchers read all the responses in text form to open questions. Next, using the Nvivo 12 program, each sentence in all the responses was coded with topics containing the meaning of each sentence. The coded topics were made into a list, and the similar topics were categorized into higher level topics (see Table 2). Finally, the decision on the abbreviation for each topic was make (Tesch, 2013).

Table 2. Coded topics regarding teachers' responses to open-question of the final report

Initial Topics	Higher Level Topics
Prior belief on lack of relevance between the process of being prepared as a teacher and the actual curriculum of department of Mathematics Education	Gap between what PSTs thought on curriculum of department of Mathematics Education
Expectations for the project while there is skepticism about the curriculum	
Prior belief on lack of relevance between university mathematics and school mathematics	No relevance between university mathematics and school mathematics
Regret over focusing only on school mathematics in the pre-assignment when preparing for the lesson.	
Prior doubt about whether university mathematics can be connected to school mathematics	
Need to strengthen teacher competency for connection between university mathematics and school mathematics.	
Benefits from connecting university mathematics to school mathematics	Benefits and effects from connecting university mathematics to school mathematics
Recommend to expand the content of university mathematics and apply it by connecting it to school mathematics.	
Recommend to promote the project utilized in this study for future pre-service teachers	
Understanding the curriculum of mathematics in secondary school	

IV. FINDINGS

Didactic Transpositions in the Pre-Assignment

To see what kind of didactic transposition the pre-service mathematics teachers attempted, the dialogues they wrote in the pre-assignment were analyzed. As a result, a total of 10 kinds of didactic transposition were shown. Among them, the representative didactic transposition commonly shown by many pre-service mathematics teachers could be summarized into four categories as follows.

Proof of relationship between roots and coefficients. The most common didactic transposition presented by pre-service mathematics teachers (27 out of 29) was to prove the relationship between roots and coefficients. The way of proof used by pre-service mathematics teachers was to obtain the two roots of the quadratic equation using the formula, and then find the values of sum and product by directly adding and multiplying the two roots. (i.e., PST_2). This proof was the way presented in the textbook, and the pre-service mathematics teachers chose to transposed the contents of the textbook to their lessons design.

Teacher: We can find the roots of a quadratic equation by quadratic formula, right? So, in Table 1, how is the sum of the two roots obtained?

Student: Hmm... I think the method to obtain that is to multiply the coefficient of x by a minus.

Teacher: Good find! Let's check the sum of the two roots in Table 2. If you use the method found a while ago, won't you get the sum of the roots? So, what method should be used to obtain the sum of the roots in Table 2?

Student: Just divide the coefficient of x by the coefficient of x^2 and then multiply by minus.

Teacher: Now, let's generalize the method we found into a formula. Suppose the quadratic equation is $ax^2 + bx + c = 0$. So how do you get two roots?

Student: Through quadratic formula!

Teacher: That's right! So, if the roots of the quadratic equation are α and β, we know $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Then, the sum of the two roots, $\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{a}$, can be confirmed, right?

Teacher: Shall we check the product of two roots? You can see $\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) \times \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{c}{a}$, right?

< The dialogues of PST_2 >

Prior to proving the relationship between roots and coefficients, pre-service mathematics teachers provided students with an opportunity to guess the relationship between roots and coefficients using some specific examples. What was noteworthy in this process was that pre-service mathematics teachers paid attention to the types of misconceptions that students who were new to the relationship between roots and coefficients might have. Two major misconceptions that students could express were expected. First, 9 pre-service mathematics teachers predicted a misconception that did not consider the coefficient of the quadratic term in estimating the relationship between roots and coefficients (i.e., PST_2). Second, two pre-service math teachers expected the misconception of considering the sum of two roots as the absolute value rather than the negative value of the first term coefficient. In the case of textbooks presented by researchers to pre-service mathematics teachers, two quadratic equations $x^2 - 4x - 5 = 0$ and $2x^2 - 5x - 3 = 0$ were presented, leading students to consider both cases where the coefficient of x^2 is 1 and 2. However, since the coefficients of x were all negative quadratic equations, it was insufficient for students to explore the second misconception.

Suggest learning objectives. The second didactic transposition exhibited by many pre-service mathematics teachers was the presentation of learning objectives. All pre-service teachers participating in this study exhibited this didactic transposition. The learning objectives presented by the pre-service mathematics teachers were mainly "Today, you will learn about the relationship between the roots and coefficients of a quadratic equation (PST_2)." In this process, some pre-service mathematics teachers started to review the previous lessons. The concepts of 'root (19 PSTs),' 'coefficient (10 PSTs),' and 'discriminant (6 PSTs)' of quadratic equations were mainly reviewed, and after reviewing these concepts, learning objectives were presented.

Application of the relationship between roots and coefficients. The third didactic transposition that many pre-service mathematics teachers focused on was the aspect of utilizing the relationship between roots and coefficients. The first case of applying the relationship between roots and coefficients was to find a quadratic equation that satisfies the value of the sum and product of the two roots. Some pre-service mathematics teachers have directly derived a quadratic equation where the coefficient of x^2 was 1 using the sum and product of two roots, as shown in textbooks. On the other hand, some pre-service mathematics teachers informed students that the quadratic equation could be expressed as $y = (x - \alpha)(x - \beta)$ by assuming that the two roots were α and β for the quadratic equation where coefficient of x^2 was 1. Then, when the quadratic equation expressed as $y = (x - \alpha)(x - \beta)$ is expanded and rearranged, $y = (x - \alpha)(x - \beta)$ becomes $x^2 - (\alpha + \beta)x + \alpha\beta = 0$, which is the general form of the quadratic equation. Therefore, the teacher showed the students that the sum of the two roots in this equation is the negative value of the x coefficient, and the product of the two roots is equal to the constant term.

The second case of using the relationship between roots and coefficients was a problem in which the values of the sum and product of the two roots of a quadratic equation were presented, and then other values were calculated based on them. For example, some teachers have suggested the problem of how $\frac{1}{\alpha} + \frac{1}{\beta}$ is obtained when the root of $ax^2 + bx + c = 0$ is α and β. It was instructed that students could use the relationship between root and coefficient in the quadratic equation to obtain the sum and product of two roots and use them to calculate the value of the new expression. This problem are the types of problems presented in textbooks.

The usefulness of the relationship between roots and coefficients. A total of 22 pre-service mathematics teachers commented on how the relationship between roots and coefficients is useful. Pre-service mathematics teachers talked about the usefulness of the relationship between roots and coefficients in two aspects. The first aspect was that by using the relationship between roots and coefficients, the sum and product of two roots could be obtained without knowing each of the two roots, and the second aspect was that the quadratic equation could be inferred using the sum and product of the two roots. As for the first aspect, in the case of PST_1 who explained the usefulness of the relationship between roots and coefficients, he explained as follows.

Teacher: In this way, the values of various expressions can be obtained using the relationship between roots and coefficients without directly obtaining the roots using the formula for roots. If you directly obtain the root with the root formula and substitute it, the irrational root and the imaginary root can come out, so the calculation will be complicated, right? However, using the relationship between roots and coefficients, the values of various expressions can be obtained much more simply.

< The dialogues of PST_1 >

As a second aspect, PST_25 and PST_7 explained that the quadratic equation could be inferred through the sum and product of the two roots, especially in the case of PST_25, analogy was mentioned.

Teacher: Why did we learn this? What was useful in learning it?
Teacher: If we know the sum and the product of two roots, we can set up a quadratic equation immediately. For example, in $ax^2 + bx + c = 0$, a is 1. And the sum of the two roots is 4 and the product is 9. What do we know then?
Student: Coefficients!
Teacher: Right! So, what's each coefficient? What about b?
Student: b is -4, and c is 9!
Teacher: Yes, you didn't forget to put minus on the sum of the two roots! Isn't this like Sherlock Holmes? Detectives like Sherlock Holmes look at the room at the scene of the incident and find out how the crime happened and tell them. It was to look at the scene of the incident and reconstruct the case. Similarly, we can find what the original equation $ax^2 + bx + c = 0$ is by looking at the scene of the incident called the sum and product of two roots.

< The dialogues of PST_25 >

Teacher: Through this concept, we can set up a quadratic equation if we know the sum and product of the two real roots of a quadratic equation. If the sum of real roots is given as i and the product of real roots as j, how do we set up a quadratic equation? First, let k be the leading coefficient because we don't know the leading coefficient. So, we can put the quadratic equation as $k(x^2 - ix + j) = 0$, right?

< The dialogues of PST_7 >

Teachers' recognition to the difference between university mathematics and school mathematics

Through the project, the pre-service mathematics teachers compared the contents of university mathematics and school mathematics regarding the relationship between roots and coefficients. As a result, pre-service mathematics teachers recognized the difference between university mathematics and school mathematics regarding the relationship between roots and coefficients, shown in Table 3. In Table 3, teachers commonly mentioned the contents of university mathematics in a total of four areas. However, it was confirmed that the contents of school mathematics compared converted from each area of university mathematics are recognized slightly differently by teachers.

Table 3. Comparison the contents of university mathematics and school mathematics regarding the relationship between roots and coefficients

University Mathematics	School Mathematics	# of PSTs	
Assuming that the two roots of the quadratic equation are x_1 and x_2 & introducing the relationship between the root and the coefficient through factorization theorem	Proving the relationship between roots and coefficients using the quadratic formula	20	26
	Proving the relationship between roots and coefficients using the quadratic formula & Finding equations using factorization theorem	5	
	Finding equations using factorization theorem	1	
Introducing the relationship between the roots and coefficients using the factor theorem after assuming that the three roots of the cubic equation are x_1, x_2, and x_3	There is no mention of this in school mathematics.	21	24
	It is not mentioned in school mathematics, but it appears in commercial workbooks and most students learn it.	2	
	School mathematics does not mention the relationship between the roots and coefficients of a cubic equation itself, but it can be introduced in an example where factorization is possible.	1	
Relationship between roots and coefficients of the *n*th degree equation	There is no mention of this in school mathematics.	24	26
	In the case of higher-degree equations, the relationship between roots and coefficients is mentioned only when factoring can be done by the synthetic division.	2	
Symmetric expression	There is no mention of this in school mathematics.	23	24
	In school mathematics, the concept of symmetric expression is not introduced, but symmetric expressions are presented.	1	

Among the comparisons between university mathematics and school mathematics, the most pre-service mathematics teachers mentioned was the introduction of the relationship between root and coefficient. 26 pre-service teachers stated that in university mathematics, x1 and x2 are assumed to be roots, and the relationship between roots and coefficients is introduced through factorization theorem. Twenty of the 26 pre-service teachers recognized that in school mathematics, unlike university mathematics, the relationship between roots and coefficients was proved only by the quadratic formula. Five out of 26 students recognized that in school mathematics, the relationship between roots and coefficients was proved through the quadratic formula, and factorization theorem was used for finding equations rather than proving the relationship between roots and coefficients.

The relationship between roots and coefficients in the *n*th degree equation dealt with in university mathematics was also mentioned by the most pre-service mathematics teachers (26 PSTs). Twenty four out of 26 recognized that school mathematics did not mention the relationship between the root and coefficient of the *n*th degree equation. In the other hand, two out of 26 recognized that in the case of higher-degree equations, the relationship between roots and coefficients is mentioned only when factorization can be done by the synthetic division.

The relationship between roots and coefficients in the cubic equation dealt with in university mathematics was mentioned by 24 pre-service mathematics teachers. Similar to the aforementioned the *n*th degree equation, most teacher (21 PSTs) stated that the relationship between roots and coefficients in the cubic equation is not mentioned in school mathematics. One pre-service mathematics teachers recognized that the relationship between roots and coefficients in the cubic equation is mentioned only where factorization is possible. On the other hand, unlike in the *n*th equation, two pre-service mathematics teachers pointed out that the relationship between the root and coefficient of the cubic equation is not dealt with in schools, but is dealt with in commercial workbooks, so most students learn this content.

Changes in Didactic Transpositions in the Post- Assignment

Through the project, the pre-service mathematics teachers conducted activities to compare the contents of university mathematics and school mathematics regarding the relationship between roots and coefficients. Based on these activities, they had the opportunity to revise the dialogues submitted in the pre-assignment. The didactic transpositions they modified in the post-assignment mainly consisted of three aspects.

Another way to introduce the relationship between roots and coefficients. The part that the most pre-service mathematics teachers focused on in revising the dialogue after the project was how to introduce the relationship between roots and coefficients. In the past, according to the contents of the presented textbook, most pre-service mathematics teachers derived the relationship between roots and coefficients by finding the two roots of a quadratic equation using the quadratic formula, and then adding and multiplying them. On the other hand, after the project, instead of directly obtaining the two roots of the quadratic equation, they started the lesson by expressing the quadratic equation as $y = (x-\alpha)(x-\beta)$ by assuming that the two roots of the quadratic equation are α and β. Then, the pre-service mathematics teacher developed the equation as $(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta = 0$, and informed the sum and product of the two roots represent a specific relationship with the coefficients. In the pre-assignment, 9 pre-service mathematics teachers introduced this introduction, but in the post- assignment, additional 9 pre-service mathematics teachers introduced this introduction as well. For example, a PST_12, added the following part in a post-assignment.

Teacher: A quadratic equation with a coefficient of $x^2$of 1 is called $x^2 + bx + c = 0$. So, if we call the two roots α & β, how do we represent the quadratic equation in any other way?

Student: It's $(x - \alpha)(x - \beta) = 0$.
Teacher: Expansion of $(x - \alpha)(x - \beta) = 0$ results in $x^2 - (\alpha + \beta) + \alpha\beta = 0$. Let me compare it with the $x^2 + bx + c = 0$. What is b?
Student: It's $-(\alpha + \beta)$.
Teacher: What is c?
Student: It's $\alpha\beta$.
Teacher: Yes, if the coefficient of the quadratic term is a, we can express $ax^2 + bx + c = 0$, and if the two roots are α & β, we can express $a(x - \alpha)(x - \beta) = 0$.. So, if we expand it and compare each other, we can reconfirm the relationship between the roots and coefficients we learned earlier.

<The dialogues of PST_12>

Pre-service mathematics teachers who added this proof in the post-assignment took a method of additionally mentioning a new proof along with it, rather than deleting the proof using the quadratic formula presented in the textbook. What many pre-service teachers mentioned as the reason for introducing the proof method of university mathematics was that it was easy to understand and accept the 'relationship between roots and coefficients'. Another reason is that the proof method of university mathematics provides an opportunity to expand mathematical thinking view and for students to think about the relationship between roots and coefficients by extending to the equations of degree 3 and higher for students themselves.

Some pre-service mathematics teachers have added explanations about the meaning of other methods of proving the 'relationship between roots and coefficients'. In other words, these pre-service mathematics teachers explained that students can draw a relationship between roots and coefficients even when they do not know or remember the quadratic formula. For example, PST_1 asked students "What is the good point of using this method [a new proof that derives a relationship between roots and coefficients]?" and then, said "When we first derived the relationship between the roots and coefficients of a quadratic equation, we derived it using the quadratic formula. However, for equations above the cubic equation, it may be difficult to use a quadratic formula or there may not be the formula for the roots of the cubic equations! (Ellipsis) But, this method can derive the relationship between the root and coefficients of various equations [a cubic or higher degree equation]."

Expansion to equations of degree 3 and higher. Similar to PST_1 above, some teachers (i.e., PST_1, PST_11, PST_13, PST_25) tried to explain why the new method of deriving the relationship between roots and coefficients is useful, considering the relationship between roots and coefficients of cubic equations or higher equations. Currently, the formula for the roots of cubic equations is not dealt in Korea's middle and high school curriculum. Therefore, when teachers want to introduce the relationship between the root coefficient for the cubic equation, they inevitably consider the fact that they cannot use the method of proving the relationship between the root and coefficient presented in the textbooks.

Pre-service mathematics teachers who introduced the relationship between roots and coefficients for equations of cubic or higher in pre- and post-projects were 3 and 22, respectively. When introducing the relationship between the roots and coefficients of a cubic equation, 6 pre-service mathematics teachers introduced only the result without proof, and 8 pre-service mathematics teachers attempted to introduce the same as in the quadratic equation. In other words, assuming that the three roots were $\alpha,\ \beta,$ and γ, the cubic expression was derived as $(x-\alpha)(x-\beta)(x-\gamma)=0$, and then the relationship between the root and the coefficient was proved as follows.

Teacher: *Then, let's find the relationship between roots and coefficients in the cubic equation, although it doesn't come out of our textbook? So far, the first method of finding the relationship between roots and coefficients in quadratic equations is to use root formulas, and the second method is to transform quadratic equation into factorization form. So what method should we use for cubic equations?*

Student: *The second method!*

Teacher: *Since we don't know the root formula for cubic equations, it's correct to use the second method, right? Now, let's express the equation. If the roots of the cubic equation* $ax^3+bx^2+cx+d=0$ *are* x_1, x_2, x_3, *how can they be expressed in the form of a factorized expression?*

Student: *It can be expressed as* $a(x-x_1)(x-x_2)(x-x_3)=0$.

Teacher: *Right, then let's expand the equation. Then the equation appears as* $ax^3-a(x_1+x_2+x_3)+a(x_1x_2+x_2x_3+x_3x_1)-ax_1x_2x_3=0$*? Now, we can express the sum of three roots, the sum of the products of two roots, and the product of three roots as coefficients. Anyone want to express it?*

Student: *The sum of the three roots is* $x_1+x_2+x_3=-\frac{b}{a}$ *, the sum of the products of the two roots* is $x_1x_2+x_2x_3+x_3x_1=\frac{c}{a}$*, and the product of the three roots is* $x_1x_2x_3=-\frac{d}{a}$.

Teacher: *That's right. That's all for today. For those who want to study more about the relationship between roots and coefficients, let's find out how the relationship between roots and coefficients appears and is expressed in 4th, 5th, and n-th order equations. Only those who want can do this~*

<The dialogues of PST_2>

One of the reasons why teachers additionally introduced the relationship with the root coefficient in the cubic equation is that it induces students' interest. Another reason is that the root and coefficient relationship in the cubic equation is openly appearing in commercial workbooks, as mentioned in the project activity comparing university mathematics and school mathematics.

Some of these (i.e., PST_2, PST_5, PST_7, PST_13, PST_15, PST_22, PST_25, PST_27) also introduced the RBRC of the *n*-th order equations. All but two of them took the logic of expanding from the RBRC of the quadratic equation to the RBRC of the cubic equation, and then expanding to the RBRC of the *n*-th equation.

Teacher: But looking at the relationship between the root and coefficient of the cubic equation, doesn't the relationship between the root and coefficient of the quadratic equation look different? What's the rule? If you write down polynomials in descending order, except for the coefficient of the highest order, what rules are there for the first, the second, and the third coefficient? For five minutes now, let's think with the partner next to each other.
(Ellipsis)
…
Teacher: Shall we drive the momentum and push for finding the relationship between the roots and coefficients of the n-th equation? If we use what we learned now, we can get it quickly! Let's try it with courage! When the n-th equation is $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 = 0$*, what does the first coefficient mean except for the leading coefficient?*
Student: It means the sum of all roots.
Teacher: What does the second coefficient mean?
(Ellipsis)
…

< The dialogues of PST_25>

Introduction to symmetric expressions. Some PSTs who have extended RBRC to equations of third or higher order have added an introduction to "symmetric expressions" in their classes. There was one pre-service mathematics teacher who mentioned symmetry in a pre-assignment. On the other hand, there were a total of 5 pre-service mathematics teachers who mentioned symmetry equations in the post- assignment. Most of these introduced the term along with the concept of symmetric expressions, but PST_18 did not directly mention the term of ‘symmetrical expressions’, but tried to mention the properties of symmetrical expressions through the following episode.

Student: Teacher! I have a question!! The problem didn't tell me what α and β are respectively, so it doesn't matter if the two change??
Teacher: That's a good question!! If we look at the linear equation $x + y = 2$ *with unknown x and y, it doesn't matter if the values of x and y change, right??*
Student: Yes!!
Teacher: And it doesn't matter if we change x and y when $xy = 5$*, right?*
Student: Yes!!
Teacher: In this way, there are cases where the answer is the same even if the two unknowns are exchanged for each other. The $x + y$ *and* xy *that you just heard are examples, and this example corresponds to b and c of the quadratic equation!! So, we can set up a quadratic equation even if we don't know what the values of α and β are!! Thank you for the good question.*

<The dialogues of PST_18>

Regarding the reason for adding the lesson about symmetrical expressions, PST_18 stated, "When the roots are given as letters, some students may question whether the two letters can be used interchangeably [whether it is possible to change the order of two letters and add them], so I summarized the concept of symmetrical expressions in an easy and simple way to explain why they are interchangeable."

Four other pre-service mathematics teachers explained that the equation holds even when the variables change in the equation representing the relationship between roots and coefficients, and referred to it as a "symmetric equation."

Teacher: *Let the three roots of the cubic equation* $f(x) = ax^3 + bx^2 + cx + d = 0$ $(a \neq 0)$ *be α, β, and γ. Then it becomes* $f(x) = a(x-\alpha)(x-\beta)(x-\gamma)$ *in the way we just described, right? Let's expand this equation and talk about the relationship between roots and coefficients?*

Student: *It's* $-\frac{b}{a} = \alpha + \beta + \gamma$, $\frac{c}{a} = \alpha\beta + \beta\gamma + \gamma\alpha$, $-\frac{d}{a} = \alpha\beta\gamma$*!*

Teacher: *Very good. By the way, the value does not change even if we choose two variables among α&β, and γ and change the positions of each variable, right? We call this 'the Elementary Symmetric Polynomia.' Even in the case of quartic equations, this Elementary Symmetric Polynomial can be used to find the relationship between roots and coefficients as follows without complex expansion* $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$, $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$, $\alpha\beta\gamma\delta = \frac{e}{a}$

<The dialogues of PST_13 >

PST_13 stated the following to explain the reason for adding RBRC for 3rd and nth degree equations in the post-assignment, and then adding the concept of symmetric equations.

"In order to ensure that [RBRC] is not limited to quadratic equations, we provide a brief introduction to symmetric equations and also provide an understanding of the relationships between roots and coefficients of cubic and quartic equations. [In addition, we hope that these contents] will be useful to students in their second-grade course in addition to expanding their thinking skills."

PSTs' impressions

After completing the project, the pre-service mathematics teachers were asked to write down their overall impressions before and after the project. Pre-service mathematics teachers shared their thoughts on three major topics. Of these three topics, two topics were about the gap between what PSTs thought before entering the Department of Mathematics Education at the College of Education and after entering. The two gaps they encountered were about the curriculum and the gap between university mathematics and school mathematics. Lastly, the topic mentioned by pre-service mathematics teachers was about the benefits and effects from connecting university mathematics to school mathematics.

First, pre-service mathematics teachers expressed their thoughts that the curriculum of the mathematics education department was somewhat different from what they expected.

While attending the department of mathematics education, I used to tell my friends in other departments what subjects I was learning. And the responses I always get back from my friends are similar, "Why are you learning something like that in the mathematics education department?" "Don't you need to learn how to teach high school math well?" (PST_8)

I was already feeling skeptical while studying math in college. Is this class really necessary for me to become a teacher, or is it just a class used as a means to pass the employment exam? I think it was a semester in which I had a lot of worries about my major. (PST_10)

I want to develop my skills to become a teacher. In fact, my friends asked me if going to the mathematics education department would help them more during private tutoring classes. When every time they asked me, I would reply that it wasn't much help. (PST_19)

They stated that they expected to take courses related to mathematics education when entering department of mathematics education, but that they experienced the gap as they mainly took courses related to university mathematics. Pre-service mathematics teachers gradually became tired of the curriculum of mathematics education departments that focused on university mathematics, or began to worry about whether a mathematics education major was right for them. Additionally, while taking university mathematics courses in the department of mathematics education, pre-service mathematics teachers had the cynical thought that they would not be helpful to them.

Second, pre-service mathematics teachers mentioned their belief that the mathematics they learned in college had no relevance to the mathematics they thought about during their school days.

While taking 'Set Theory,' the first major subject after entering university, I have studied with questions such as "This is not the mathematics I thought, " "It is really different from the mathematics I have done so far," and "What can I use this for?" (PST_4)

When I entered university, I felt that the mathematics I was learning was a completely different field. When studying isn't as easy as it seems, I wonder, "How will the university mathematics I'm learning now apply when I become a teacher?" I also even had the question, "Isn't it better to just study high school mathematics well?" (PST_14)

Prior to the project, while learning about university mathematics, I thought there was little connection with secondary mathematics, and I was contemplating the necessity of learning university mathematics to become a secondary teacher. (PST_20)

Lastly, pre-service mathematics teachers mentioned that project connecting university mathematics to school mathematics are helpful for cultivating the professionalism as the mathematics teachers.

In the process of transposing university mathematics including many advanced terms to

> *school mathematics, I personally thought this project was very meaningful in that it led the concern about the connection between university mathematics and school mathematics to the idea of how best to proceed with the class in terms of content and composition when I came to the in-service mathematics teacher later on. (PST_29)*
>
> *Through this project, my attitude toward taking major classes has changed. Rather than thinking that major classes are about difficult mathematics, I found myself taking classes thinking, "Ah! I think it would be good to teach these contents in __ subjects of __ grade later on." Even if the mathematics content becomes more difficult as the grade goes up, I will learn university mathematics hardly without being afraid, thinking the students who will be taught when I stand in the future teaching profession. (PST_5)*

V. DISCUSSIONS

In order to overcome the double discontinuity of mathematics teachers, this study conducted a project dealing with didactic transposition to examine what changes occurred. In particular, among the various contents of algebra, cases dealing with 'relationship between roots and coefficients' were explored, and based on this, implications for developing the curriculum of teacher education institutions were suggested. Among the problems of double-discontinuity defined by Klein et al. (2016), this study focused on the problem of disconnection experienced by pre-service mathematics teachers after graduating from college and entering the school field. In addition, among the three areas of double disconnection presented by Winsløw and Grønbæk (2014), this study focused on the ‘mathematical content’ area.

According to the results of this study, the change in the attitude of pre-service teachers regarding to the didactic transformation was characterized by a change from passive to active. Pre-service mathematics teachers basically tried didactic transpositions based on textbooks in the pre-assignment. However, in the post-assignment, after the didactic transposition project, pre-service mathematics teachers tried to convert the contents from university mathematics that were not covered in school mathematics, and apply them to their high school classrooms. For example, it is about cubic-higher equation, symmetric, and new proofs, etc.

The didactic transposition that showed the biggest change was the process of proving the relationship between roots and coefficients. Prior to didactic transposition project, most pre-service mathematics teachers used the method presented in textbooks to prove the relationship between roots and coefficients. On the other hand, after the project, along with the proof method presented in the textbook, the pre-service mathematics teachers tried to additionally explain the proof presented in the university mathematics textbook. The proof presented in secondary school textbooks is a method of directly obtaining two roots using the quadratic formula and then calculating their sum and product. On the other hand, the proof presented in university mathematics is a method of proving by setting the two roots as variables instead of directly obtaining the two roots, expressing them as a factor theorem, and expanding them. The purpose of exploring the relationship between roots and coefficients is to find out that a specific relationship exists between the coefficient of an equation and the sum or product of two roots without knowing the two roots. The proof method presented in secondary school textbooks does not reflect the meaning well because it derives the relationship between the root and the coefficient by directly obtaining the two roots.

If this algebraic meaning or context is not understood, university mathematics education for pre-service mathematics teachers has no choice but to remain limited to arithmetic exercises. When teaching the relationship between roots and coefficients of quadratic equations in secondary education, the mathematics teachers' knowledge directly affects students' learning (Hiebert & Grouws, 2007; Hill et al., 2005). In other words, in recognizing the relationship between the roots and coefficients of the quadratic equation explored in this study, mathematics teachers may approach it as a way to examine the structure of the equation, or may stop at performing algorithmic calculations using the relationship between roots and coefficients of the quadratic equation [RRCQE]. Which of these two ways a mathematics teacher recognizes the relationship between roots and coefficients will make a big difference in student learning. In that respect, at least mathematics teachers should know the context and background of the content of school mathematics, and this is knowledge that can be obtained through university mathematics. In this context, the fact that pre-service mathematics teachers introduced the approach of university mathematics in proving the relationship between the roots and coefficients of quadratic equations can be interpreted as an approach to apply the algebraic structure in school mathematics classes.

On the other hand, there were also one teacher who wanted to intentionally introduce a relationship with the root coefficient as a textbook centered approach rather than a university mathematics approach after the project. The teacher judged that the textbook centered approach was easier. As such, the pedagogical approach teachers take to the certain concept may differ. Teachers may have different perspectives on whether or not it is better to apply the university mathematics approach according to a concept. Therefore, pre-service mathematics teachers will have to discuss deeply during university education to derive the most appropriate pedagogical approach in the process of transferring university mathematics to school mathematics (Lee & Kim, 2016). And the curriculum of university education will have to be reorganized into a process that enables this discussion.

According to the results of this study, pre-service mathematics teachers attempted to make additional didactic transpositions by focusing on the differences they recognized through activities comparing the content between university mathematics and school mathematics. The three contents that pre-service mathematics teachers tried to make additional pedagogical changes when revising the class dialogues were consistent with what they commonly pointed out in the content comparison between university mathematics and school mathematics. In other words, the difference between the contents of university mathematics and school mathematics indicated out by pre-service mathematics teachers might be a point of double discontinuities that they can recognize when teaching students at school. In this study, pre-service mathematics teachers were able to have the opportunity to connect the content of university mathematics and school mathematics by focusing on the differences between the contents of university mathematics and school mathematics, and making additional didactic transpositions. Moreover, it is significant that, rather than just connecting university mathematics and school mathematics, pre-service mathematics teachers made efforts to convert the content of university mathematics, which does not appear in school mathematics, into the content of school mathematics. Even though the content of university mathematics is not included in the school mathematics curriculum, it is expected that through the process of attempting didactic transposition, pre-service mathematics teachers would have had the opportunity to think about the context and background of mathematics content.

Finally, as a result of analyzing the impression of pre-service mathematics teachers on this project, it was confirmed that the problem of double discontinuity that teachers face was significantly resolved through this project. As confirmed in the impression, most pre-service mathematics teachers were skeptical about why they were learning university mathematics in preparation for future teachers. Several studies (Eichler & Isaev, 2023; Hwang, 2007; Lee & Kim 2016) have also mentioned this issue. As Gueudet et al. (2016)'s research also mentioned why this problem arises, the main cause of this problem was that the connection between university mathematics and school mathematics is not well addressed in the university education for pre-service teachers. In other word, the obstacle to understanding the connection between university mathematics and school mathematics is the university curriculum that separates university mathematics and school mathematics (Eichler & Isaev, 2023). Even, pedagogical lectures conducted in university education actually mainly deal with education based on school mathematics, and there is little about how university mathematics is converted or applied to school mathematics. In fact, there are also few university lecture books where pre-service mathematics teachers can learn university mathematics by considering the connection to and applicability with school mathematics (Lee & Kim, 2016; Park, 2009). Most lectures on university mathematics usually use books that focus on mathematics academically (Lee & Kim, 2016). Admittedly, school mathematics are limited in comparison to mathematics that may have been acquired at university (Grenier-Boley & Robert, 2024). It may be difficult to directly draw on the university mathematics in secondary school (Grenier-Boley & Robert, 2024). However, many teachers said in their impressions that the activities that tried to make a didactic transposition through the connection between university mathematics and school mathematics helped them realize the value and usefulness of learning university mathematics. Therefore, it is expected that the problem of double disconnection of many teachers will be resolved if lectures on didactic transposition by considering the connection between university mathematics and school mathematics are given in university lectures (Eichler & Isaev, 2023). In addition, it will be urgent in the future to develop university books on the connection between university mathematics and school mathematics for these lectures to be conducted well in practice.

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