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Use of signifiers in discourse on the derivative in Korean and English speaking mathematics classroom

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ABSTRACT

Based on the observation that there is a common signifier "derivative" in English used for both the derivative at a point and the derivative function and that there are two phonetically and semantically different signifiers for those objects in Korean, we explored potential differences in classroom teaching discourses aimed to teach canonic mathematical discourses about the mathematical objects that those terms signify and their relations in Korean and English. Our analysis of one Korean teacher's and one American teacher's classroom discourse about the derivative using the commognitive approach uncovered differences in their teaching discourse in terms of addressing metarules about the terms and connections to colloquial discourse that exist in English, but not in Korean. We also found differences in how the American teacher made connections between the derivative at a point and the derivative function and how the Korean teacher connected the two signifiers in Korean - the American teacher extended a realization of the first defined object to define the second object, and after that, made shifts in a way that precludes simultaneous use of the common signifier for both objects whereas, in Korean teacher's discourse, no extensions were observed and the first defined object gained additional realizations.

Keywords Commognition, Derivative, Calculus, Mathematical object, Realization tree

Introduction

The impact of the language in which mathematics is practiced on mathematical discourse has been emphasized in the mathematics education literature with a long history. Examining students', teachers', or textbooks' uses of mathematical words is an important part in this literature (Choi, 2022; Han & Ginsburg, 2001; Kim et al., 2005; Kim, 2023). Our study explores uses of terms in mathematical discourse in different languages by examining the teacher's classroom discourse aimed to help students learning about those terms. We focused on the discourse about the derivative in Korean and American English (English, here after), which provides a useful context for the study. The derivative is a crucial concept in Calculus and commonly, within introductory calculus, derivatives can be separated into the derivative at a point and the derivative function. The motivation for our study is the observation that in English, the word "derivative" is included

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in “the derivative at a point” and “the derivative function” and the word “derivative” alone is often used for these objects, while in Korean the phonetically and semantically different terms “mi-bun-gye-su” and “do-ham-su” are used respectively for the corresponding objects and there is no common term corresponding to “derivative” that could be used for either object. By “the derivative function” we mean a function obtained by differentiating another function. Studies have addressed challenges students face distinguishing or relating these objects (Font & Contreras, 2008; Park, 2013).

We take a commognitive perspective (Sfard, 2008) to examine discourse on the derivative at a point, the derivative function, and the narratives connecting these objects in two languages based on teachers' classroom discourses. Commognition views learning mathematics as “the process in which students extend their discursive repertoire by individualizing the historically established discourse called mathematics” (Sfard, 2018, p. 222) and teaching as “the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse” (Tabach & Nachlieli, 2016, p. 303). Individualizing a discourse essentially means developing one's ability to communicate with others and oneself according to the rules of the discourse. We will refer to the historically established discourse called mathematics as the canonic discourse. As Kim et al. (2012) noted, from the commognitive perspective, one should not expect mathematical discourse in different languages to be homeomorphic and thus one should expect differences in the canonic discourses in different languages. From this perspective, differences in canonic discourses are differences in what students are trying to individualize. Consequently, the difference between English and Korean about use of terminologies about the derivative is a difference in the canonic discourses of the two languages. Thus, we speculate to see the differences in teachers' actions that aim to help students learn about those terminologies.

While the non-homeomorphic nature of mathematical discourse in different languages has been investigated from the student perspective, to our knowledge, our study is the first to focus on differences from the teaching perspective except for our previous paper addressing the differences canonic discourse about the derivative in Korean and English (Park & Rizzolo, 2022). For example, several studies examined linguistic differences between languages to explain its potential influences on teaching and learning mathematics. Their aims range from attempts to explain students' mathematical performance based on linguistic features (Han & Ginsburg, 2001; Miller & Stigler, 1987) to students' development of mathematical discourse based on the relevant colloquial discourse (Kim et al., 2012; Paik & Mix, 2003) to students' or teacher's engagement of language in bilingual settings (Favilli et al., 2013; Ní Ríordáin & Flanagan, 2020). In contrast, we consider a different aspect of language-dependent discourse in which one language has a common term for two different but related objects and the other does not. This is an important difference to investigate because several researchers have suggested that one term/notation being used for multiple objects creates challenges for teaching and learning about those objects in different mathematical discourses (e.g., inverse in algebra or discrete mathematics, Thompson & Rubenstein, 2000; tangent line in geometry and analysis, Biza & Zachariades, 2010). Especially, multiple uses of one term in the same discourse are hard to communicate with students (e.g., “limit” as a number and “limit” as a process, Güçler, 2013).

To investigate how the differences in the canonic discourses was addressed in teaching, we analyzed the discourse of one teacher in each language and address the following research question:

How does the existence of a common term in English and two terms in Korean manifest (or not manifest) in teachers' classroom discourses aimed to help students learn about mathematical objects those terms signify?

1. Theoretical Background

(1) Differences in Mathematical Discourses in Different Languages

The commognitive framework has been used in several settings to examine the dependence of mathematical discourse on language. This conceptual and discourse-analysis framework combines cognition and communication. It asserts that thinking is communicating with oneself and views mathematics as a type of discourse. Since languages are the medium for communicating, commognition makes the language-dependent nature of mathematics almost self-evident (Kim et al., 2012), which makes it a natural framework for studying such dependence.

Language studies using the commognitive framework have focused on the discourse of students. For example, Kim et al. (2012) used this framework to study differences between students' discourses about infinity in Korean and English and hypothesized that "there is no such thing as perfect homeomorphism between linguistically distinct versions of 'the same discourse'" (p. 87). Kim et al. (2012) supported their hypothesis by analyzing Korean and American students' discourse about infinity based on phonetic and semantic disconnections between "Korean formal mathematical discourse in infinity and its informal, colloquial predecessor" and "the cohesiveness of the infinity discourse" in colloquial and formal discourse in English (p. 95). Their results showed that American students' discourse developed continuously from their informal processual discourse on infinity whereas Korean students' infinity discourse was formal but "built of scraps of textbook expressions…[and] isolated from any other discourse they knew" (p. 105).

Ní Ríordáin (2013) adopts the commognitive framework, discussing how, theoretically, the syntactical structures of Irish "lend itself to easier interpretation of mathematical meaning in comparison to English" (pp. 1581-1582). This was supported with a later study using this framework (Ní Ríordáin & Flanagan, 2020), which found that bilingual students demonstrated more competency and shifts towards more formal discourse when using Irish rather than English.

Outside of the commognitive literature, if one views the ability to correctly solve problems as a proxy for fluency in a discourse, there is a long history of investigating phenomena similar to what Ní Ríordáin and Flanagan (2020) investigated. For example, some of these studies considered the feature of Asian language terms stating mathematical concepts clearly, such as part-whole relations in fraction words, which is not a feature in other languages (e.g., English) as an explanation for Asian-language speaking students' higher performance regarding those concepts (Han & Ginsburg, 2001; Miller & Stigler, 1987; Miura, 1987; Miura et al., 1999).

These studies have led to results that are potentially useful for teaching. For example, in their study investigating the differences described above in Korean and English, Paik and Mix, (2003) related American children's higher performance on matching tasks between fractions and pictorial representations with words involving parts (e.g., "of four parts, one" instead of ¼) than without them to their familiarity with "parts" in colloquial/informal discourse. Although this phenomenon itself only involves English and, in principle could have been investigated outside of the language comparison context, as a historical matter it was investigated in an effort to understand the impact of linguistic differences. In our study, we look at teachers' classroom discourses aimed to help students learn about mathematical objects signified by terms that are different in nature in each language.

(2) Importance of Teachers' Discourse in Commognition

Studies using the commognitive framework argue that the learning-teaching agreement between students and teachers is a condition for learning (Sfard, 2007). One component of the learning-teaching agreement is that "those who have agreed to be teachers feel responsible for the change in the learners' discourse, and those who have agreed to learn show confidence in the leaders' guidance and are genuinely willing to follow in the expert participants' discursive footsteps" (Sfrad, 2007, p.607). Based on this, within the commognitive framework, teaching can be defined as "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (Tabach & Nachlieli, 2016, p.303).

Within higher-mathematics, which features extreme objectification and rigor that is often unfamiliar to students, new students' learning "begins with an exposure to" instructors' discursive practices (Sfard, 2014, p. 202). Then, students start "collaborating across communicational conflict" due to the differences between this new discourse and their old discourse "by observing, and then imitating, the expert's moves while also trying to figure out the reasons", which might be the only way students "come to grips with the objects" at the abstract level (Sfard, 2014, p.202). A consequence of this is that as students start learning higher-mathematics, their discourse will include imitations of experts' discursive moves and, just as importantly, will not include moves from the canonical discourse that they have not observed. Both of these consequences have been documented in the context of calculus concepts. For example, in a study comparing Korean and American students' discourse on infinity, Kim et al. (2012) stated that Korean calculus students' discourse could "best be described as phrase-driven and ritualized, built of scraps of textbook expressions" (p.105). Specifically, in the context of the derivative, taken together, the three studies (Park, 2013; 2015; 2016) show the difficulty that calculus students have making moves that they have not directly observed their instructor or textbook making.

Our study focuses on teacher discourse because differences in canonic discourses in different languages that can be observed in teacher discourse (Park & Rizzolo, 2022) that is relevant for student learning. Specifically, the motive of instructor discourse is to bring students' discourse closer to the canonic discourse and the moves that students make when they start learning are imitations of the moves they observe.

(3) Derivative as a Mathematical Object

Our study focuses on discourse about two mathematical objects, the derivative at a point and the derivative function. In the commognitive framework, mathematical discourse can be divided into two levels, object-level discourse and metalevel discourse. Object-level discourse is discourse about mathematical objects. Metalevel discourse is discourse about the object-level discourse. For example, object-level discourse includes narratives such as "the derivative of $sin(\chi^2)$ is $2\chi\ cos(\chi^2)$", which is a narrative about mathematical objects that can be endorsed (or labeled as true) or rejected based on the rules of mathematics. Metalevel discourse includes statements like "the derivative of a composition of differentiable functions can be found using the chain rule". This is a statement about how a participant in the discourse could produce and substantiate the object-level statement about the derivative of $sin(\chi^2)$. This type of narrative is called a metarule. More generally, "metarules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p.201). In the example, the pattern of activity defined by the metarule is the use of the chain rule to find derivatives of compositions of differentiable functions.

To analyze discourse about mathematical objects, we must be precise about what mathematical objects are. Commognition treats mathematical objects as discursively constructed objects, thus comprised of aspects of the existing discourse. For example, when the derivative at a point is introduced to students it must be done using elements of the existing discourse, such as functions, limits and numbers. In general, constructing the derivative starts with applying the limit process on a difference quotient (e.g., $\frac{f(1+h)-f(1)}{h}$ as h approaches 0). When the limit exists, the limit as the product of this process is defined as "the derivative at a point", which is the signifier of this limit i.e. the phrase or symbol used to communicate about this product. The signifier is just the words, separate from their interpretation. This is an example of reifying, which is "introducing a noun or pronoun" that turns narratives about processes on some objects into narratives about objects or relations between objects (Sfard, 2008, p. 43). Specifically, the limit process is reified to a pair (e.g., "(1,$\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$)" or "(1,$f'(1)$)") which both become realizations of "the derivative at a point". Other realizations (e.g., the slope of the tangent line) can be introduced. Those realizations must be constructed at some point and thus are also signifiers. For example, "$f'(1)$" is introduced as being equal to, or the same as "$\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$". Although these two symbols are different, they are considered the same in the canonic mathematical discourse because "there is a closed set of true statements in which the two signifiers are exchangeable" (Sfard, 2012, p. 4). Thus, starting with a signifier, we can create a tree where the signifier is a node and its realizations are connected nodes. Each realization is a signifier with its own realizations, which are also added to the tree. This process repeats until one gets down to simple discursive objects (such as words for physically existing things, like "apple"), out of which all discourse is constructed. This tree is called the realization tree of the signifier. In Commognition, "a mathematical object" is defined as "a mathematical signifier together with its realization tree" (Sfard, 2012, p. 4). Thus, the mathematical object signified by "the derivative at a point" is the pair of the signifier "the derivative at a point" and its realization tree.

Once the "the derivative at a point" is defined, it can be used to construct "the derivative function." One way to do it is encapsulating, which is "assigning a signifier to a set of objects and using this signifier in singular when talking about a property of all of the set members taken together" (Sfard, 2008, p. 170). Once "the derivative at a point" is defined, one can consider, for any number a, the pair (a,$f'(a)$). A set of all these pairs can be encapsulated into a function called "the derivative function". Another way is through reification by expanding the x values on which the "derivative" is considered (i.e., the domain of "the derivative"), which is originally defined at a point. For example, by viewing (1,$f'(1)$) as a function whose domain is {1}, one can expand it to (1,$f'(1)$), (2,$f'(2)$) and view this as a function whose domain is {1,2}. One can tell the story of continuing the process until all the points at which f is differentiable are in the domain, and the end product of this story is called "the derivative function". This approach is typically carried out graphically, for example, by graphing the derivative of a function that is given as a graph by drawing a curve whose values match the slopes of tangent lines as x changes and calling the resulting curve "the derivative function". We call this way of constructing "the derivative function" expanding (Park, 2015; 2016). These are not the only ways to construct "the derivative function", but adopted by widely-used textbooks and instructors (including those in our study) (Park, 2015; 2016; Stewart, 2016).

Once the derivative function has been constructed, one can evaluate it at a specific input and find a value. Evaluating is part of the discourse on functions that students must be familiar with before learning about derivatives because function evaluation is used in the definition of the derivative at a point. In calculus, one of the important steps is saming (which is the process of considering two objects to be the same in the sense described above) the derivative at a point with the result of applying the evaluation process at a point to the derivative function. The existing research suggested that making this connection can be challenging for students (Park, 2013).

Methodology

1. Participants

One Korean high school teacher and one American Advanced Placement (AP) calculus teacher were recruited using their affiliation with a Korean institution and a U.S. institution through a graduate program or professional development that they were involved. We recruited teachers using purposive sampling (Cohen et al., 2017; Mertens, 2005), aiming to gather data on teachers' discourses that manifest differences in canonic discourses about the derivative in Korean and English in classroom teaching contexts. We sought teachers with expertise in teaching mathematics, particularly calculus, for this purpose. As shown in Table 1 (with pseudonyms), both teachers had extensive experience teaching calculus. Mrs. Kim had actively participated in professional development and facilitated workshops for mathematically gifted students. Mr. William had extensive professional development experience, a graduate degree, and had participated in the creation of state-wide mathematics tests. In our initial meetings, both teachers emphasized the importance of sense-making and rigor in mathematics classes. Based on these qualifications, we determined that Mrs. Kim and Mr. William were expert teachers who could teach canonic discourse about the derivative rigorously in a high school calculus class, providing data to examine classroom discourses that manifest the use of the common term, the derivative, in English and the use of the two terms, mi-bun-gye-su and do-ham-su in Korean. In the U.S., AP courses are high school courses for which students can earn college credit through scoring well on a standardized test. The Korean class we observed was for students who intend to study STEM fields in college. The teaching formats were similar in both classes, consisting of teacher's explanations of key words with several examples on the board, followed by students' individual or group work on problems, and then a whole class discussion led by the teacher with students' participation.

Table 1. Participants' information

	Mrs. Kim (Korean teacher)	Mr. William (U.S. teacher)
Degree	B.S. in Mathematic, Secondary mathematics education certificate	B.S. and M.S. in Mathematics
Teaching experience	12 years	10 years
Calculus teaching	4 times	7 times
Major teaching tools	Blackboard and chalk	SMARTBoard (A whiteboard with touch) detection
Students	32	34
Video-recording	Seven 50-minute classes	Six 90-minute classes
Students' ages	17-18	17-18
Grade	12	12

The different education level could have impacted their teaching but the numbers of nodes and complexity in their realization trees were compatible. There is a difference in length of the recorded lessons. Mrs. Kim's discourse stayed related to the mathematical objects of the lesson whereas Mr. William's class included various non-mathematical discourses and reviewing previous topics irrelevant to this study. Approximately 400 minutes of relevant discourse were recorded in Mr. William's class.

2. Data Collection/Analysis

We video-recorded classes where teachers laid the groundwork for introducing the key signifiers, introduced them, explained their connections, and added other realizations, which we transcribed. For analysis, we first separated lessons into several episodes based on the type of teachers' activities such as (a) constructing a mathematical object from existing objects, (b) connecting a key word (signifier) with its realizations, (c) connecting previously constructed mathematical objects, and (d) setting up and solving a problem. In this paper, we abbreviate canonic mathematical discourse in English as AMD and canonic mathematical discourse in Korean as KMD and view our data as an example of classroom discourse aimed to help students learn AMD about the derivative and an example of classroom discourse aimed to help students learn KMD about "mi-bun-gye-su" and "do-ham-su".

In each episode, we identified the teacher's use of signifiers and the contexts where they are used, which is an important when studying objectification (e.g., Nachlieli & Tabach, 2012). Mr. William made an explicit distinction between two uses of the single word “derivative” as “a number” and a “function”. Since “function” and “a number” cannot replace each other in most endorsed narratives in AMS, we categorized his use of “derivative” alone into two usages, as “the derivative at a point” or as “the derivative function” based on the surrounding context.

Episodes that included both objects signified by “the derivative at a point” and “the derivative function” in Mr. William’s discourse, or “mi-bun-gye-su” and “do-ham-su” in Mrs. Kim’s discourse, were viewed as potential manifestations of having a common signifier in English and two signifiers in Korean. Episodes in Mr. William’s discourse where “derivative” is used as, or in, a signifier for both objects were characterized as manifestations while the remaining potential manifestations were characterized as non-manifestations as shown in Figure 1. In Mrs. Kim’s discourse, a potential manifestation was characterized as a manifestation if “mi-bun-gye-su” and “do-ham-su” were both used as a non-manifestation otherwise as shown in Figure 2.

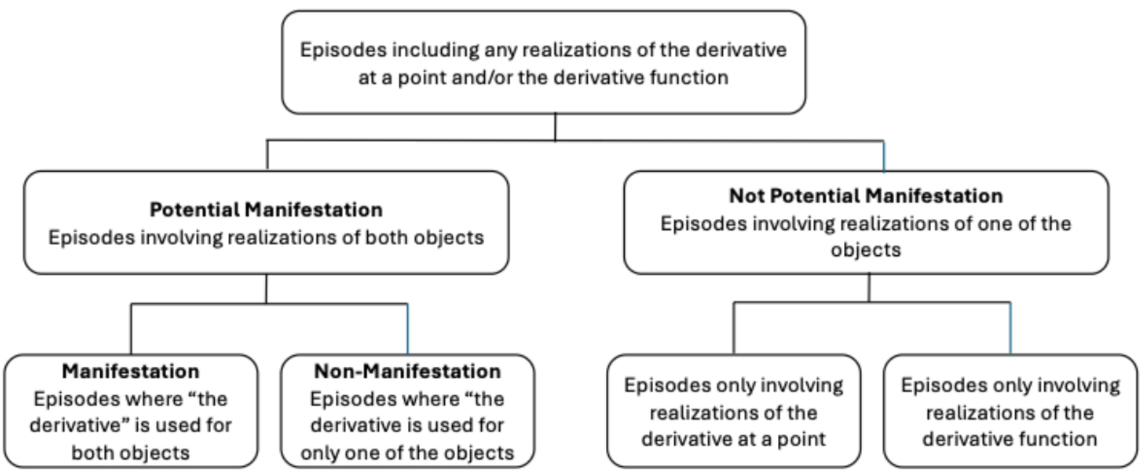

Figure 1. Classification of episodes from Mr. William’s class in terms of manifestation.

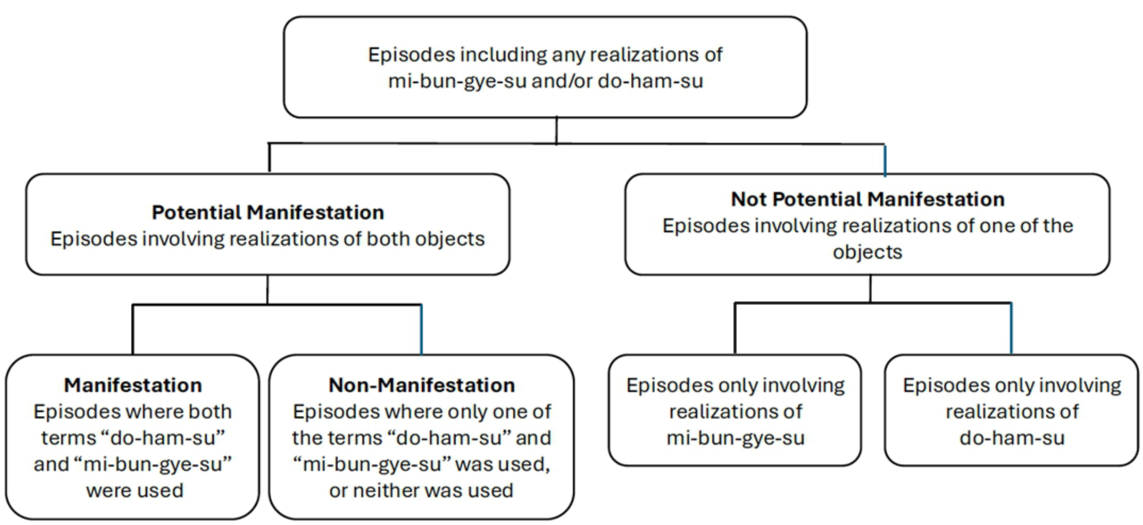

Figure 2. Classification of episodes from Mrs. Kim’s class in terms of manifestation.

For each episode, we created a tree consisting of a key signifier and its realizations, and edges and arrows showing their connections (e.g., evaluating, encapsulating). We then combined these into a single realization tree. Since we are interested in connections between the objects signified by “the derivative at a point” as shown in DR-PT tree on the left in Figures 3, 4 and “the derivative function” as shown in DR-FT tree on the right in Figures 3 and 4, we separated their realization trees. Although the former tree can be part of the second tree because when pairs of numbers are encapsulated to a function, those pairs of numbers become realizations of the function, we kept them separate to show their connections. Similarly, we created two realization trees for “mi-bun-gye-su” and “do-ham-su” for Mrs. Kim’s discourse and connected them with arrows. Following the principle of verbal fidelity and alternating insider and outsider perspectives of the discourse (Sfard, 2008), the English translation of Korean data and realization trees was carefully reviewed by another Korean-English speaking commognitive researcher, and English-translated Korean data and English data were analyzed through discussions with native English speaker who is also familiar with commognitive framework.

Results

The realization trees that we created from the two classes are shown in Figures 3, 4. Those trees provide an overview of the signifiers, their realizations, and their relations. In Results, we will discuss features of the two teacher’s classroom discourses about the derivative referring to those trees.

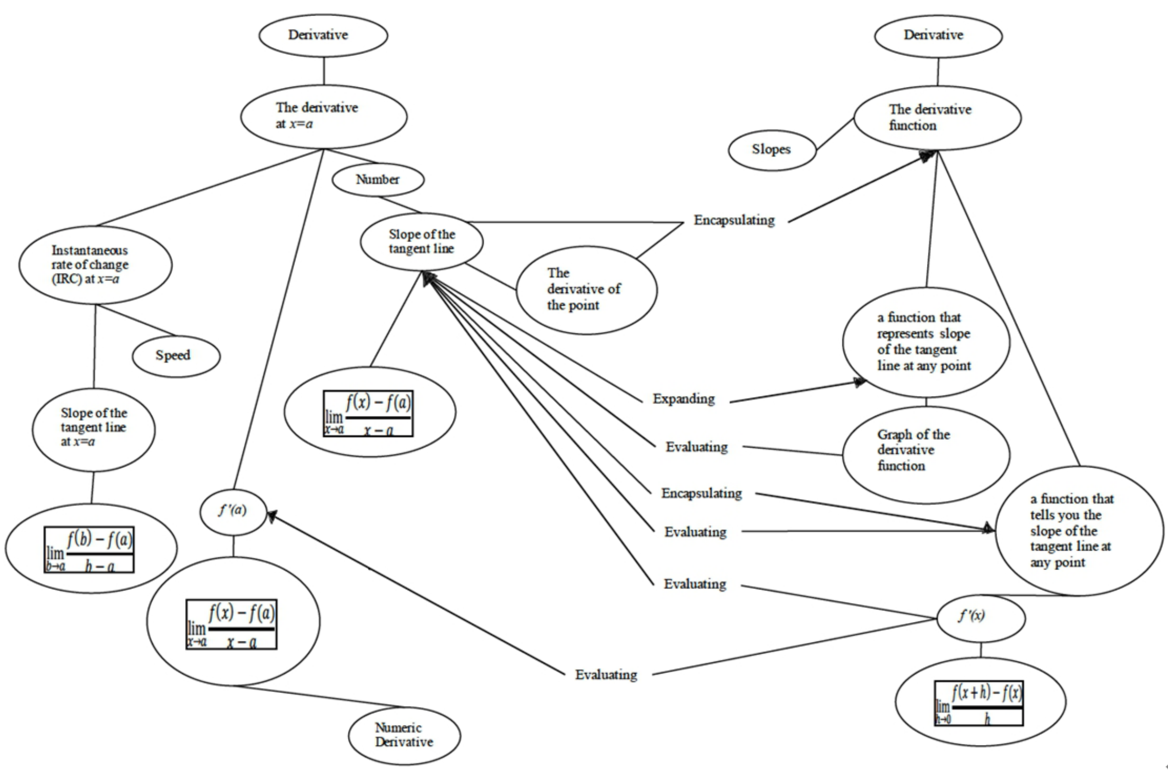

Figure 3. Realization tree from Mr. William’s classroom discourse.

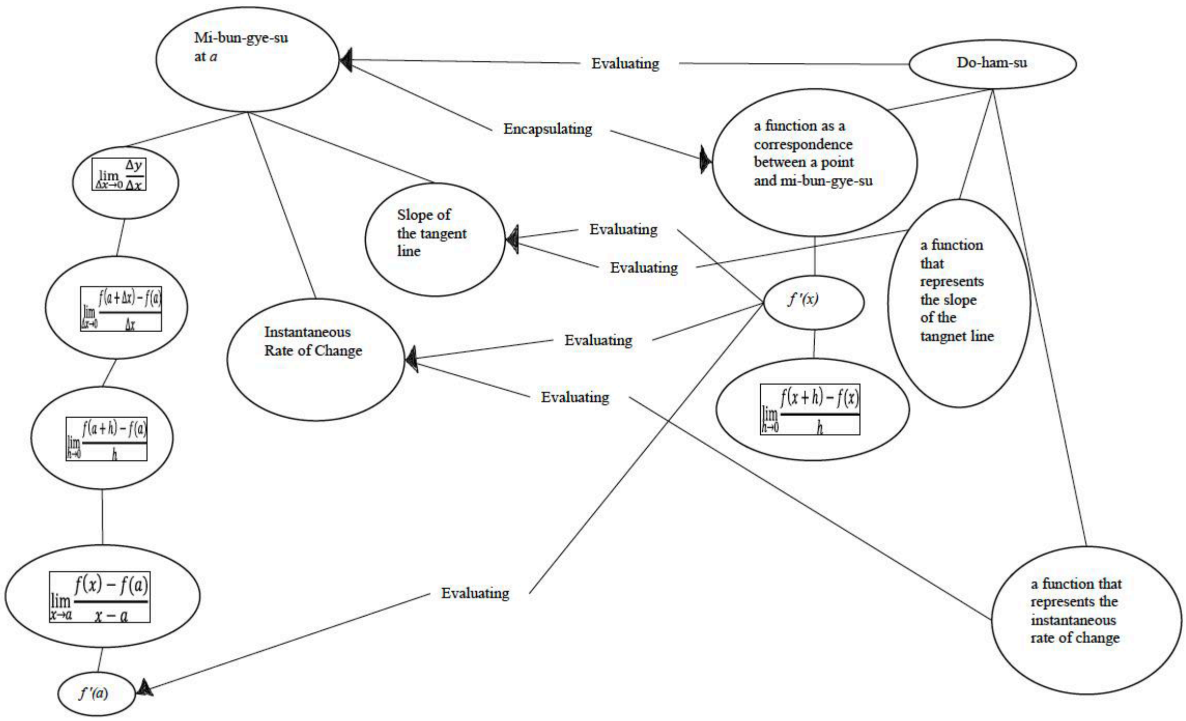

Figure 4. Realization tree from Mrs. Kim's classroom discourse.

1. Manifestations and Non-manifestations

As shown in Table 2, the number of potential manifestations, manifestations, and non-manifestations that occurred in each class shows that a much higher percentage of potential manifestations were manifestations in the Korean class. Specifically, Mrs. Kim used both "mi-bun-gye-su" and "do-ham-su" in 12 out of 20 episodes (60%) that included both objects whereas Mr. William used the common term "derivative" for both the derivative at a point and the derivative function only in 5 out of 27 such episodes (19%). Within the manifestations in Mr. William's class, our analysis uncovered his explicit discussion about metarules and connections to colloquial discourse that exist in English, but not in Korean. We also found differences in how Mr. William made connections between the derivative at a point and the derivative function and how Mrs. Kim connected the two signifiers in Korean, which we will detail below.

Table 2. Manifestations in American and Korean classes

	Potential manifestations	Manifestations	Non-manifestations
American class	27	5	22
Korean class	20	12	8

(1) Discussing Metarules of Use of the Key Signifiers

One of the main differences between the two classes we observed was whether the teacher explicitly addressing the rules for using the key signifiers about the derivative. In Mr. Williams' class, several manifestations revealed narratives about how to distinguish the two uses of the signifier "derivative" For example, when constructing the derivative function, he explained,

[W1] The *derivative* can be thought of as a number, which represents the slope of the tangent line at a point. But, notice that I can talk about slope of the function at every point along this curve ⋯ and the slope is always changing. Now, what we get then, is a function that represents the slope of the tangent line at any point on here, as a function of *x*. Same as the original function, as a function of *x*. And, that's called the derivative too. From the context, you understand what I'm talking about. ⋯ just remember now we're talking about numbers and we're talking about functions, and we have to keep them straight. Then we get into the other half of Calc 1, and talk about integrals. There will be the same thing. There will be some integrals that are numbers, some integrals that are functions⋯we use the same words because the concepts are related. Alright? (Day 2, 6 minutes)

Here, Mr. William explicitly used the signifier "derivative" for both the derivative at a point and the derivative function as shown on top of both DR-PT and DR-FT trees in Figure 3 and explained a rule for distinguishing the two uses. If the context indicates that the object signified by "derivative" is a number, then "derivative" signifies the derivative at a point as shown in "number" in DR-PT tree in Figure 3. Conversely, if the context indicates that the object signified by "derivative" is a function, then "derivative" signifies the "derivative function" as shown in DR-FT tree in Figure 3. This metarule of the AMD discourse has no counterpart, as there is no signifier in KMD that requires such distinction. The discussion of integrals is also noteworthy. In KMD, the terminology around integrals is similar to AMD, there is a signifier for "definite integrals" and signifier for "indefinite integrals" and a signifier like "integral" that can be used for either. As shown in this excerpt, in AMD, the usage of the terms around integrals can be explained by analogy with the terminology around derivatives, but such analogies do not exist in KMD. We coded this as expansion as shown in the first "Expanding" arrow from the left to the right in Figure 3 because he constructed derivative function by tracing the graph of the original function using the stick to indicate the slope of the tangent line. Thus, the set of points where the value of the derivative had been indicated expanded until it included all the points.

(2) Use of a Familiar Object with Similar the Discursive Feature

Another difference that we observed between the two classes was the teachers' use of another object with the similar discursive feature. Mr. William used changing numbers on a speedometer to motivate the construction of the derivative function as shown in "speed" on DR-PT tree in Figure 3, and then defined the derivative function similarly. He first discussed how speed changes over time using a graph as shown in Figure 5:

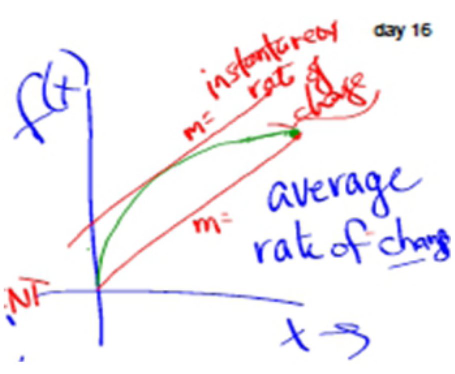

Figure 5. Mr. William's visual mediation of the instantaneous rate of change (IRC) at a point.

[W2] As I start out on my trip, how fast am I going (pointing to (0, 0) on the curve) (Students: Pretty fast). Pretty fast! (hand showing the tangent line at (0,0)) ⋯ As I travel along, what is happening to the slope? (moving a pen imitating tangent lines from (0, 0) along the curve) Or my instantaneous velocity sort of thing. Instantaneous rate of change? It is going down right? ⋯ Let's say⋯I have a flea, following this line. The flea has a rate of change meter on. It's the speedometer! He's traveling along the line, this number is changing all the time. Now ⋯ I– (slamming hands against the Smartboard) squash the flea at one point. That number is frozen ⋯That number I'm gonna use as the slope, and that line (drawing the tangent line at the point and writing “m”)⋯ is what we call the tangent line. The tangent line⋯that keeps visible that slope or that instantaneous rate of change at that point (Day 1, 37min).

Here, “IRC” was realized as what mediates “how fast” and the slope on the curve at a point, and then expanded over the interval as shown in “IRC” and “Slope of the tangent line at “$x=a$” on DR-PT tree in Figure 3, which was also realized as changing numbers on a rate of change meter. A specific number for “the slope” and “IRC” at a specific point was realized as a number on the meter when it is stopped at a specific point over a curve on this interval. Then, after he introduced the term “derivative” as “IRC” and “the slope of the tangent line”, he used the similar expansion to address the “derivative” as a function (See [W1]). Then, he used another graph to discuss the “derivative” as a function as shown in Figure 6:

[W3] If I come back here, what is ***the slope of that tangent line***? (showing a tangent line with the stick near the left bottom of the curve) Somebody guess ***that number***. (Students: Three). Three, okay⋯So, here, I start out with three or something. Now, as I move on... ***it gets smaller, and smaller*** until–(moving the tangent line to the right and stop at the first max point). What's the slope? (Students: Zero). ***It's gonna be zero*** cause I have a horizontal line (continues the conversation for the rest of the graph)⋯If I wanted the graph, sort of the picture of ***the derivative***, I have a ***fairly constant function***, up until here⋯***if I'm graphing a derivative function***, I'd come along here then I'd have to ***start dipping down⋯So I start up here*** *(creating the shape of the derivative function graph using the end of the stick)*, ***I come down and at this point, you know, I'm gonna have to be at zero...***You can follow the, the ball here, and graph or trace out on a graph, ***what the slopes are to get a function of this.***

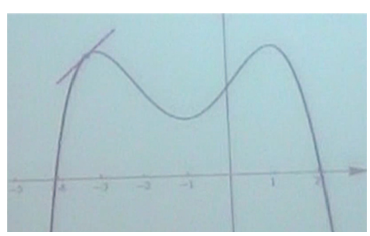

Figure 6. Graphs of a function and a tangent line (Day 3, 8 minutes).

Similar to the excerpt about the speed ([W2]), the “derivative” was realized as the “slope of the tangent line” that is changing along the curve (“smaller and smaller”). Two types of gestures mediated this: (a) multiple tangent lines to the original function with the word “slope” and numbers approximating their slopes and (b) a curve imitating the graph of the derivative function whose y value is those estimated numbers as a process of constructing the graph of the derivative function and then the product as shown in “Expanding” arrow in Figure 3.

Mrs. Kim did not include any similar colloquial discourse about IRC changing over an interval while constructing the do-ham-su and, in fact, did not use expansion as a method of constructing the do-ham-su. Though Mr. Kim defined “mi-bun-gye-su” with similar algebraic realizations and graphical mediator of the “slope” of the tangent line at a point that Mr. William used and also mentioned the IRC as a realization of “mi-bun-gye-su” as shown in “IRC” in DR-PT tree in Figure 4, she did not further use the “IRC” or “slope” to expand the use of “mi-bun-gye-su.” As shown in Figure 4, there is no “Expending” arrow. In Mrs. Kim's use, “mi-bun-gye-su” stayed as an object at a point. In and of itself, this appears to be the teachers making different didactic choices as Mr. William's discourse in this case can be faithfully translated into Korean. However, the parallels in Mr. William's colloquial discourse ([W2]) and construction of the derivative function ([W3]) suggest a modification of Mr. William's discourse along the lines of his discourse on the integral ([W1]) that would still make sense in English but could not be faithfully translated to Korean, which we will detail in Discussion.

(3) Encapsulating to Define the Derivative Function

In both classes, the episodes where the derivative function was first constructed were manifestations using encapsulating as shown in "Encapsulating" arrows in Figures 3, 4. However, despite broad similarities, there was a difference in the use of signifiers. Specifically, the following excerpt shows Mrs. Kim's encapsulation of the mi-bun-gye-su when first constructing the do-ham-su as shown in Figure 7:

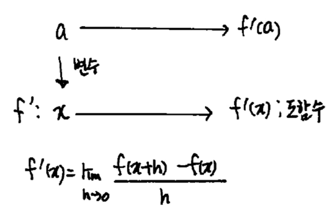

Figure 7. Mrs. Kim's writing about do-ham-su (변수 means a variable. 도함수 is do-ham-su.).

[K1] If a certain value of x,a is determined (writing "$a\rightarrow$" in Figure 7), and when there is a differentiable function, $f(x)$ (writing "$y=f(x)$)⋯the value of mi-bun-gye-su, $f'(a)$ (Writing "$a\rightarrow f'(a)$")⋯as a limit, only one value exists. When x is determined as a certain value a, there is one value of y corresponding to that [pointing to $f'(a)$ in "$a\rightarrow f'(a)$"] ⋯ A function represents a corresponding relation between two sets, but⋯each element in X corresponds to one element in Y ⋯. When x is determined as a certain a value, it corresponds to one y value as the value of mi-bun-gye-su ⋯ it can be a function because when a is determined, what it corresponds to is also determined by the only one. Now we consider the value of a as a variable. What do we generally use for a variable? (Students: x) x, so we can see this as a certain function that corresponds to the value of $f'(x)$. $f'(x)$ is a function ⋯ and we call it "do-ham-su". Let's write the expression (Writing $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$) (Day 3, 25 minutes).

Mrs. Kim encapsulated the relation between a and mi-bun-gye-su to a function. No other realizations previously used for mi-bun-gye-su were spoken and $f'(a)$ was the only symbolic realization used.

Mr. William's encapsulation was similar in general. However, after using "derivative" as the signifier for the derivative at a point, he switched to using "the slope of the tangent line" as the signifier for the derivative at a point. This is shown in Figure 3 with "Derivative at x=a" to "slope of the tangent line" in DR-PT tree, and then "Encapsulating" arrow to "The derivative function" on DR-FT tree:

[W4] Let's say I have this idea, a and that's supposed to be tangent. ... I can move that a somewhere else and I have another value of the derivative. For any point a (pointing to "a") I pick, in the right function, I have a different number that tells me what the slope of the tangent line is. What is this idea? ⋯ It's a function of any point I give you here, I can tell you what the slope of the tangent line is there.

He then wrote $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x}$,$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and said,

[W5] f prime of x equals the- it has to be a limit again, but now it's around this point x ⋯ this is a definition for a derivative (Day 1, 82 minutes).

In this way, Mr. William's discourse did not directly connect "derivative" as a signifier for the derivative at a point to “derivative” as a signifier for the derivative function. Rather, he transitioned to using “the slope of the tangent line” as the signifier for the derivative at a point and used this in the construction of the derivative function, which he then used “derivative” as a signifier for.

(4) Evaluation of the Derivative Function

Many of the potential manifestations occurred when the derivative function was evaluated at a point. In the Korean class, many of those were characterized as manifestations (i.e., both “do-ham-su” and “mi-bun-gye-su” were involved), while in the American class most of them were not - rather “slope of the tangent line” was typically used as the signifier for the object obtained by the evaluation as shown in Figure 3, particularly three of four “Evaluating” arrows to “Slope of the tangent line”. As seen in [W4], as part of the definition, Mr. William said, “it’s a function of any point I give you here, I can tell you what the slope of the tangent line is there”. This connection was highlighted in several problems that he solved in Days 3 and 4 computing “the slope of the tangent line” at a point given the equation for $f(x)$ by computing $f'(x)$ (e.g., $f'(x)=6x$ using $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and evaluated it at a point (e.g., $f'(3)=18$). In this transition, only the expression $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ were involved as realizations the derivative function, and its value at a point from evaluation was realized with “the slope of the tangent line” and a symbol f' with a number (e.g., $f'(3)$). Through this process, the two objects $-f'(x)$ evaluated at) and the slope of the tangent line at a (i.e., $f'(a)$) – were samed.

In contrast, once introduced, Mrs. Kim immediately connected “do-ham-su” evaluated at a point directly to “mi-bun-gye-su” as shown in Figure 4, particularly the first “Evaluating” arrow. She used the textbook statement, “Mi-bun-gye-su of $f(x)$ at $x=a$, $f'(a)$ is the value when $x=a$ is substituted in the expression of do-ham-su $f'(x)$”(Hwang et al., 2009, p. 123) and reiterated it multiple times. Then, she solved 13 problems about the transition from “do-ham-su” to “mi-bun-gye-su” similarly to Mr. William - finding the equation of “do-ham-su” and substituting a specific number for x. Most of these problems involved the word “mi-bun-gye-su” and also its realizations such as IRC, slope, which were also often used as part of a realization of “do-ham-su” in these problems as shown in Figure 4, particularly the rest five “Evaluating” arrows. Thus, narratives were provided saming “do-ham-su” evaluated at a specific point directly with “mi-bun-gye-su” and prior realizations of “mi-bun-gye-su”. The following excerpt shows an example involving both “do-ham-su” and “mi-bun-gye-su” with IRC as shown in Figure 8:

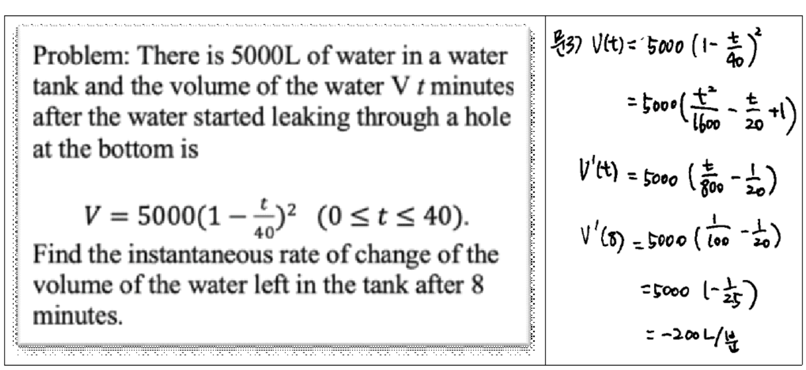

Figure 8. IRC problem and Mrs. Kim’s writing (Day 4, 36 minutes).

[K2] What is the concept of instantaneous rate of change? It is mi-bun-gye-su. So, we need to find mi-bun-gye-su at what value of t? when it is 8. My goal is to find the value of $V'(8)$. What do we need to find first? We need to find do-ham-su that informs the rate of change.

Note that, even when the problem starts with “instantaneous rate of change”, Mrs. Kim transitions to using “mi-bun-gye-su” before transitioning to “do-ham-su”.

(5) Shift in Use of the Signifier

In Mr. William’s class, there was a noticeable shift in the use of the word “derivative” depending on whether the derivative at a point and derivative function were both being discussed in the same episode. No such shift was observed in Mrs. Kim’s class. Specifically, Mr. William used the word “derivative” 131 times during 6 days of observation, and in most cases (103, 79%), he used the word alone. In episodes only involving realizations of the derivative at a point (thus, not a potential manifestation), “derivative” was the most dominantly used realization for the derivative at a point compared to other realizations - “IRC”, “slope of the tangent line”, and “$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$” (24 among 35 realizations, 69%). However, when episodes involved both objects (i.e., potential manifestations) he exclusively used “derivative” for the derivative function, and rarely used “derivative” for the derivative at a point in potential manifestations. The only times that he used the word for both objects were when he directly addressed the metarules for distinguishing its uses (e.g., [W1]), or in an episode which originally was not a potential manifestation (i.e., Mr. William solving review problems about “the derivative at a point” on Day 6), but students asked questions using the “derivative” as the derivative function and Mr. William adopted the student use to respond. This shows that potential manifestations were characterized as non-manifestations largely because he used a different signifier for the derivative at a point in episodes involving both objects. In non-manifestations, Mr. William typically used “derivative” for the derivative function and “the slope of the tangent line” for the derivative at a point, as in the following excerpt:

> [W6] Your goal … is to figure out which one is the original function, and which one is the derivative, just from the graphs (pointing to the graphs in Figure 9). (Students presenting their choices). Eddy claims that, the curve A is the original function because this slope is negative something, and so is that point (pointing to about $(0,-\frac{1}{2})$ on curve B)…Here, if I look at this part of the curve (pointing to the right-hand side of curve B where it is increasing), the slope of the tangent line is gonna be positive, positive, positive (imitating multiple tangent lines with a stick). If curve A’s the derivative, it would have to be up here, and it’s not. It’s down here.

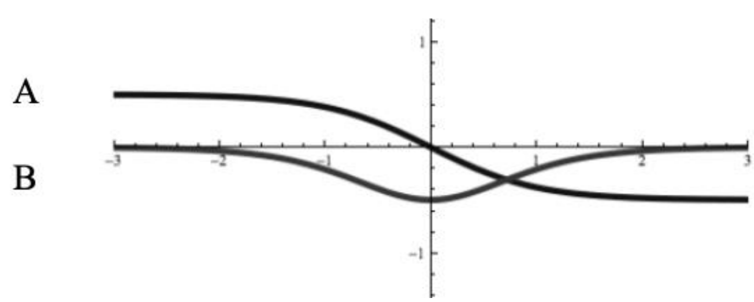

Figure 9. Curves A and B (Day 5, 33 minutes).

The result of the shift in Mr. William’s discourse, and the lack of such a shift in Mrs. Kim’s discourse, can be seen in the realization trees as shown in Figures 3, 4. As seen in the realization tree for Mr. William, connections from the derivative function to the derivative at a point occur using only “the slope of the tangent line” and “$f'(a)$” as signifiers for the derivative at a point. In contrast, Mrs. Kim directly connected “do-ham-su” and “mi-bun-gye-su” in 12 of the 20 potential manifestations we recorded.

Discussion and Conclusion

In our results, we showed that the differences between the teachers' classroom discourse aimed to teach the canonic mathematical discourse in English (AMD) and the canonic mathematical discourse in Korean (KMD) extend beyond the observation that Mr. William talked about the metarules for distinguishing the two uses of a signifier "derivative" in AMD while such discourse was not present in Mrs. Kim's class because KMD does not have common signifier for both objects. Although not every instructor in AMD will necessarily include these metarules, this does not change that they are part of the canonic discourse that those teachers aim to help students learn. Additionally, the common signifier "derivative" in AMD seemed to allow the use of narratives about a familiar object (speed) upon which one can develop the discourse about the new object "derivative". Pedagogically, this supports students' tendency of relying on familiar objects and their narratives instead of directly dealing with a new object (Kim et al., 2012; Nachlieli & Tabach, 2012; Sfard, 2008). It also seemed to relate to Mr. William's emphasis on how to separate the uses as a number and function, and to limit connections between the two objects due to the fact that only one of the two objects is likely to be signified by the word "derivative" in one context for communication purposes, which lead to a discursive shift. In Mrs. Kim's classroom discourse, two different signifiers "mi-bun-gye-su" and "do-ham-su" in KMD seemed to allow for more connections between their two realization trees but to make it harder to rely on familiar narratives e.g., about "speed" in the same way one could do in AMD. Also, in contrast to Mr. William's classroom discourse, in Mrs. Kim's class, there was no emphasis on "mi-bun-gye-su" being a number and "do-ham-su" being a function, which is consistent with studies about linguistic features that convey mathematical concepts clearly through semantic use of the terms (e.g., Han & Ginsburg 2001; Ní Ríordáin, 2013; Paik & Mix, 2003).

We believe that our results have practical instructional consequences. For example, there are many students from Korea and Japan (signifiers for derivatives in Japanese and Korean are homeomorphic) who go to the USA for graduate school and teach calculus. Although these students may have encountered English textbooks, they might not be prepared to engage with elements of the canonical discourse in English that do not exist in the canonical discourses of their native languages, which is likely how they first learned mathematics before being exposed to English terms. As a result, they may struggle to teach their students about these elements. Furthermore, in AMD, these metarules can be used as an analogy to explain the metarules around the use the signifier "integral" later in the course. In contrast, in KMD the use of signifiers around integration are similar to those in AMD given that one word could be also used for a number and a function, but cannot be explained by analogy with signifiers around derivatives. This further supports Kim et al.'s (2012) conclusion that "teachers need to be cognizant of those language-specific features of the discourse that may support learning and of those that may hinder successful participation" (p. 106). If we consider the reverse case of someone from a country where a word for "derivative" is used for both a number and a function teaching (e.g., the U.S.) in a country where two separate terms exist for those objects (e.g., Korea), the person may not see a need to connect those two terms given that such connections was never needed in their own language. Such hypothetical situations provide implications for teaching and teacher education. Explicit discussions about a metarules of how to distinguish the two uses of a mathematical word and potentially why using one term would make sense could help students understand the differences in those uses and the connections between them in a language like English. On the other hand, explicit discussions about why two terms are used for two objects when one is built up on the other and how they are related would help students understand the merit of using two separate terms and properties of each of the objects, which enables their mathematical connection. Given that these are subtle linguistic differences that could be easily dismissed with intensive use of algebraic symbols in mathematics, addressing these in teacher education or professional development for mathematics teachers would benefit preservice and in-service teachers' building of a strong knowledge about teaching mathematics especially for uses of terms in mathematics.

Additionally, the similarity between Mr. William's analogy of speed changing over an interval using a speedometer and his construction of the derivative function through expansion suggests a potential story that could be used in an English class to connect the colloquial and canonical discourses, but that could not be used in a Korean class. In particular, "speed" in colloquial discourse (in both Korean and English) behaves like "derivative" in AMD in the sense that it can be used as a number or as an (informal) function of time. As seen in Mr. William's class, in an English class, a teacher could explicitly recall this dual use of speed and then explain that "derivative" works in the same way students are familiar with "speed" working and, from context, one can determine if it is being used as a number or a changing quantity. This narrative could be used in English in support of the construction of the derivative through expanding, which involves reification. Although we did not directly observe this story in our data, as in (Nachlieli & Tabach, 2019), we find it valuable to explore scenarios that reasonably could have been observed based on our data. Although one can use speed to motivate the derivative in Korean, this particular narrative about how to use "derivative" - one use of the derivate can be expanded to another use of the derivative - has no direct counterpart in Korean. This supports Kim et al.'s (2012) hypothesis that "different language may differ in the means and degree of support they give to the encapsulation and reification" (p. 106).

The conclusions above are independent of the didactic choices of the teachers we observed. We learned about them by examining teacher discourse, but they are results about canonic discourses and how they connect to colloquial discourses. Although the discursive shift in the American teacher's discourse does depend on the teacher's didactic choices, we believe that it is interesting for instructional reasons and merits further theoretical investigation for the following reasons. The literature has shown that the connection between the derivative at a point and the derivative function can challenging for students (Font & Contreras, 2008; Park, 2013) and implicit shifts in instructor discourse have been tied to student difficulties (Güçler, 2013).

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Conflict of Interest

The authors declare that they have no competing interests.

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