Korean J. Math. **32** (2024), No. 4, pp. 593–614
https://dx.doi.org/10.11568/kjm.2024.32.4.593

ON CLASSES OF INDEFINITE β-KENMOTSU STATISTICAL MANIFOLD

Shagun Bhatti and Jasleen Kaur*

Abstract. This paper introduces the notion of lightlike hypersurfaces for a novel class of manifolds known as an indefinite nearly β-Kenmotsu statistical manifold and explores the associated geometric properties. It establishes results on the screen totally geodesic and screen totally umbilical lightlike hypersurfaces. It delineates the structure of the recurrent, Lie-recurrent and nearly recurrent structure tensor fields of lightlike hypersurfaces of an indefinite nearly $\beta-$Kenmotsu statistical manifold. Additionally, the geometry of leaves of integrable distributions of lightlike hypersurfaces in an indefinite β-Kenmotsu statistical manifold tangent to the structure vector field has been researched.

1. Introduction

In the realm of contact manifolds, various classes of almost contact metric manifolds have been periodically examined. [24] presented a new class of almost contact metric structure, known as trans-Sasakian structure and explored its geometry. [10] introduced the $\mathbb{C}_5$ and $\mathbb{C}_6$-structures, two subclasses of trans-Sasakian structures that contain the Sasakian and Kenmotsu structures, respectively. Further, a nearly trans-Sasakian structure of type (α, β) was established by [16], which generalizes the trans-Sasakian structure . A nearly trans-Sasakian of type (α, β) is nearly-Sasakian or nearly Kenmotsu or nearly cosymplectic if $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$, respectively. Therefore, a class of almost contact manifolds known as nearly Kenmotsu manifolds was introduced by [26]. Various researchers studied the theory of nearly Kenmotsu manifolds extensively in [1], [8], [4] and [19]. Afterwards, the indefinite nearly trans-Sasakian manifolds were investigated by [18] wherein the geometry of lightlike hypersurfaces was explored and results for recurrent, nearly recurrent and Lie recurrent structure tensor fields developed.

Lightlike hypersurfaces, being an intriguing branch of geometry, have numerous applications in various branches of mathematics and physics. [11] formulated the theory

Received April 8, 2024. Revised October 5, 2024. Accepted October 7, 2024.
2010 Mathematics Subject Classification: 53D10, 53D15, 58A05, 58B20.
Key words and phrases: Lightlike hypersurfaces, leaves, indefinite nearly β-Kenmotsu statistical manifold.
∗ Corresponding author.

of a lightlike hypersurfaces of a proper semi-Riemannian manifold which was subsequently investigated by various geometers [27], [12]. Thereafter, the lightlike hypersurfaces of various almost contact metric manifolds such as indefinite Kenmotsu and trans-Sasakian manifolds were studied by [21], [22], [23] and [17].

The study of geometric structures on a set of certain probability distributions resulted in the formation of an interesting branch of manifolds known as statistical manifolds which have been investigated by [2], [3], [20] et.al. Furuhata [13] made significant contributions to the initiation of the geometry of hypersurfaces in statistical manifolds. The concept of Sasakian statistical manifold and Kenmotsu statistical manifold were introduced by [15] and [14], respectively. They constructed certain results related to the real hypersurfaces and warped product of statistical manifolds. [5] introduced the lightlike hypersurfaces of an indefinite Sasakian statistical manifold. Further [6], [7] studied the induced geometric objects and developed the curvature identities on lightlike hypersurfaces of statistical manifold.

From this perspective, the present research work introduces the concept of an indefinite nearly $\beta-$Kenmotsu statistical manifold and delves into its geometric properties. The theory of lightlike hypersurfaces within the framework of the indefinite nearly $\beta-$Kenmotsu statistical manifold has been investigated. Some assertions for a lightlike hypersurface to be totally umbilical and screen totally umbilical have also been provided. The structure of the recurrent, Lie-recurrent and nearly recurrent structure tensor fields of lightlike hypersurfaces in the indefinite nearly $\beta-$Kenmotsu statistical manifold has been characterized. Furthermore, the lightlike theory of leaves of integrable distributions in indefinite Kenmotsu statistical manifold has been examined and several structural theorems pertaining to its geometry have been formulated.

2. Preliminaries

Consider a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $(2n+1)$. If $\bar{g}$ is a semi-Riemannian metric, ϕ is a $(1,1)$ tensor field, ν is a characteristic vector field and η is a 1-form, such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \qquad \bar{g}(\nu,\nu) = 1, \tag{1}$$

$$\phi^2(X) = -X + \eta(X)\nu, \quad \bar{g}(X,\nu) = \eta(X), \quad \bar{g}(\phi X, Y) + \bar{g}(X, \phi Y) = 0 \tag{2}$$

which follows that $\phi\nu = 0$ and $\eta o \phi = 0$ for all $X, Y \in \Gamma(T\bar{M})$, then $(\phi, \nu, \bar{g})$ is called an almost contact metric structure on $\bar{M}$.

Definition 2.1. [18] An almost contact metric structure on $\bar{M}$ is called an indefinite nearly β-Kenmotsu structure if

$$(\hat{\bar{\nabla}}_X \phi)Y + (\hat{\bar{\nabla}}_Y \phi)X = -\beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \tag{3}$$

holds for any $X, Y \in \Gamma(T\bar{M})$, where $\hat{\bar{\nabla}}$ is a Levi-Civita Connection and β is a smooth function. Therefore, $(\bar{M}, \phi, \nu, \eta, \bar{g})$ is called an indefinite nearly β-Kenmotsu manifold.

We note that, if $\beta = 0$, then $\bar{M}$ is said to be an indefinite nearly cosymplectic manifold.

Let (M, g) be a hypersurface of $(\bar{M}, \bar{g})$ with $g = \bar{g} \mid M$. If the induced metric g on M is degenerate, then M is called a lightlike or degenerate hypersurface of $\bar{M}$. There exists a vector field $\xi \neq 0$ on M such that $g(\xi, X) = 0$, for all $X \in \Gamma(TM)$. The null space or radical space of $T_x(M)$ at each point $x \in M$ is a subspace $Rad(T_xM)$ defined as

$$Rad(T_xM) = \{\xi \in T_x(M) : g_x(\xi, X) = 0 \; for \; all \; X \in \Gamma(TM)\}$$

whose dimension is called the nullity degree of g.

Since g is degenerate and any null vector is perpendicular to itself, therefore $T_xM^\perp$ is also null and

$$Rad(T_xM) = T_xM \cap T_xM^\perp.$$

For a hypersurface M, dimension of $T_xM^\perp$ equals 1 which implies that the dimension of $Rad(T_xM)$ is also 1 and $Rad(T_xM) = T_xM^\perp$. Here $Rad(TM)$ is called a radical distribution of M.

Consider $S(TM)$, screen distribution, as a complementary vector bundle of $Rad(TM)$ in TM, such that

$$TM = Rad(TM) \perp S(TM) \tag{4}$$

It follows that $S(TM)$ is a non-degenerate distribution. Thus,

$$T\bar{M}_{|M} = S(TM) \perp S(TM)^\perp$$

where $S(TM)^\perp$, known as screen transversal vector bundle, is the orthogonal complement to $S(TM)$ in $T\bar{M}_{|M}$.

Theorem 2.2. [11] *Let (M, g) be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Then there exists a unique vector bundle $tr(TM)$ known as lightlike transversal vector bundle of rank 1 over M, such that for any non-zero local normal section ξ of $Rad(TM)$, there exist a unique section N of $tr(TM)$ satisfying*

$$\bar{g}(N, \xi) = 1 \tag{5}$$

$$\bar{g}(N, N) = 0, \quad \bar{g}(N, V) = 0 \;\; for \; all \;\; V \in \Gamma(S(TM)).$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M}_{|M} = S(TM) \perp (TM^\perp \oplus tr(TM)) = TM \oplus tr(TM).$$

Following are some basic concepts related to lightlike hypersurfaces of an indefinite statistical manifold.

Definition 2.3. A pair $(\bar{\nabla}, \bar{g})$, where $\bar{g}$ is a semi-Riemannian metric of constant index $q \geq 1$, is called an indefinite statistical structure on $\bar{M}$, if $\bar{\nabla}$ is torsion free and

$$(\bar{\nabla}_X \bar{g})(Y, Z) = (\bar{\nabla}_Y \bar{g})(X, Z) \tag{6}$$

holds for any $X, Y, Z \in \Gamma(T\bar{M})$.

Moreover, there exists $\bar{\nabla}^*$ which is a dual connection of $\bar{\nabla}$ with respect to $\bar{g}$, satisfying

$$X\bar{g}(Y,Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}^*_X Z) \ \ for\ all\ X, Y, Z \in \Gamma(T\bar{M})$$

If $(\bar{M}, \bar{g}, \bar{\nabla})$ is an indefinite statistical manifold, then so is $(\bar{M}, \bar{g}, \bar{\nabla}^*)$. Hence the indefinite statistical manifold is denoted by $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$.

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g})$. Then the Gauss and Weingarten formulae with respect to dual connections as given by [13], [7] are as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y)$$
$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \bar{\nabla}^*_X N = -A^*_N X + \nabla^{\perp *}_X N,$$

for X, $Y \in \Gamma(TM)$, $N \in \Gamma(trTM)$, where $\nabla_X Y, \nabla^*_X Y, A_N X, A^*_N X \in \Gamma(TM)$ and $h(X,Y)$, $h^*(X,Y)$, $\nabla^{\perp}_X N, \nabla^{\perp *}_X N \in \Gamma(tr(TM))$.

Here ∇, ∇^* are called induced connections on M and A_N, A^*_N are called shape operators with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ respectively. The second fundamental forms with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ are denoted by B and B^*, respectively. Then

$$B(X,Y) = \bar{g}(h(X,Y), \xi), \quad B^*(X,Y) = \bar{g}(h^*(X,Y), \xi),$$
$$\tau(X) = \bar{g}(\nabla^{\perp}_X N, \xi), \quad \tau^*(X) = \bar{g}(\nabla^{\perp *}_X N, \xi).$$

It follows that

$$h(X,Y) = B(X,Y)N, \quad h^*(X,Y) = B^*(X,Y)N$$
$$\nabla^{\perp}_X N = \tau(X)N, \quad \nabla^{\perp *}_X N = \tau^*(X)N$$

Hence,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \bar{\nabla}^*_X Y = \nabla^*_X Y + B^*(X,Y)N \tag{7}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \bar{\nabla}^*_X N = -A^*_N X + \tau^*(X)N \tag{8}$$

As per [7] showed that the relation between dual connections using the Gauss formula as follows:

$$Xg(Y,Z) = g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}^*_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) + B(X,Y)\theta(Z) + B^*(X,Z)\theta(Y)$$

where θ is a 1-form such that $\theta(X) = \bar{g}(X, N)$.

From the above equation, it is concluded that the induced connections ∇ and ∇^* are not dual connections and a lightlike hypersurface of a statistical manifold need not a statistical manifold. Also, the induced connections ∇ and ∇^* and the second fundamental forms B and B^* are symmetric.

Further, using Gauss and Weingarten formulae, we have:

$$(\nabla_X g)(Y,Z) + (\nabla^*_X g)(Y,Z) = B(X,Y)\theta(Z) + B^*(X,Z)\theta(Y) + B^*(X,Y)\theta(Z) + B(X,Z)\theta(Y)$$

Let P denote the projection morphism of TM on $S(TM)$ with respect to the decomposition (4). Then

$$\nabla_X PY = \nabla'_X PY + h'(X, PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X, PY),$$
$$\nabla_X \xi = -A'_{\xi} X + \nabla'^{\perp}_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_{\xi} X + \nabla^{*\prime\perp}_X \xi$$

holds for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\nabla'_X PY, \nabla^{*\prime}_X PY, A'_\xi X$ and $A^{*\prime}_\xi X \in \Gamma(S(TM))$. Also $\nabla', \nabla^{*\prime}$ and $\nabla'^{\perp}, \nabla^{*\prime\perp}$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. Here $h', h^{*\prime}$ and $A', A^{*\prime}$ are respectively called screen second fundamental forms and screen shape operators of $S(TM)$.

The local second fundamental forms of $S(TM)$ are defined as

$$C(X, PY) = \bar{g}(h'(X, PY), N), \quad C^*(X, PY) = \bar{g}(h^{*\prime}(X, PY), N),$$

$$\epsilon(X) = g(\nabla'^{\perp}_X \xi, N), \quad \epsilon^*(X) = g(\nabla^{*\prime\perp}_X \xi, N) \quad \textit{for all } X, Y \in \Gamma(TM).$$

Therefore,

$$h'(X, PY) = C(X, PY)\xi, \quad h^{*\prime}(X, PY) = C^*(X, PY)\xi,$$

$$\nabla'^{\perp}_X \xi = -\tau(X)\xi, \quad \nabla^{*\prime\perp}_X \xi = -\tau^*(X)\xi,$$

$$\begin{aligned} \nabla_X PY &= \nabla'_X PY + C(X, PY)\xi, \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + C^*(X, PY)\xi, \\ \nabla_X \xi &= -A'_\xi X - \tau(X)\xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X - \tau^*(X)\xi \quad \textit{for all } X, Y \in \Gamma(TM) \end{aligned} \tag{9}$$

where $\epsilon(X) = -\tau(X)$.

Using above equation, the induced objects are related as:

$$\begin{aligned} B(X, \xi) + B^*(X, \xi) = 0, \quad g(A_N X + A^*_N X, N) = 0, \\ C(X, PY) = g(A^*_N X, PY), \quad C^*(X, PY) = g(A_N X, PY). \end{aligned} \tag{10}$$

From the equations (5), (6),(7),(8) and (9), the following propositions hold:

Proposition 2.4. *[25] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$. Then the second fundamental forms B and B^* are related to the shape operators $A'_\xi X$ and $A^{*\prime}_\xi X$ of $S(TM)$ as follows:*

$$g(A'_\xi X, PY) = B^*(X, PY), \quad g(A^{*\prime}_\xi X, PY) = B(X, PY). \tag{11}$$

Therefore, equation (11) gives,

$$B(A^{*\prime}_\xi X, Y) = B(X, A^{*\prime}_\xi Y), \quad B^*(A'_\xi X, Y) = B^*(X, A'_\xi Y).$$

Proposition 2.5. *[7] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$. Then the shape operator of any screen distribution of a lightlike hypersurface is symmetric with respect to the second fundamental form of M.*

Proposition 2.6. *[7] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$. Then the second fundamental forms B and B^* are not degenerate.*

Also for the dual connections, following holds:

$$B(X, Y) = g(A^{*\prime}_\xi X, Y) - B^*(X, \xi)\theta(Y)$$

$$B^*(X, Y) = g(A'_\xi X, Y) - B(X, \xi)\theta(Y).$$

Using above equations, $A^{*\prime}_\xi \xi + A'_\xi \xi = 0$.

3. Indefinite nearly β-Kenmotsu statistical manifold

Following [15], we consider a Levi-Civita connection $\hat{\bar{\nabla}}$ with respect to $\bar{g}$ such that $\hat{\bar{\nabla}} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$. For a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$, the difference $(1,2)$ tensor K of a torsion free affine connection $\bar{\nabla}$ and Levi-Civita connection $\hat{\bar{\nabla}}$ is defined as

$$K(X,Y) = K_X Y = \bar{\nabla}_X Y - \hat{\bar{\nabla}}_X Y. \tag{12}$$

Since $\bar{\nabla}$ and $\hat{\bar{\nabla}}$ are torsion free, then

$$K_X Y = K_Y X, \qquad \bar{g}(K_X Y, Z) = \bar{g}(Y, K_X Z) \tag{13}$$

holds for any $X, Y, Z \in \Gamma(T\bar{M})$.

Moreover, $K(X,Y) = \hat{\bar{\nabla}}_X Y - \bar{\nabla}^*_X Y$ implies $K(X,Y) = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y)$.

Definition 3.1. Let $(\bar{g}, \phi, \nu)$ be an indefinite nearly $\beta-$Kenmotsu structure on $\bar{M}$. A quadruplet $(\bar{\nabla} = \hat{\bar{\nabla}} + K, \bar{g}, \phi, \nu)$ is known as an indefinite nearly $\beta-$Kenmotsu statistical structure on $\bar{M}$ if $(\bar{\nabla}, \bar{g})$ is a statistical structure on $\bar{M}$ and the condition

$$K_X \phi Y + K_Y \phi X = -2\phi K_X Y \tag{14}$$

holds for any $X, Y \in \Gamma(T\bar{M})$.

Then $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \nu)$ is called an indefinite nearly $\beta-$Kenmotsu statistical manifold.

If $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \nu)$ is an indefinite nearly $\beta-$Kenmotsu statistical manifold, then so is $(\bar{M}, \bar{\nabla}^*, \bar{g}, \phi, \nu)$.

Theorem 3.2. *Let $(\bar{M}, \bar{\nabla}, \bar{\nabla}^*, \bar{g})$ be an indefinite statistical manifold with an almost contact metric structure $(\bar{g}, \phi, \nu)$. Then $(\bar{M}, \bar{\nabla}, \bar{\nabla}^*, \bar{g}, \phi, \nu)$ is said to be an indefinite nearly β-Kenmotsu statistical manifold $\bar{M}$ if and only if*

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y + \bar{\nabla}_Y \phi X - \phi \bar{\nabla}^*_Y X = -\beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \tag{15}$$

holds for all $X, Y \in \Gamma(T\bar{M})$ on $\bar{M}$.

Proof. Let $\bar{M}$ be an indefinite nearly $\beta-$Kenmotsu statistical manifold. Then, (3) and (12) implies

$$\bar{\nabla}_X \phi Y - K_X \phi Y - \phi(K_X Y + \bar{\nabla}^*_X Y) + \bar{\nabla}_Y \phi X - K_Y \phi X - \phi(K_Y X + \bar{\nabla}^*_Y X) = -\beta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$

Now using (14), above equation becomes

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y + \bar{\nabla}_Y \phi X - \phi \bar{\nabla}^*_Y X = -\beta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$

for all $X, Y \in \Gamma(TM)$.

Conversely, replacing $\bar{\nabla}$ by $\bar{\nabla}^*$ in above and subtracting obtained equation from (15), we get

$$(\bar{\nabla}_X \phi Y - \bar{\nabla}^*_X \phi Y) - \phi(\bar{\nabla}^*_X Y - \bar{\nabla}_X Y) + (\bar{\nabla}_Y \phi X - \bar{\nabla}^*_Y \phi X) - \phi(\bar{\nabla}^*_Y X - \bar{\nabla}_Y X) = 0$$

Hence (14) follows from (12) and (13). □

REMARK 3.3. Let $(\bar{g}, \phi, \nu)$ be an indefinite nearly $\beta-$Kenmotsu structure on $\bar{M}$. So, by setting

$$K_X Y = \eta(X)\eta(Y)\nu$$

for any $X, Y \in \Gamma(T\bar{M})$ such that K satisfies (13) and (14), an indefinite nearly $\beta-$Kenmotsu statistical structure $(\bar{\nabla}^{\lambda} = \hat{\bar{\nabla}} + \lambda K, \bar{g}, \phi, \nu)$ is obtained on $\bar{M}$ for $\lambda \in \mathbb{C}^{\infty}(\bar{M})$.

Inspired from [22], we present the following example.

EXAMPLE 3.4. Let $\bar{M}$ be a 7-dimensional manifold defined by $\bar{M} = \{z \in \mathbb{R}^7 : z_7 \neq 0\}$, where $z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ are the standard coordinates in $\mathbb{R}^7$. Consider, the vector fields $\{l_1, l_2, l_3, l_4, l_5, l_6, l_7\}$, linearly independent at each point of $\bar{M}$, defined as

$$l_1 = z_7\frac{\partial}{\partial z_1},\ l_2 = z_7\frac{\partial}{\partial z_2},\ l_3 = z_7\frac{\partial}{\partial z_3},\ l_4 = z_7\frac{\partial}{\partial z_4},$$
$$l_5 = -z_7\frac{\partial}{\partial z_5},\ l_6 = -z_7\frac{\partial}{\partial z_6}, l_7 = -z_7\frac{\partial}{\partial z_7}.$$

Let $\bar{g}$ be the semi-Riemannian metric defined as $\bar{g}(l_i, l_j) = 0$, for all $i \neq j, i, j = 1, 2, ..., 7$ and $\bar{g}(l_k, l_k) = 1$, for all $k = 1, 2, 3, 4, 7$; $\bar{g}(l_m, l_m) = -1$, for all $m = 5, 6$. Also, let η be the 1-form defined by $\eta(X) = \bar{g}(X, l_7)$, for any $X \in \mathfrak{X}(\bar{M})$, where $\mathfrak{X}(\bar{M})$ is the set of all differentiable vector fields on $\bar{M}$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi l_1 = -l_2,\ \phi l_2 = l_1,\ \phi l_3 = -l_4,\ \phi l_4 = l_3,\ \phi l_5 = -l_6,\ \phi l_6 = l_5,\ \phi l_7 = 0.$$

Using the linearity of ϕ and $\bar{g}$, we get $\eta(l_7) = 1$, $\phi^2 X = -X + \eta(X)l_7$, $\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y)$, for any $X, Y \in \mathfrak{X}(\bar{M})$. Thus, for $l_7 = \nu$, $(\bar{M}, \bar{g}, \phi, \nu, \eta)$ is an almost contact metric manifold.

Let $\hat{\bar{\nabla}}$ be the Levi-Civita connection with respect to the metric $\bar{g}$ given by Koszul's formula as

$$2\bar{g}(\hat{\bar{\nabla}}_X Y, Z) = X\bar{g}(Y, Z) + Y\bar{g}(X, Z) - Z\bar{g}(X, Y) - \bar{g}(X, [Y, Z]) - \bar{g}(Y, [X, Z]) + \bar{g}(Z, [X, Y])$$

Now, taking $l_7 = \nu$ and using Koszul's formula, following expressions are derived:

$$\hat{\bar{\nabla}}_{l_1} l_1 = -\nu, \quad \hat{\bar{\nabla}}_{l_2} l_2 = -\nu, \quad \hat{\bar{\nabla}}_{l_3} l_3 = -\nu,$$
$$\hat{\bar{\nabla}}_{l_4} l_4 = -\nu, \quad \hat{\bar{\nabla}}_{l_5} l_5 = \nu, \quad \hat{\bar{\nabla}}_{l_6} l_6 = \nu,$$
$$\hat{\bar{\nabla}}_{l_1} l_7 = l_1, \quad \hat{\bar{\nabla}}_{l_2} l_7 = l_2, \quad \hat{\bar{\nabla}}_{l_3} l_7 = l_3,$$
$$\hat{\bar{\nabla}}_{l_4} l_7 = l_4, \quad \hat{\bar{\nabla}}_{l_5} l_7 = l_5, \quad \hat{\bar{\nabla}}_{l_6} l_7 = l_6.$$

Therefore, using above, it is concluded that $(\bar{M}, \bar{g}, \phi, \nu, \eta)$ is an indefinite nearly Kenmotsu manifold (for $\beta = 1$).

Taking $\lambda = 1$ in remark (3.3), we have

$$\bar{\nabla}_{l_1} l_1 = -\nu, \quad \bar{\nabla}_{l_2} l_2 = -\nu, \quad \bar{\nabla}_{l_3} l_3 = -\nu,$$
$$\bar{\nabla}_{l_4} l_4 = -\nu, \quad \bar{\nabla}_{l_5} l_5 = \nu, \quad \bar{\nabla}_{l_6} l_6 = \nu,$$
$$\bar{\nabla}_{l_1} l_7 = l_1, \quad \bar{\nabla}_{l_2} l_7 = l_2, \quad \bar{\nabla}_{l_3} l_7 = l_3,$$
$$\bar{\nabla}_{l_4} l_7 = l_4, \quad \bar{\nabla}_{l_5} l_7 = l_5, \quad \bar{\nabla}_{l_6} l_7 = l_6.$$

Now $(\bar{\nabla}, \bar{g})$ is a statistical structure and $K_{l_i}\phi l_j + K_{l_j}\phi l_i = -2\phi K_{l_i} l_j$ holds for all $i, j = 1, 2, ..., 7$. Hence, $(\bar{\nabla}, \bar{g}, \phi, \eta, \nu)$ becomes an indefinite nearly Kenmotsu statistical structure on $\bar{M}$. Similarly, the above equations for dual connection $\bar{\nabla}^*$ can be obtained using $\bar{\nabla}^*_X Y = \hat{\hat{\bar{\nabla}}}_X Y - \eta(X)\eta(Y)\nu$.

Thus, $(\bar{\nabla} = \hat{\hat{\bar{\nabla}}} + K, \bar{g}, \phi, \nu)$ defines an indefinite nearly Kenmotsu statistical structure on $\bar{M}$.

3.1. Lightlike hypersurfaces of an indefinite nearly $\beta-$Kenmotsu statistical manifold.

Let (M, g) be a lightlike hypersurface of an indefinite nearly $\beta-$Kenmotsu statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*, \phi, \nu)$, where g is the degenerate metric induced on M. Therefore, for any $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltrTM)$, using (1) and (2), following holds:

$$\bar{g}(\xi, \nu) = 0, \quad \bar{g}(N, \nu) = 0 \tag{16}$$

$$\phi^2\xi = -\xi, \quad \phi^2 N = -N.$$

The above proposition leads to the following decomposition:

$$S(TM) = \{\phi Rad(TM) \oplus \phi ltr(TM)\} \perp L_o \perp <\nu> \tag{17}$$

where L_o is non-degenerate and ϕ-invariant distribution.

If distributions on M are denoted by

$$L = Rad(TM) \perp \phi Rad(TM) \perp L_o, \quad L' = \phi ltr(TM), \tag{18}$$

then L is invariant and L' is anti-invariant distributions under ϕ. Also,

$$TM = L \oplus L' \perp \nu. \tag{19}$$

Consider two null vector fields U and W such that

$$U = -\phi N, \quad W = -\phi\xi \tag{20}$$

and their corresponding 1-forms

$$u(X) = \bar{g}(X, W), \quad w(X) = \bar{g}(X, U).$$

Denote by S, the projection morphism of $T\bar{M}$ on the distribution L. Then,

$$X = SX + u(X)U \tag{21}$$

for any $X \in \Gamma(T\bar{M})$. Applying ϕ to (21), we have

$$\phi X = \bar{\phi}X + u(X)N \tag{22}$$

where $\bar{\phi}$ is a tensor field of type $(1, 1)$ defined on M by $\bar{\phi}X = \phi SX$.

Using (2),

$$\bar{\phi}^2 X = -X + \eta(X)\nu + u(X)U. \tag{23}$$

Since $\bar{\phi}U = 0$, we obtain $\bar{\phi}^3 + \bar{\phi} = 0$ from (23), which shows that $\bar{\phi}$ is an f-structure on M.

Using (16) and (20),

$$\bar{g}(U, W) = 1$$

which implies that $< U > \oplus < W >$ is non-degenerate vector bundle of $S(TM)$ with rank 2. From (17) and (18), the following decompositions hold:

(24)
$$S(TM) = \{U \oplus W\} \perp L_o \perp < \nu >, \quad L = Rad(TM) \perp < W > L_o, \quad L' = < U > .$$

Let P and Q be two projections of TM into L and L', respectively. Then $X = PX + QX + \eta(X)\nu$, for any $X \in \Gamma(TM)$. Therefore (1), (2), (22) and (24) gives

$$\bar{\phi}^2 X = -X + \eta(X)\nu + u(X)U \tag{25}$$

where $QX = u(X)U$ and $\phi PX = \bar{\phi}X$. Also, using (22), following identities hold:

$$g(\bar{\phi}X, \bar{\phi}Y) = g(X, Y) - \eta(X)\eta(Y) - u(X)w(Y) - u(Y)w(X), \tag{26}$$

$$g(\bar{\phi}X, Y) = -g(X, \bar{\phi}Y) - u(Y)\theta(X) - u(X)\theta(Y), \tag{27}$$

$$\bar{\phi}\nu = 0, \quad g(\bar{\phi}X, \nu) = 0 \quad for\ all\ X, Y \in \Gamma(TM). \tag{28}$$

As, $\bar{\phi}^2 X = -X + \eta(X)\nu + u(X)U$, then by applying ∇_X to $\bar{\phi}\xi = -W$, that is $\nabla_X \bar{\phi}\xi = -\nabla_X W$ and using (9) implies,

$$(\nabla_X \bar{\phi})\xi = -\nabla_X W + \bar{\phi}A'_\xi X - \tau(X)W, \tag{29}$$

$$(\nabla^*_X \bar{\phi})\xi = -\nabla^*_X W + \bar{\phi}A^{*\prime}_\xi X - \tau^*(X)W. \tag{30}$$

Also, applying ∇_X to $\bar{\phi}W = \xi$ that is $\nabla_X \bar{\phi}W = \nabla_X \xi$ implies,

$$(\nabla_X \bar{\phi})W = -\bar{\phi}\nabla_X W - A'_\xi X - \tau(X)\xi, \tag{31}$$

$$(\nabla^*_X \bar{\phi})W = -\bar{\phi}\nabla^*_X W - A^{*\prime}_\xi X - \tau^*(X)\xi. \tag{32}$$

EXAMPLE 3.5. Following example (3.4), consider a hypersurface M defined by

$$M = \{z \in \bar{M} : z_5 = z_2\}$$

of an indefinite nearly Kenmotsu statistical manifold $(\bar{M}, \bar{g}, \phi, \eta, \nu)$ for $\beta = 1$.

The tangent space TM is spanned by $\{Z_i\}$, where $Z_1 = l_1,\ Z_2 = l_2 - l_5,\ Z_3 = l_3,\ Z_4 = l_4,\ Z_5 = l_6,\ Z_6 = \nu$. Further, $E = l_2 - l_5$ spans the distribution $TM^\perp$ of rank 1. Therefore, $TM^\perp \subset TM$ and M is a 6-dimensional lightlike hypersurface of $\bar{M}$. The transversal bundle $ltr(TM)$ is spanned by $N = \frac{1}{2}(l_2 + l_5)$. From decomposition (17) and the almost contact structure of $\bar{M}$, L_o is spanned by $\{H, \phi H\}$, where $H = Z_3,\ \phi H = -Z_4$ and the distributions ν, $\phi Rad(TM)$ and $\phi ltr(TM)$ are spanned by ν, $\phi E = Z_1 + Z_5$ and $\phi N = \frac{1}{2}(Z_1 - Z_5)$, respectively.

Hence, M is a lightlike hypersurface of an indefinite nearly Kenmotsu statistical manifold $\bar{M}$.

THEOREM 3.6. *Let M be a lightlike hypersurface of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$. Then*

$$\nabla_X \nu = \beta X - \{\beta\eta(X) - \eta(\bar{\nabla}_X \nu) - \eta(\bar{\nabla}_\nu X)\}\nu - \phi\nabla^*_\nu \phi X + B^*(\nu, \phi X)U - \nabla_\nu X$$

for all $X, Y \in \Gamma(T\bar{M})$ on $\bar{M}$.

Proof. Consider (15) and replace Y by ν. Then

$$-\phi\bar{\nabla}_X\nu + \bar{\nabla}^*_\nu\phi X - \phi\bar{\nabla}_\nu X = -\beta(\phi X)$$

Applying ϕ on both sides and using (2), we get

$$\bar{\nabla}_X\nu = \beta X - \{\beta\eta(X) - \eta(\bar{\nabla}_X\nu) - \eta(\bar{\nabla}_\nu X)\}\nu - \phi\nabla^*_\nu\phi X + B^*(\nu,\phi X)U - \nabla_\nu X - B(\nu,X)N$$

Therefore, using (7) in above and then comparing tangential and normal parts, we have the desired result.

□

Theorem 3.7. *Let M be a lightlike hypersurface of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$ with $\nu \in \Gamma(TM)$. Then for any $X, Y \in \Gamma(TM)$, following holds:*

$$\begin{aligned} &\nabla_X\bar{\phi}Y - \bar{\phi}\nabla^*_X Y + \nabla_Y\bar{\phi}X - \bar{\phi}\nabla^*_Y X \\ &= -\beta\{\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y\} + u(Y)A_N X + u(X)A_N Y - 2B^*(X,Y)U. \end{aligned} \tag{33}$$

Proof. In an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$, the condition $\phi X = \bar{\phi}X + u(X)N$ and the symmetric character of B^* implies

$$\bar{\nabla}_X\bar{\phi}Y + \bar{\nabla}_X u(Y)N - \phi\nabla^*_X Y + 2B^*(X,Y)U + \bar{\nabla}_Y\bar{\phi}X + \bar{\nabla}_Y u(X)N - \phi\nabla^*_Y X$$
$$= -\beta\{\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y + \eta(Y)u(X)N + \eta(X)u(Y)N\}$$

This leads to

$$\begin{aligned} &\nabla_X\bar{\phi}Y + B(X,\bar{\phi}Y)N + u(Y)\tau(X)N + Xu(Y)N - \phi\nabla^*_X Y \\ &+\nabla_Y\bar{\phi}X + B(Y,\bar{\phi}X) + u(X)\tau(Y)N + Yu(X)N - \phi\nabla^*_Y X \\ &= -\beta\{\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y + \eta(Y)u(X)N + \eta(X)u(Y)N\} \\ &\quad +u(Y)A_N X + u(X)A_N Y - 2B^*(X,Y)U \end{aligned} \tag{34}$$

Therefore, using (22) and comparing tangential parts, (33) is obtained.

□

Corollary 3.8. *Let M be a lightlike hypersurface of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$. Then following hold:*

$$B(X,\bar{\phi Y}) + u(Y)\tau(X) + (\nabla^*_X u)Y + B(Y,\bar{\phi X}) + u(X)\tau(Y) + (\nabla^*_Y u)X = -\beta\{\eta(Y)u(X) + \eta(X)u(Y)\}$$

for all $X, Y \in \Gamma(TM)$.

Proof. On comparing the transversal parts in (34), we get the required assertion.

□

Theorem 3.9. *For a lightlike hypersurface M of an indefinite nearly $\beta-$Kenmotsu statitsical manifold $\bar{M}$ following assertion holds:*

(i) $B(U,\nu) + B^*(U,\nu) + C(W,\nu) + C^*(W,\nu) = 0,$
(ii) $B(U,W) + B^*(U,W) - C(W,W) - C^*(W,W) = 0,$
(iii) $B(U,U) + B^*(U,U) + C(W,U) + C^*(W,U) = 0,$
(iv) $w(\nabla_U W) + w(\nabla^*_U W) = -\tau(U) - \tau^*(U),$
(v) $2[C(W,\nu) + C^*(W,\nu)] + [C(\nu,W) + C^*(\nu,W)] = 0.$

Proof. On replacing ∇ by ∇^* in (33), we obtained

$$\nabla^*_X\bar{\phi}Y - \bar{\phi}\nabla_X Y + \nabla^*_Y\bar{\phi}X - \bar{\phi}\nabla_Y X$$

$$= -\beta\{\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y\} + u(Y)A^*_N X + u(X)A^*_N Y - 2B(X,Y)U.$$

Now adding above equation to (33), we get

$$\nabla_X\bar{\phi}Y - \bar{\phi}\nabla^*_X Y + \nabla_Y\bar{\phi}X - \bar{\phi}\nabla^*_Y X + \nabla^*_X\bar{\phi}Y - \bar{\phi}\nabla_X Y + \nabla^*_Y\bar{\phi}X - \bar{\phi}\nabla_Y X \tag{35}$$

$$= -2\beta\{\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y\} + u(Y)(A_N X + A^*_N X) + u(X)(A_N Y + A^*_N Y)$$

$$-2[B^*(X,Y) + B(X,Y)]U.$$

Now, taking $X = U$ and $Y = W$ in (35), we have

(36)

$$\nabla_U\xi - \bar{\phi}\nabla^*_U W - \bar{\phi}\nabla^*_W U + \nabla^*_U\xi - \bar{\phi}\nabla_U W - \bar{\phi}\nabla_W U = A_N W + A^*_N W - 2[B^*(U,W) + B(U,W)]U.$$

Taking scalar product of above with ν, W, U and N respectively, and using (10), (11), (20) and (5) we get

$$g(\nabla_U\xi,\nu) + g(\nabla^*_U\xi,\nu) = g(A_N W,\nu) + g(A^*_N W,\nu),$$

$$-g(A'_\xi U,W) - g({A^*}'_\xi U,W) = g(A_N W,W) + g(A^*_N W,W) - 2[B(U,W) + B^*(U,W)],$$

$$-g(A'_\xi U,U) - g({A^*}'_\xi U,U) = C^*(W,U) + C(W,U), \text{ and}$$

$$\tau(U)g(\xi,N) + g(\nabla^*_U W,U) + g(\nabla^*_W U,U) + \tau^*(U)g(\xi,N) - g(\nabla_U W,U) + g(\nabla_W U,U) = 0$$

which leads to $(i) - (iv)$.

On replacing X by U and Y by ν in (35) and then using (22), we get

$$\bar{\phi}\nabla^*_U\nu + \bar{\phi}\nabla^*_\nu U + \bar{\phi}\nabla_U\nu + \bar{\phi}\nabla_\nu U = -A_N\nu - A^*_N\nu + 2[B^*(U,\nu) + B(U,\nu)]U.$$

Now, taking scalar product of above with W, we get

$$2[B^*(U,\nu) + B(U,\nu)] - C(\nu,W) - C^*(\nu,W) = 0.$$

Further, from (i), we have $2[C(W,\nu) + C^*(W,\nu)] + C(\nu,W) + C^*(\nu,W) = 0.$

□

Definition 3.10. Let $\bar{M}$ be an indefinite nearly $\beta-$Kenmotsu statistical manifold.

Then, **(i)** A lightlike hypersurface M of $\bar{M}$ is said to be totally tangentially umbilical with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if

$$B(X,Y) = cg(X,Y) \quad (respectively,\ B^*(X,Y) = c^*g(X,Y))\ for\ all\ X,Y \in \Gamma(TM)$$

where c and c^* are smooth functions.

(ii) A lightlike hypersurface M of $\bar{M}$ is said to be totally normally umbilical with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if

$${A^*}'_\xi X = cPX \quad (respectively,\ A'_\xi X = cPX)\ for\ all\ X,Y \in \Gamma(TM).$$

In case, if $c = 0$ and $c^* = 0$, then M is totally geodesic.

Definition 3.11. **(i)** A lightlike hypersurface M of $\bar{M}$ is said to be screen totally umbilical with respect to $\bar{\nabla}$ if

$$A_N^* X = dPX \quad or \quad \text{equivalently, } C(X, PY) = dg(X, PY) \; for \; all \; X, Y \in \Gamma(S(TM))$$

where d is a smooth function.

(ii) A lightlike hypersurface M of $\bar{M}$ is said to be screen totally umbilical with respect to $\bar{\nabla}^*$if

$$A_N X = d^* PX \quad or \quad \text{equivalently, } C^*(X, PY) = d^* g(X, PY) \; for \; all \; X, Y \in \Gamma(S(TM))$$

where d^* is a smooth function.

In case, if $d = 0$ and $d^* = 0$, then M is screen totally geodesic.

Remark 3.12. For a lightlike hypersurface M of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$, the integral curve of ν is a spacelike geodesic of both $\bar{M}$ and M with respect to the connections $\bar{\nabla}$ and ∇ (respectively $\bar{\nabla}^*$ and ∇^*), where $\nu \in \Gamma(TM)$.

Proof. Taking $X = Y = \nu$ in (15), we get $\bar{\nabla}^*_\nu \nu = 0$ and $\bar{\nabla}_\nu \nu = 0$. Thus,

$$\nabla_\nu \nu = 0, \quad B(\nu, \nu) = 0, \quad C(\nu, \nu) = 0, \tag{37}$$

$$\nabla^*_\nu \nu = 0, \quad B^*(\nu, \nu) = 0, \quad C^*(\nu, \nu) = 0. \tag{38}$$

by Gauss formula.

As, $\bar{\nabla}_\nu \nu = 0, \nabla_\nu \nu = 0, \bar{\nabla}^*_\nu \nu = 0$ and $\nabla^*_\nu \nu = 0$, then the integral curve of ν is a spacelike geodesic of both $\bar{M}$ and M with respect to the connection $\bar{\nabla}$ and ∇ (respectively $\bar{\nabla}^*$ and ∇^*).

□

Theorem 3.13. *For a lightlike hypersurface M of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$ where ν is tangent to M, the following assertions hold:*
***(i)** If M is totally umbilical, then M is totally geodesic.*
***(ii)** If M is screen totally umbilical, then M is screen totally geodesic.*

Proof. **(i)** On taking, $X = Y = \nu$ in definition (3.10), we have $B(\nu, \nu) = cg(\nu, \nu) = c$ which implies $c = 0$ using (37) and similarly $B^*(\nu, \nu) = c^* g(\nu, \nu) = c^*$ which implies $c^* = 0$ using (38). Therefore, M is totally geodesic.

(ii) If M is screen totally umbilical, then $C(X, PY) = dg(X, PY)$ and $C^*(X, PY) = d^* g(X, PY)$ from definition (3.11). Now, on taking $X = PY = \nu$ in above, we get $C(\nu, \nu) = dg(\nu, \nu) = d$ which implies $d = 0$ using (37) and similarly $C^*(\nu, \nu) = d^* g(\nu, \nu) = d^*$ which implies $d^* = 0$ using (38). Therefore, M is screen totally geodesic.

□

3.2. Recurrent and Lie recurrent structure tensor field.

Definition 3.14. An almost contact metric manifold equipped with an indefinite statistical structure is called an indefinite almost contact metric statistical manifold.

Definition 3.15. The structure tensor field $\bar{\phi}$ of a lightlike hypersurface M is said to be recurrent if there exists a $1-$form μ on M such that

$$(\nabla_X \bar{\phi})Y = \mu(X)\bar{\phi}Y \quad (\text{resp. } (\nabla^*_X \bar{\phi})Y = \mu^*(X)\bar{\phi}Y). \tag{39}$$

THEOREM 3.16. *For a lightlike hypersurface M of an indefinite almost contact metric statistical manifold $\bar{M}$ such that ν is tangent to M, if $\bar{\phi}$ is recurrent, then $\mu(X) + \mu^*(X) = 0$.*

Proof. Replacing Y by W in (39) and then from (31), (32) we have

$$\bar{\phi}\nabla_X W + \bar{\phi}\nabla^*_X W + A'_\xi X + {A^*}'_\xi X + \{\tau(X) + \tau^*(X) + \mu(X) + \mu^*(X)\}\xi = 0. \quad (40)$$

Similarly, replacing Y by ξ in (39) and then using (29), (30) we have

$$\nabla_X W + \nabla^*_X W - \bar{\phi}A'_\xi X - \bar{\phi}{A^*}'_\xi X + \{\tau(X) + \tau^*(X) - \mu(X) - \mu^*(X)\}W = 0. \quad (41)$$

Now, taking scalar product of above with W we get

$$u(\nabla_X W) + u(\nabla^*_X W) + g(A'_\xi X, \bar{\phi}W) + g({A^*}'_\xi X, \bar{\phi}W) + [u(A'_\xi X) + u({A^*}'_\xi X)]\theta(W)$$
$$+ u(W)[\theta(A'_\xi X) + \theta({A^*}'_\xi X)] = 0$$

which implies, $u(\nabla_X W)+u(\nabla^*_X W) = 0$ as $A'_\xi X, {A^*}'_\xi X \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$.

Now, taking scalar product with ν to (41), we have

$$\eta(\nabla_X W) + \eta(\nabla^*_X W) + g(A'_\xi X, \bar{\phi}\nu) + g({A^*}'_\xi X, \bar{\phi}\nu) + [u(A'_\xi X) + u({A^*}'_\xi X)]\theta(\nu)$$
$$+ u(\nu)[\theta(A'_\xi X) + \theta({A^*}'_\xi X)] = 0$$

which implies $\eta(\nabla_X W) + \eta(\nabla^*_X W) = 0$.

By applying $\bar{\phi}$ to (40), we get

$$-\nabla_X W - \nabla^*_X W + \bar{\phi}A'_\xi X + \bar{\phi}{A^*}'_\xi X - \{\tau(X) + \tau^*(X) + \mu(X) + \mu^*(X)\}W = 0$$

which implies $\mu(X) + \mu^*(X) = 0$. Hence, the result.

□

DEFINITION 3.17. The structure tensor field $\bar{\phi}$ of M is said to be Lie recurrent if there exists a $1-$form ψ on M such that

$$(\mathbb{L}_X\bar{\phi})Y = \psi(X)\bar{\phi}Y \quad (42)$$

where, $\mathbb{L}_X$ stands for the Lie derivative on M w.r.t X that is,

$$(\mathbb{L}_X\bar{\phi})Y = [X, \bar{\phi}Y] - \bar{\phi}[X, Y]. \quad (43)$$

If $(\mathbb{L}_X\bar{\phi})Y = 0$, then $\bar{\phi}$ is called Lie parallel.

THEOREM 3.18. *Let M be a lightlike hypersurface of an indefinite almost contact metric statistical manifold $\bar{M}$ such that ν is tangent to M. If the structure tensor field $\bar{\phi}$ is Lie recurrent, then $\bar{\phi}$ is Lie parallel.*

Proof. Since ∇ and ∇^* are torsion free,

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \text{and} \quad \nabla^*_X Y - \nabla^*_Y X = [X, Y].$$

From (42) and (43), we have $\psi(X)\bar{\phi}Y = [X, \bar{\phi}Y] - \bar{\phi}[X, Y]$ which implies

$$\psi(X)\bar{\phi}Y = (\nabla_X\bar{\phi})Y - \nabla_{\bar{\phi}Y}X + \bar{\phi}\nabla_Y X. \quad (44)$$

Similarly, we have

$$\psi(X)\bar{\phi}Y = (\nabla^*_X\bar{\phi})Y - \nabla^*_{\bar{\phi}Y}X + \bar{\phi}\nabla^*_Y X. \quad (45)$$

Adding (44) and (45), then replacing Y by W, we have

$$2\psi(X)\xi = (\nabla_X\bar{\phi})W - \nabla_\xi X + \bar{\phi}\nabla_W X + (\nabla^*_X\bar{\phi})W - \nabla^*_\xi X + \bar{\phi}\nabla^*_W X. \tag{46}$$

Using (31) and (32) in (46), we have

$$\nabla_\xi X + \nabla^*_\xi X = -\bar{\phi}(\nabla_X W + \nabla^*_X W - \nabla_W X - \nabla^*_W X) - A'_\xi X - {A^*}'_\xi X - \{\tau(X) + \tau^*(X) + 2\psi(X)\}\xi. \tag{47}$$

Adding (44) and (45), then replacing Y by ξ, we have

$$-2\psi(X)W = (\nabla_X\bar{\phi})\xi + \nabla_W X + \bar{\phi}\nabla_\xi X + (\nabla^*_X\bar{\phi})\xi + \nabla^*_W X + \bar{\phi}\nabla^*_\xi X. \tag{48}$$

Using (29) and (30), we obtain

$$\begin{aligned}\bar{\phi}[\nabla_\xi X + \nabla^*_\xi X] = {} & \nabla_X W + \nabla^*_X W - \nabla_W X - \nabla^*_W X - \bar{\phi}(A'_\xi X + {A^*}'_\xi X)\\ & + \{\tau(X) + \tau^*(X) - 2\psi(X)\}W.\end{aligned} \tag{49}$$

Now, taking scalar product of (49) with W, we have

$$u(\nabla_X W + \nabla^*_X W - \nabla_W X - \nabla^*_W X) = 0. \tag{50}$$

Similarly, taking scalar product of (49) with ν, we have

$$\eta(\nabla_X W + \nabla^*_X W - \nabla_W X - \nabla^*_W X) = 0. \tag{51}$$

On applying $\bar{\phi}$ to (47) and using (50), (51), we have

$$\bar{\phi}[\nabla_\xi X + \nabla^*_\xi X] = \nabla_X W + \nabla^*_X W - \nabla_W X - \nabla^*_W X - \bar{\phi}(A'_\xi X + {A^*}'_\xi X) + \{\tau(X) + \tau^*(X) + 2\psi(X)\}W.$$

Comparing above equation with (49), we have $\psi(X) = 0$ which implies $\psi = 0$. Therefore, $\bar{\phi}$ is Lie parallel.

□

Theorem 3.19. *Let M be a lightlike hypersurface M of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$. If one of the following three conditions is satisfied:*
(i) $(\nabla_X\bar{\phi})Y + (\nabla_Y\bar{\phi})X + (\nabla^*_X\bar{\phi})Y + (\nabla^*_Y\bar{\phi})X = 0,$
(ii) *$\bar{\phi}$ is parallel w.r.t induced connection ∇ and ∇^* on M,*
(iii) *$\bar{\phi}$ is recurrent,*
then $\beta = 0$. Thus, $\bar{M}$ is an indefinite nearly cosymplectic statistical manifold.

In this case, the shape operators $A'_\xi, {A^*}'_\xi, A_N$ and A^*_N satisfy:

$$A'_\xi W + {A^*}'_\xi W = 0, \quad A_N W + A^*_N W = 0, \quad A_N\nu + A^*_N\nu = 0, \quad A_N\xi + A^*_N\xi = 0, \tag{52}$$

$$A'_\xi\nu + {A^*}'_\xi\nu = 0, \quad A_N X + A^*_N X = [u(A_N X) + u(A^*_N X)]U. \tag{53}$$

Proof. **(i)** Let $(\nabla_X\bar{\phi})Y + (\nabla_Y\bar{\phi})X + (\nabla^*_X\bar{\phi})Y + (\nabla^*_Y\bar{\phi})X = 0$. Considering (33) and replacing ∇ by ∇^* and then adding the obtained equation to (33), we get

$$\begin{aligned}&(\nabla_X\bar{\phi})Y + (\nabla_Y\bar{\phi})X + (\nabla^*_X\bar{\phi})Y + (\nabla^*_Y\bar{\phi})X\\ &= -2\beta[\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y] - 2[B^*(X,Y) + B(X,Y)]U\\ &\quad + u(Y)[A_N X + A^*_N X] + u(X)[A_N Y + A^*_N Y]\end{aligned} \tag{54}$$

which implies

$$0 = -2\beta[\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y] - 2[B^*(X,Y) + B(X,Y)]U + u(Y)[A_N X + A^*_N X] + u(X)[A_N Y + A^*_N Y]. \tag{55}$$

Taking the scalar product with N of above, we have

$0 = -2\beta[\eta(Y)w(X)+\eta(X)w(Y)]+u(Y)[g(A_N X+A^*_N X, N)]+u(X)[g(A_N Y+A^*_N Y, N)]$ which implies, $\beta[\eta(Y)w(X) + \eta(X)w(Y)] = 0$, using (10).

On taking $Y = W$ in $\beta[\eta(Y)w(X) + \eta(X)w(Y)] = 0$, $\beta\eta(X) = 0$ is obtained.

Now, putting $X = \nu$ in $\beta\eta(X) = 0$, we obtain $\beta = 0$.

Therefore, $\bar{M}$ is an indefinite nearly cosymplectic statistical manifold.

As $\beta = 0$, (55) becomes

(56) $$u(Y)[A_N X + A^*_N X] + u(X)[A_N Y + A^*_N Y] - 2[B^*(X, Y) + B(X, Y)]U = 0.$$

Taking scalar product of (56) with W, we have

(57) $$2[B^*(X, Y)+B(X, Y)] = u(Y)[u(A_N X)+u(A^*_N X)]+u(X)[u(A_N Y)+u(A^*_N Y)]$$

Taking $Y = W$ in (57) and using (10), we derive

$$2[B^*(X, W) + B(X, W)] = u(X)[C^*(W, W) + C(W, W)]$$

Replacing X by U in above equation, we have

(58) $$2[B^*(U, W) + B(U, W)] = C^*(W, W) + C(W, W)$$

On comparing (58) with (ii) of theorem (3.9), we have $C^*(W, W) + C(W, W) = 0$ which implies $B^*(U, W) + B(U, W) = 0$ and hence

(59) $$B^*(X, W) + B(X, W) = 0.$$

Since $S(TM)$ is non-degenerate and B, B^* are symmetric, therefore using (11) in (59) gives $A'_\xi W + {A^*}'_\xi W = 0$.

Now taking $X = U$ and $Y = W$ in (56) implies

$$A_N W + A^*_N W = 0.$$

Taking scalar product of the above equation with ν, we get

(60) $$C^*(W, \nu) + C(W, \nu) = 0.$$

Using (60) in (i) and (v) of theorem (3.9) gives $B(U, \nu)+B^*(U, \nu) = 0$ and $C(\nu, W)+C^*(\nu, W) = 0$, respectively.

Also, taking $X = U$ and $Y = \nu$ in (56), we get

(61) $$A_N\nu + A^*_N\nu = 0.$$

Similarly, taking $X = U$ and $Y = \xi$ in (56) gives $A_N\xi + A^*_N\xi = 0$.

Replacing Y by ν in (57) and using (61), we get $B^*(X, \nu) + B(X, \nu) = 0$. Since, B and B^* are symmetric then using (11), we have $A'_\xi\nu + {A^*}'_\xi\nu = 0$.

Now taking $Y = U$ in (57), we obtain

(62) $$2[B^*(X, U) + B(X, U)] = [u(A_N X) + u(A^*_N X)] + u(X)[u(A_N U) + u(A^*_N U)].$$

Replacing Y by U in (56) and using (62), we have

(63)
$$A_NX+A_N^*X+u(X)[A_NU+A_N^*U-u(A_NU)U-u(A_N^*U)U]-u(A_NX)U-u(A_N^*X)U=0.$$

Taking $X=U$ to (63), we have $A_NU+A_N^*U=[u(A_NU)+u(A_N^*U)]U$ and hence, $A_NX+A_N^*X=[u(A_NX)+u(A_N^*X)]U$.

(ii) If $\bar{\phi}$ is parallel w.r.t ∇ and ∇^*, then $(\nabla_X\bar{\phi})Y+(\nabla_Y\bar{\phi})X=0$ and $(\nabla_X^*\bar{\phi})Y+(\nabla_Y^*\bar{\phi})X=0$, respectively. Hence $(\nabla_X\bar{\phi})Y+(\nabla_Y\bar{\phi})X+(\nabla_X^*\bar{\phi})Y+(\nabla_Y^*\bar{\phi})X=0$.

Thus, $\beta=0$ and all the required equations are satisfied using (i).

(iii) If $\bar{\phi}$ is recurrent, then from theorem (3.16), $\bar{\phi}$ is parallel w.r.t ∇ and ∇^*. Thus, $\beta=0$ and the equations in (52) and (53) hold by (ii).

□

DEFINITION 3.20. The structure tensor field $\bar{\phi}$ of the lightlike hypersurface M is s.t.b nearly recurrent if there exists a $1-$form μ on M such that

$$(\nabla_X\bar{\phi})Y+(\nabla_Y\bar{\phi})X=\mu(X)\bar{\phi}Y+\mu(Y)\bar{\phi}X, \quad (64)$$
$$\text{(resp. } (\nabla_X^*\bar{\phi})Y+(\nabla_Y^*\bar{\phi})X=\mu^*(X)\bar{\phi}Y+\mu^*(Y)\bar{\phi}X.)$$

THEOREM 3.21. *Let M be a lightlike hypersurface of an indefinite nearly $\beta-$Kenmotsu statistical manifold $\bar{M}$. If $\bar{\phi}$ is nearly recurrent, then $2\beta\eta=-(\mu+\mu^*)$ on M. In this case, the shape operators $A'_\xi, {A^*}'_\xi, A_N$ and A_N^* satisfies all the equations from the theorem (3.19).*

Proof. If $\bar{\phi}$ is nearly recurrent, then using (64) and (54), we have

$$[\mu(X)+\mu^*(X)+2\beta\eta(X)]\bar{\phi}Y+[\mu(Y)+\mu^*(Y)+2\beta\eta(Y)]\bar{\phi}X \quad (65)$$
$$=u(Y)[A_NX+A_N^*X]+u(X)[A_NY+A_N^*Y]-2[B^*(X,Y)+B(X,Y)]U$$

Taking the scalar product of above equation with N and then from (10), we have

$$[\mu(X)+\mu^*(X)+2\beta\eta(X)]w(Y)+[\mu(Y)+\mu^*(Y)+2\beta\eta(Y)]w(X)=0 \quad (66)$$

Taking $X=Y=W$ in above equation, we get

$$\mu(W)+\mu^*(W)=0. \quad (67)$$

Replacing Y by W in (66) and using (67), we get $\mu(X)+\mu^*(X)=-2\beta\eta(X)$.

Taking scalar product of (65) with W, we obtain

$$2[B^*(X,Y)+B(X,Y)]=u(Y)[u(A_NX)+u(A_N^*X)]+u(X)[u(A_NY)+u(A_N^*Y)]$$

which is (57) of theorem (3.19). Therefore, by the course of the proof of (i) of theorem (3.19), we have all equations of shape operators.

□

4. Lightlike geometry of leaves in indefinite $\beta-$Kenmotsu statistical manifold

This section deals with study of the lightlike geometry of leaves of screen distribution S(TM) and $\{\phi(TM^{\perp}) \oplus \phi(tr(TM))\} \perp L_o$, where L_o is ϕ-invariant distribution in an indefinite $\beta-$Kenmotsu statistical manifold.

Definition 4.1. [9] Let $(\bar{g}, \phi, \nu)$ be an almost contact metric structure defined on an indefinite statistical manifold $(\bar{M}, \bar{\nabla}, \bar{\nabla}^*, \bar{g})$. Then $(\bar{\nabla}, \bar{\nabla}^*, \bar{g}, \phi, \nu)$ is said to be an indefinite $\beta-$Kenmotsu statistical struture on $\bar{M}$ if and only if:

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y = \beta\{\bar{g}(\phi X, Y)\nu - \eta(Y)\phi X\} \quad and \tag{68}$$

$$\bar{\nabla}_X \nu = \beta\{X - \eta(X)\} + \mu(X)\nu \tag{69}$$

hold for all the vector fields X, Y on $\bar{M}$, where $\mu(X) = \eta(\bar{\nabla}_X \nu) = -\eta(\bar{\nabla}^*_X \nu) = \eta(K(\nu, \nu))\eta(X)$.

Theorem 4.2. *For a lightlike hypersurface M of an indefinite nearly $\beta-$Kenmotsu statistical manifold,*

$$B(X, U) + B^*(X, U) = C(X, W) + C^*(X, W), \ for\ all\ X \in \Gamma(TM). \tag{70}$$

Proof. From (8), we have

$$\bar{\nabla}_X N + \bar{\nabla}^*_X N = -A_N X - A^*_N X + \tau(X)N + \tau^*(X)N$$

Since $\bar{M}$ is an indefinite $\beta-$Kenmotsu statistical manifold, therefore on applying ϕ in above and using (22), we have

$$-(\bar{\nabla}^*_X U + \bar{\nabla}_X U) - B(X, U)N - B^*(X, U)N - 2\beta w(X)\nu$$
$$= -\bar{\phi} A_N X - u(A_N X)N - \bar{\phi} A^*_N X - u(A^*_N X)N - \tau(X)U - \tau^*(X)U$$

On comparing transversal parts, we get the desired result. □

Let the screen distribution $S(TM)$ of lightlike hypersurface M be integrable and M' be a leaf of $S(TM)$. Using (7) and (9), we get

$$\bar{\nabla}_X Y = \nabla'_X Y + C(X, Y)\xi + B(X, Y)N, \tag{71}$$

$$\bar{\nabla}^*_X Y = \nabla^{*\prime}_X Y + C^*(X, Y)\xi + B^*(X, Y)N \tag{72}$$

for any $X, Y \in \Gamma(TM')$.

Therefore, we obtain

$$\hat{\bar{\bar{\nabla}}}_X Y = \nabla^o_X Y + h^0(X, Y) \tag{73}$$

where, ∇^o and h^o are the Levi-Civita connection and the second fundamental form of M' in $\bar{M}$ respectively.

So, for any $X, Y \in \Gamma(TM')$,

$$\nabla^o_X Y = \frac{1}{2}(\nabla'_X Y + \nabla^{*\prime}_X Y), \tag{74}$$

$$h^o(X, Y) = C(X, Y)\xi + C^*(X, Y)\xi + B(X, Y)N + B^*(X, Y)N. \tag{75}$$

Theorem 4.3. *If $(M, g, S(TM))$ is a screen integrable lightlike hypersurface of an indefinite $\beta-$Kenmotsu statistical manifold $\bar{M}$ with $\nu \in \Gamma(TM)$ and M' is a leaf of $S(TM)$, then*

$$2\nabla^o_X U = -w(A^*_N X)\xi - w(A_N X)\xi - w({A^*}'_\xi X)N - w(A'_\xi X)N - 2\beta w(X)\nu + \phi A_N X + \phi A^*_N X + \tau(X)U + \tau^*(X)U \tag{76}$$

for $X \in \Gamma(TM')$.

Proof. Using (8), we obtain

$$\bar{\nabla}_X N + \bar{\nabla}^*_X N = -A_N X - A^*_N X + \tau(X)N + \tau^*(X)N \tag{77}$$

On applying ϕ to (77) and using the fact that $\bar{M}$ is an indefinite $\beta-$Kenmotsu statistical manifold, we have

$$-(\bar{\nabla}^*_X U + \bar{\nabla}_X U) - 2\beta w(X)\nu = -\phi A_N X - \phi A^*_N X - \tau(X)U - \tau^*(X)U$$

Therefore, we get the desired result from (71) and (72).

□

Theorem 4.4. *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite $\beta-$Kenmotsu statistical manifold $\bar{M}$ with $\nu \in \Gamma(TM)$ and M' be a leaf of $S(TM)$. Then,*

$$2\nabla^o_X W = -u(A^*_N X)\xi - u(A_N X)\xi - u({A^*}'_\xi X)N - u(A'_\xi X)N - 2\beta u(X)\nu + \phi A'_\xi X + \phi {A^*}'_\xi X - \tau(X)W - \tau^*(X)W \tag{78}$$

for $X \in \Gamma(TM')$.

Proof. Considering (7), we have

$$\bar{\nabla}_X Y + \bar{\nabla}^*_X Y = \nabla_X Y + \nabla^*_X Y + B(X,Y)N + B^*(X,Y)N$$

On replacing Y by ξ in above equation, we get

$$\bar{\nabla}_X \xi + \bar{\nabla}^*_X \xi = -A'_\xi X - {A^*}'_\xi X - \tau(X)\xi - \tau^*(X)\xi \tag{79}$$

By applying ϕ to (79) and using the concept of indefinite $\beta-$Kenmotsu statistical manifold, we derive

$$-(\bar{\nabla}^*_X W + \bar{\nabla}_X W) - 2\beta u(X)\nu = -\phi A'_\xi X - \phi {A^*}'_\xi X + \tau(X)W + \tau^*(X)W$$

Hence the result follows using (71) and (72).

□

Theorem 4.5. *Let $\bar{M}$ be an indefinite $\beta-$Kenmotsu statistical manifold and $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of $\bar{M}$. Then, the vector field U is parallel with respect to the Levi-Civita connection ∇^o on the leaf M' of $S(TM)$ if and only if*

$$A_N X + A^*_N X = u(A_N X)U + u(A^*_N X)U, \; for \; all \; X \in \Gamma(TM'),$$

w and $(\tau + \tau^*)$ vanish on M'.

Proof. Let, U be parallel with respect to ∇^o on M'. Then from (76), we have

$$\phi A_N X + \phi A_N^* X = w(A_N X)\xi + w(A_N^* X)\xi + w(A'_\xi X)N + w({A^*}'_\xi X)N - \tau(X)U - \tau^*(X)U + 2\beta w(X)\nu$$

Now $\phi X = \bar{\phi} X + u(X)N$ implies

$$\bar{\phi} A_N X + u(A_N X)N + \bar{\phi} A_N^* X + u(A_N^* X)N = w(A_N X)\xi + w(A_N^* X)\xi + w(A'_\xi X)N + w({A^*}'_\xi X)N + 2\beta w(X)\nu - \tau(X)U - \tau^*(X)U$$

Applying $\bar{\phi}$ on both sides of the above equation and using (23), we have

$$A_N X + A_N^* X = w(A_N X)W + w(A_N^* X)W + w(A'_\xi X)U + w({A^*}'_\xi X)U.$$

Then from (70),

$$A_N X + A_N^* X = w(A_N X)W + w(A_N^* X)W + u(A_N X)U + u(A_N^* X)U. \tag{80}$$

Using (80) in (76), we get

$$w(A'_\xi X)N + w({A^*}'_\xi X)N - u(A_N X)N - u(A_N^* X)N + 2\beta w(X)\nu - \tau(X)U - \tau^*(X)U = 0.$$

On comparing tangential and transversal parts, we conclude that

$$w(X) = 0, \ for\ all\ X \in \Gamma(TM')$$
$$\text{and} \quad \tau(X) + \tau^*(X) = 0$$

which implies $A_N X + A_N^* X = u(A_N X)U + u(A_N^* X)U$.

□

THEOREM 4.6. *Let $(M, g, S(TM))$ be a screen integrable ligtlike hypersurface of an indefinite $\beta-$Kenmotsu statistical manifold $(\bar{M}, \bar{g})$ and M' be a leaf of $S(TM)$. Then, the vector field W is parallel with respect to the Levi-Civita connection ∇^o on M' if and only if*

$$A'_\xi X + {A^*}'_\xi X = w(A'_\xi X)W + w({A^*}'_\xi X)W, \ for\ all\ X \in \Gamma(TM'),$$

u and $(\tau + \tau^*)$ vanish on M'.

Proof. Suppose that W is parallel with respect to ∇^o on M'. Then using (78), we obtain

$$\phi A'_\xi X + \phi {A^*}'_\xi X = u(A_N X)\xi + u(A_N^* X)\xi + u(A'_\xi X)N + u({A^*}'_\xi X)N + \tau(X)W + \tau^*(X)W + 2\beta u(X)W$$

Since $\phi X = \bar{\phi} X + u(X)N$, so the above equation implies

$$\bar{\phi} A'_\xi X + u(A'_\xi X)N + \bar{\phi} {A^*}'_\xi X + u({A^*}'_\xi X)N = u(A_N X)\xi + u(A_N^* X)\xi + u(A'_\xi X)N + u({A^*}'_\xi X)N + 2\beta u(X)\nu + \tau(X)W + \tau^*(X)W$$

On applying $\bar{\phi}$ on both sides and using (23), we drive

$$A'_\xi X + {A^*}'_\xi X = u(A_N X)W + u(A_N^* X)W + u(A'_\xi X)U + u({A^*}'_\xi X)U - \tau(X)\xi - \tau^*(X)\xi$$

From (70), we have

$$A'_\xi X + {A^*}'_\xi X = w(A'_\xi X)W + w({A^*}'_\xi X)W + u(A'_\xi X)U + u({A^*}'_\xi X)U - \tau(X)\xi - \tau^*(X)\xi.$$

Now comparing tangential and transversal parts, we have

$$A'_\xi X + {A^*}'_\xi X = w(A'_\xi X)W + w({A^*}'_\xi X)W + u(A'_\xi X)U + u({A^*}'_\xi X)U \tag{81}$$
$$\text{and} \quad \tau(X) + \tau^*(X) = 0.$$

Now, from (81), (78) and (22), we get

$$2\beta u(X)N - 2(\tau(X) + \tau^*(X))W = 0.$$

which implies

$$u(X) = 0, \ for \ all \ X \in \Gamma(TM').$$

Hence $A'_\xi X + {A^*}'_\xi X = w(A'_\xi X)W + w({A^*}'_\xi X)W$.

□

Let K be an element of $\phi(TM^\perp)\oplus\phi(tr(TM))$ which is non-degenrate vector subbundle of $S(TM)$ of rank 2. Then there exists non-zero function a and b such that

$$K = aW + bU. \tag{82}$$

We see that $a = w(K)$ and $b = u(K)$. Let κ be a 1−form locally defined by $\kappa(X) = g(K, X)$.

Theorem 4.7. *Let $\bar{M}$ be an indefinite $\beta-$Kenmotsu statistical manifold with $\nu \in \Gamma(TM)$ and M' be a leaf of $S(TM)$. Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of $\bar{M}$. Then*

$$\kappa(\nabla^o_X Y) = \beta\kappa(X)\eta(Y) - \kappa(\phi h^o(X, \bar{\phi}Y)),$$
$$\kappa([X, Y]) = \beta(\kappa(X)\eta(Y) - \kappa(Y)\eta(X)) - \kappa(\phi h^o(X, \bar{\phi}Y)) + \kappa(\phi h^o(Y, \bar{\phi}X))$$

for any $X, Y \in \Gamma(TM')$.

Proof. Since $\hat{\bar{\nabla}} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$, then for any $X, Y \in \Gamma(TM')$, using (73) and (82), we obtain

$$\kappa(\nabla^o_X Y) = \frac{1}{2}[\ g(\bar{\nabla}_X Y, aW + bU) + g(\bar{\nabla}^*_X Y, aW + bU)\ ]$$

which implies

$$\kappa(\nabla^o_X Y) = \frac{1}{2}[\ w(K)u(\bar{\nabla}_X Y) + u(K)w(\bar{\nabla}_X Y) + w(K)u(\bar{\nabla}^*_X Y) + u(K)w(\bar{\nabla}^*_X Y)\ ] \tag{83}$$

On applying ϕ on (68) and using (2) and (22), we get

$$\phi\nabla_X\bar{\phi}Y + \phi(\nabla_X u(Y)N) - B(X, \bar{\phi}Y)U + \bar{\nabla}^*_X Y - \eta(\nabla^*_X Y)\nu = \beta X\eta(Y) - \beta\eta(X)\eta(Y)\nu$$

which implies

$$\begin{aligned}\bar{\nabla}^*_X Y = \beta X\eta(Y) - \beta\eta(X)\eta(Y)\nu + \eta(\nabla^*_X Y)\nu + B(X, \bar{\phi}Y)U - \phi(\nabla_X u(Y)N) \\ +C(X, \bar{\phi}Y)W - u({\nabla'}_X\bar{\phi}Y)N - \bar{\phi}({\nabla'}_X\bar{\phi}Y).\end{aligned} \tag{84}$$

Similarly we have

$$\begin{aligned}\bar{\nabla}_X Y = \beta X\eta(Y) - \beta\eta(X)\eta(Y)\nu + \eta(\nabla_X Y)\nu + B^*(X, \bar{\phi}Y)U - \phi(\nabla^*_X u(Y)N) \\ +C^*(X, \bar{\phi}Y)W - u({\nabla^*}'_X\bar{\phi}Y)N - \bar{\phi}({\nabla^*}'_X\bar{\phi}Y).\end{aligned} \tag{85}$$

Using (84) and (85) in (83), we get

$$\begin{aligned}\kappa(\nabla^o_X Y) = \frac{1}{2}w(K)[B(X, \bar{\phi}Y) + B^*(X, \bar{\phi}Y)] + \frac{1}{2}u(K)[C(X, \bar{\phi}Y) + C^*(X, \bar{\phi}Y)] \\ +\beta[w(K)u(X) + u(K)w(X)]\eta(Y)\end{aligned}$$

which leads to

$$\kappa(\nabla^o_X Y) = \beta\kappa(X)\eta(Y) - \kappa(\phi h^o(X, \bar{\phi}Y)). \tag{86}$$

Considering $\kappa([X,Y]) = \kappa(\nabla^o_X Y) - \kappa(\nabla^o_Y X)$, and then using (86), we get the desired result.

□

5. Scope and Relevance

This paper initiates the theory of lightlike hypersurfaces of indefinite nearly $\beta-$ Kenmotsu statistical manifold. So, it is proposed that there is a scope for further exploration into the geometric characteristics of such hypersurfaces. Also, this research work has analyzed the lightlike geometry of leaves of integrable distributions in $\beta-$ Kenmotsu statistical manifold which can be subsequently investigated for various classes of an almost contact metric manifold combined with the statistical structure. The study of indefinite contact metric statistical manifolds is efficacious in the field of lightlike geometry which hold relevance in general relativity. Moreover, the diverse geometric attributes of hypersurfaces discussed in this paper can also be researched for complex manifolds.

References

[1] M. Ahmad and J. B. Jun, *On semi-invariant submanifolds of a nearly Kenmotsu manifold with a semi-symmetric non-metric connection*, J. Chungcheong Math. Soc. **23** (2) (2010), 257–257.
https://dx.doi.org/10.14403/jcms.2010.23.2.257

[2] S. Amari, *Differential-geometrical methods in statistics*, Lecture Notes in Statistics, New York, Springer, 1985.

[3] S. I. Amari, O. E. Barndorff-Nielsen, R. E. Kass, S. L. Lauritzen and C. R. Rao, *Differential geometry in statistical inference*, 1987.

[4] M. Atçeken and S. Dirik, *Pseudo-slant submanifolds of nearly Kenmotsu manifold*, Serdica Math. J. **41** (2-3) (2015), 243–262.

[5] O. Bahadr, *On lightlike geometry of indefinite Sasakian statistical manifolds*, AIMS Math. **6** (11) (2021), 12845–12862.
https://dx.doi.org/10.3934/math.2021741

[6] O. Bahadr, A. N. Siddiqui, M. Glbahar and A. H. Alkhaldi, *Main curvatures identities on lightlike hypersurfaces of statistical manifolds and their characterizations*, Mathematics **10** (13) (2022), 2290.
https://dx.doi.org/10.3390/math10132290

[7] O. Bahadr and M. M. Tripathi, *Geometry of lightlike hypersurfaces of a statistical manifold*, WSEAS Trans. Math. **22** (2023), 466–474.
https://dx.doi.org/10.37394/23206.2023.22.52

[8] N. Behzad and N. H. Kashani, *On nearly Kenmotsu manifolds*, Turk. J. Math. **37** (6) (2013), 1040–1047.
https://dx.doi.org/10.3906/mat-1207-38

[9] S. Bhatti and J. Kaur, *Contact CR and SCR lightlike submanifolds of an indefinite Kenmotsu statistical manifold*, arXiv:2303.17194v1, 30 Mar 2023.
https://dx.doi.org/10.48550/arXiv.2303.17194

[10] D. Chinea and C. Gonzalez, *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl. **156** (1990), 15–36.

[11] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Dordrecht, Kluwer Academic, 1996.
https://dx.doi.org/10.1007/978-94-017-2089-2

[12] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Springer Science & Business Media, 2011.

[13] H. Furuhata, *Hypersurfaces in statistical manifolds*, Differ. Geom. Appl. **27** (3) (2009), 420–429.
https://dx.doi.org/10.1016/j.difgeo.2008.10.019
[14] H. Furuhata, I. Hasegawa, Y. Okuyama and K. Sato, *Kenmotsu statistical manifolds and warped product*, J. Geom. **108** (3) (2017), 1175–1191.
https://dx.doi.org/10.1007/s00022-017-0403-1
[15] H. Furuhata, I. Hasegawa, Y. Okuyama, K. Sato and M. H. Shahid, *Sasakian statistical manifolds*, J. Geom. Phys. **117** (2017), 179–186.
https://dx.doi.org/10.1016/j.geomphys.2017.03.010
[16] C. Gherghe, *Harmonicity on nearly trans-Sasaki manifolds*, Demonstr. Math. **33** (1) (2000), 151–157.
https://dx.doi.org/10.1515/dema-2000-0118
[17] D. H. Jin, *Lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection*, Bull. Korean Math. Soc. **53** (6) (2016), 1771–1783.
https://dx.doi.org/10.4134/BKMS.b150972
[18] D. H. Jin, *Lightlike hypersurfaces of an indefinite nearly trans-Sasakian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **2** (2019), 195–210.
[19] T. Khan, *On non-invariant hypersurfaces of a nearly Kenmotsu manifold*, IOSR-JM **15** (6) (2019), 30–34.
[20] S. L. Lauritzen, *Statistical manifolds*, Differential geometry in statistical inference **10** (1987), 163–216.
https://dx.doi.org/10.1214/lnms/1215467061
[21] F. Massamba, *On semi-parallel lightlike hypersurfaces of indefinite Kenmotsu manifolds*, J. Geom. **95** (2009), 73–89.
https://dx.doi.org/10.1007/s00022-010-0021-7
[22] F. Massamba, *Lightlike geometry of leaves in indefinite Kenmotsu manifolds*, CUBO **13** (3) (2011), 69–89.
https://dx.doi.org/10.2478/s12175-012-0012-2
[23] F. Massamba, *Lightlike hypersurfaces in indefinite trans-Sasakian manifolds*, Results Math. **63** (1-2) (2013), 251–287.
https://dx.doi.org/10.1007/s00025-011-0197-7
[24] J. A. Oubina, *New class of almost contact metric structure*, Publ. Math. Debr. **32** (1985), 187–193.
[25] V. Rani and J. Kaur, *On the structure of lightlike hypersurfaces of an indefinite Kaehler statistical manifold*, Differ. Geom.-Dyn. Syst. **23** (2021), 221–234.
[26] A. N. Shukla, *Nearly trans-Sasakian manifolds*, KJSE **23** (1996), 139–144.
[27] E. Yasar, A. C. Çoeken and A. Yuecesan, *Lightlike hypersurfaces of semi-Riemannian manifold with semi-symmetric metric connection*, KJSE **34** (2007), 11–24.

Shagun Bhatti
Department of Mathematics, Punjabi University, Patiala, Punjab, India 147002.
E-mail: shagun.bhatti82@gmail.com

Jasleen Kaur
Department of Mathematics, Punjabi University, Patiala, Punjab, India 147002.
E-mail: jasleen2381@gmail.com