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# PAIR MEAN CORDIAL LABELING OF CERTAIN LADDER GRAPHS

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Abstract. Consider a $(p,q)$ graph $G=(V,E)$. Define

$$\varrho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $\Upsilon = \{\pm 1, \pm 2, \ldots, \pm \varrho\}$ referred to as the label set. Let us consider a mapping $\varphi : V \to \Upsilon$, where, for every even $p$, distinct labels are assigned to the various elements of $V$ in $\Upsilon$, and for every odd $p$ , distinct labels are assigned to the $p-1$ elements of $V$ in $\Upsilon$, with a repeating label for the remaining one vertex. After that $\varphi$ is referred to as a pair mean cordial labeling (PMC-labeling) if for every edge $\mu\nu$ in $G$, there is a label for $\frac{\varphi(\mu)+\varphi(\nu)}{2}$ if $\varphi(\mu)+\varphi(\nu)$ is even and $\frac{\varphi(\mu)+\varphi(\nu)+1}{2}$ if $\varphi(\mu)+\varphi(\nu)$ is odd such that $|\bar{\mathbb{S}}_{\varphi_1} - \bar{\mathbb{S}}_{\varphi_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\varphi_1}$ and $\bar{\mathbb{S}}_{\varphi_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1 respectively. A pair mean cordial graph (PMC-graph) is defined as a graph $G$ with PMC-labeling. In this paper, we investigate the pair mean cordial labeling behavior of open ladder, triangular ladder, diagonal ladder, slanting ladder, circular ladder and diamond ladder.

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## 1. Introduction

A simple undirected and finite graph are considered for our research and All terminology and notations used here are as in [6]. In graph theory, one of the most extensively researched topics is graph labeling. The process is obtained by assigning integers to the elements of a graph, subject to specific limitations. Gallian [5] periodically updates a dynamic survey on graph labeling and the

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concept of cordial labeling was introduced in [3]. In this work, we examine the PMC labeling behavior of several ladder-related graphs, including as open ladder, triangular ladder, diagonal ladder, slanting ladder, circular ladder and diamond ladder.

## 2. Preliminaries

**Definition 2.1.** *[18] The ladder graph $L_r$ is defined as $L_r = P_r \times K_2$, where $K_2$ is a complete graph with two vertices, $P_r$ is a path with $r$ vertices, and $\times$ indicates the cartesian product. Evidently, the ladder graph $L_r$ consists of $3r-2$ edges and $2r$ vertices.*

**Definition 2.2.** *[18] The open ladder $O(L_r)$, $r \geq 2$ is a graph that is created by taking two paths of length $r$ with $V(O(L_r)) = \{\mu_s, \nu_s : 1 \leq s \leq r\}$ and $E(O(L_r)) = \{\mu_s\mu_{s+1}, \nu_s\nu_{s+1} : 1 \leq s \leq r-1\} \cup \{\mu_{s+1}\nu_{s+1} : 1 \leq s \leq r-2\}$. Evidently, an open ladder graph $O(L_r)$ consists of $3r-4$ edges and $2r$ vertices.*

**Definition 2.3.** *[18] The triangular ladder $TL_r$ is a graph that is created by taking the ladder $L_r$ by include the edges $\mu_s\nu_{s+1}, 1 \leq s \leq r-1$ where $\mu_s$ and $\nu_s, 1 \leq s \leq r$ are the vertices of $L_r$ such that $\mu_1, \mu_2, \ldots, \mu_r$ and $\nu_1, \nu_2, \ldots, \nu_r$ are two paths of length $r$ in $L_r$. Evidently, its vertices total $2r$, and its edges total $4r$–3.*

**Definition 2.4.** *[18] The open triangular ladder $O(TL_r)$, $r \geq 2$ is a graph that is created by taking an open ladder $O(L_r)$ by adding the edges $\mu_s\nu_{s+1}$ for $1 \leq s \leq r-1$. Evidently, the open triangular ladder graph $O(TL_r)$ consists of $4r-5$ edges and $2r$ vertices.*

**Definition 2.5.** *[18] The diagonal ladder $DL_r$ is a graph that is created by taking the ladder $L_r$ by include the edges $\mu_s\nu_{s+1}, \mu_{s+1}\nu_s$, $1 \leq s \leq r-1$ where $\mu_s$ and $\nu_s, 1 \leq s \leq r$ are the vertices of $L_r$ such that $\mu_1, \mu_2, \ldots, \mu_r$ and $\nu_1, \nu_2, \ldots, \nu_r$ are two paths of length $r$ in $L_r$. Evidently, the diagonal ladder $DL_r$ consists of $5r-4$ edges and $2r$ vertices.*

**Definition 2.6.** *[18] The open diagonal ladder $O(DL_r)$, $r \geq 2$ is a graph that is created by taking the diagonal ladder graph $DL_r$ through the elimination of the edges $\mu_s\nu_s$ for $s = 1$ and $r$. Evidently, an open diagonal ladder $O(DL_r)$ consists of $5r-6$ edges and $2r$ vertices.*

**Definition 2.7.** *The antiprism graph $A_r$, $r \geq 3$ contains an outer and inner cycles $C_r$, while the two cycles joined by lines $\nu_s\mu_s$, for $1 \leq s \leq r$ and $\nu_s\mu_{s+1}$, for $1 \leq s \leq r-1$.*

**Definition 2.8.** *[18] The slanting ladder $SL_r$ is a graph that is created by taking two paths $\mu_1\mu_2\ldots\mu_r$ and $\nu_1\nu_2\ldots\nu_r$ by joining each $\mu_s$ with $\nu_{s+1}$, $1 \leq s \leq r-1$. Evidently, the slanting ladder $SL_r$ consists of $3r-2$ edges and $2r$ vertices.*

**Definition 2.9.** *[19] The circular ladder graph $CL_r$, $r \geq 3$ is defined by $CL_r = C_r \times P_2$ where the path $P_2$ has two vertices, the cycle $C_r$ has n vertices, and $\times$ indicates the cartesian product. Evidently, the circular ladder graph $CL_r$ consists of $3r$ edges and $2r$ vertices.*

**Definition 2.10.** *[19] The Mobius ladder graph $M_r$ is a graph obtained from the ladder $P_r \times P_2$ by joining the opposite end points of the two copies of $P_r$. Evidently, its edges total $3r$, and its vertices $2r$.*

**Definition 2.11.** *[18] The connected graph $Dl_r$, which is a diamond ladder graph, has a vertex set $V(Dl_r) = \{\mu_s, \nu_s, \omega_s, \kappa_s \backslash 1 \leq s \leq r\}$ and an edge set $E(Dl_r) = \{\mu_s\nu_s, \mu_s\omega_s, \mu_s\kappa_s, \nu_s\omega_s, \nu_s\kappa_s \backslash 1 \leq s \leq r\} \cup \{\mu_s\mu_{s+1}, \nu_s\nu_{s+1}, \kappa_s\omega_{s+1} \backslash 1 \leq s \leq r-1\}$. Evidently, the diamond ladder graph $Dl_r$ consists of $8r-3$ edges and $4r$ vertices.*

## 3. Main Theorems

**Theorem 3.1.** *The ladder $L_r$ is PMC-graph for every $r \geq 3$ [8].*

**Theorem 3.2.** *The open triangular ladder $O(TL_r)$ is PMC-graph for every $r \geq 2$.*

*Proof.* Consider the open triangular ladder $O(TL_r)$. Assign the labels $2, 3, \dots, r$ on $\mu_1, \mu_2, \dots, \mu_{r-1}$ respectively and put the vertex $\mu_r$ with label $-r$. Also, we designate the label 1 to the vertex $\nu_1$ and put the vertices $\nu_2, \nu_4, \dots, \nu_r$ to the labels $-1, -2, \dots, -r+1$, respectively. Subsequently, $\bar{\mathbb{S}}_{\varphi_1} = 2r-3$ and $\bar{\mathbb{S}}_{\varphi_1^c} = 2r-2$. □

**Example 3.1.** *Figure 1 illustrates the PMC labeling of the open triangular ladder $O(TL_5)$.*

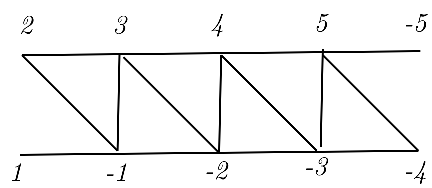


*Figure 1*

**Theorem 3.3.** *The open ladder $O(L_r)$ is PMC-graph for every $r \geq 2$.*

*Proof.* Let us consider an open triangular graph $O(TL_r)$. We have two following cases:

**Case** $A$: For odd $r$

Put labels $3, 5, \dots, r$ on $\mu_1, \mu_3, \dots, \mu_{r-2}$, accordingly. For each of $\mu_2, \mu_4, \dots, \mu_{r-1}$, assign the labels $-2, -4, \dots, -r+1$. Assign the vertex $\mu_r$ with label 1. Subsequently, allocate the labels $-1, -3, \dots, -r$ to $\nu_1, \nu_3, \dots, \nu_r$ and allocate the labels $2, 4, \dots, r-1$ to $\nu_2, \nu_4, \dots, \nu_{r-1}$.

**Case** $B$: For even $r$

At $\mu_1, \mu_3, \ldots, \mu_{r-3}$, allocate the labels $3, 5, \ldots, r-1$, accordingly. Allocate the labels $-2, -4, \ldots, -r$ to $\mu_2, \mu_4, \ldots, \mu_r$ in that order. Designate the vertex $\mu_{r-1}$ with label $r$. Subsequently, allocate the labels $-1, -3, \ldots, -r+1$ to $\nu_1, \nu_3, \ldots, \nu_{r-1}$, accordingly and allocate the labels $2, 4, \ldots, r-2$ correspondingly to $\nu_2, \nu_4, \ldots, \nu_{r-2}$. Additionally, assign the vertex $\nu_r$ with label 1. Moreover, explain the results in Table 1.

□

| $r$ | $\bar{\mathbb{S}}_{\varphi_1}$ | $\bar{\mathbb{S}}_{\varphi_1^c}$ |
|---|---|---|
| For odd $r$ | $\frac{3r-5}{2}$ | $\frac{3r-3}{2}$ |
| for even $r$ | $\frac{3r-4}{2}$ | $\frac{3r-4}{2}$ |

**Table 1**

**Theorem 3.4.** *The triangular ladder $TL_r$ is not PMC-graph for every $r \geq 2$.*

*Proof.* Suppose the triangular ladder $TL_r$ is PMC-graph. Therefore, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu)+\varphi(\nu) = 1$ and $\varphi(\mu)+\varphi(\nu) = 2$ can be utilized. The maximum number of edges labeled with one will be always $2r-3$. That's $\bar{\mathbb{S}}_{\varphi_1} \leq 2r-3$ and hence $\bar{\mathbb{S}}_{\varphi_1^c} \geq 2r$. Subsequently $\bar{\mathbb{S}}_{\varphi_1^c} - \bar{\mathbb{S}}_{\varphi_1} \geq 2r-(2r-3) = 3 > 1$, a contradiction.

□

**Theorem 3.5.** *The diagonal ladder $DL_r$ is not PMC-graph for every $r \geq 2$.*

*Proof.* Now suppose the diagonal triangular ladder $DL_r$ is PMC-graph. Therefore, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu)+\varphi(\nu) = 1$ and $\varphi(\mu)+\varphi(\nu) = 2$ can be utilized. The maximum number of edges labeled with 1 will be always $2r-3$. That's $\bar{\mathbb{S}}_{\varphi_1} \leq 2r-3$ and $\bar{\mathbb{S}}_{\varphi_1^c} \geq 3r-1$. Subsequently $\bar{\mathbb{S}}_{\varphi_1^c} - \bar{\mathbb{S}}_{\varphi_1} \geq 3r-1-(2r-3) = r+2 \geq 4 > 1$, a contradiction. □

**Theorem 3.6.** *The open diagonal ladder $O(DL_r)$ is not PMC-graph for every $r \geq 2$.*

*Proof.* Suppose the triangular ladder $O(DL_r)$ is PMC-graph. Therefore, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu)+\varphi(\nu) = 1$ and $\varphi(\mu)+\varphi(\nu) = 2$ can be utilized. The maximum number of edges labeled with 1 will be always $2r-3$. That's $\bar{\mathbb{S}}_{\varphi_1} \leq 2r-3$ and $\bar{\mathbb{S}}_{\varphi_1^c} \geq 3r-3$. Subsequently $\bar{\mathbb{S}}_{\varphi_1^c} - \bar{\mathbb{S}}_{\varphi_1} \geq 3r-3-(2r-3) = r \geq 2 > 1$, a contradiction.

□

**Theorem 3.7.** *The anti prism graph $A_r$ is not PMC-graph for every $r \geq 3$.*

*Proof.* Let $V(A_r) = \{\mu_s, \nu_s \backslash 1 \leq s \leq r\}$ and $E(A_r) = \{\mu_s\nu_s, \nu_r\nu_1, \mu_r\mu_1, \mu_r\nu_1 \backslash 1 \leq s \leq r\} \cup \{\mu_{s+1}\nu_s, \mu_s\mu_{s+1}, \nu_s\nu_{s+1} \backslash 1 \leq s \leq r-1\}$. Evidently, the anti prism graph $A_r$ consists of $4r$ edges and $2r$ vertices. Suppose $A_r$ is PMC-graph. Accordingly, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu)+\varphi(\nu) = 1$ and

$\varphi(\mu)+\varphi(\nu)=2$ can be utilized. The maximum number of edges labeled with 1 will be always $2r-3$. That's $\bar{\mathbb{S}}_{\lambda_1} \leq 2r-3$. Also $\bar{\mathbb{S}}_{\varphi_1^c} \geq 4r-(2r-3)=2r+3$. Subsequently $\bar{\mathbb{S}}_{\varphi_1^c}-\bar{\mathbb{S}}_{\varphi_1} \geq 2r+3-(2r-3)=6>1$, a contradiction.

□

**Theorem 3.8.** *The slanting ladder $SL_r$ is PMC-graph for every $r \geq 2$.*

*Proof.* Let us now consider the slanting ladder $SL_r$. We have two following cases:
**Case** *A*: For odd $r$
Put labels $3,5,\ldots,r$ on $\mu_1,\mu_3,\ldots,\mu_{r-2}$ correspondingly and allocate the labels $-1,-3,\ldots,-r+2$ according to $\mu_2,\mu_4,\ldots,\mu_{r-1}$. Fix the label $-r$ with $\mu_r$. Put label 1 on $\nu_1$. More over, allocate the labels $-2,-4,\ldots,-r+1$ corresponding to $\nu_2,\nu_4,\ldots,\nu_{r-1}$ and $2,4,\ldots,r-1$ to $\nu_3,\nu_5,\ldots,\nu_r$ accordingly.
**Case** *B*: For even $r$
Put labels $3,5,\ldots,r-1$ on $\mu_1,\mu_3,\ldots,\mu r-3$ correspondingly and allocate the labels $-1,-3,\ldots,-r+1$ according to $\mu_2,\mu_4,\ldots,\mu_r$. Designate the label $r$ with $\mu_{r-1}$. Fix label 1 with $\nu_1$. Additionally, allocate the labels $-2,-4,\ldots,-r$ corresponding to $\nu_2,\nu_4,\ldots,\nu_r$ and $2,4,\ldots,r-2$ to $\nu_3,\nu_5,\ldots,\nu_{r-1}$ accordingly. Moreover, explain the results in Table 2.

□

| $r$ | $\mathbb{S}_{\varphi_1}$ | $\mathbb{S}_{\varphi_1^c}$ |
|---|---|---|
| For odd $r$ | $\frac{3r-3}{2}$ | $\frac{3r-3}{2}$ |
| For even $r$ | $\frac{3r-4}{2}$ | $\frac{3r-2}{2}$ |

**Table 2**

**Example 3.2.** *Figure 2 illustrates the PMC labeling of the slanting ladder $SL_5$.*

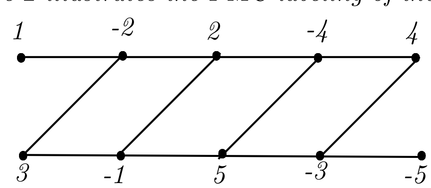


*Figure 2*

**Theorem 3.9.** *The circular ladder $CL_r$ is PMC-graph iff $r \geq 5$*

*Proof.* Let $E(CL_r)=\{\mu_s\nu_s,\mu_t\mu_{t+1},\nu_t\nu_{t+1},\mu_r\mu_1,\nu_r\nu_1 \backslash 1 \leq s \leq r, 1 \leq t \leq r-1\}$ and $V(CL_r)=\{\mu_s,\nu_s \backslash 1 \leq s \leq r\}$. Evidently, the circular ladder $CL_r$ consists of $3r$ edges and $2r$ vertices.
**Case** *A*: $r \equiv 0 \pmod 4$
**Subcase** *a*: $r=4$
Suppose that $CL_4$) is PMC-graph. Therefore, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu)+\varphi(\nu)=1$ and $\varphi(\mu)+\varphi(\nu)=2$ can be utilized. The

maximum number of edges labeled with 1 will be always 5. That's $\bar{\mathbb{S}}_{\varphi_1} \leq 5$ and $\bar{\mathbb{S}}_{\varphi_1^c} \geq 7$. Subsequently, $\bar{\mathbb{S}}_{\varphi_1^c} - \bar{\mathbb{S}}_{\varphi_1} \geq 7 - 5 = 2 > 1$, a contradiction.

**Subcase** $b$: $r > 4$

Put labels $-1, -3, \ldots, \frac{-r-2}{2}$ on $\mu_1, \mu_3, \ldots, \mu_{\frac{r+2}{2}}$. subsequently, allocate the labels $\frac{r+6}{2}, \frac{r+10}{2}, \ldots, r-2$ according to $\mu_{\frac{r+8}{2}}, \mu_{\frac{r+12}{2}}, \ldots, \mu_{r-1}$ and $3, 5, \ldots, r-1$ to $\mu_2, \mu_4, \ldots, \mu_{r-2}$ in that order. Designate label 1 with $\mu_r$. More over, Allocate the labels $2, 4, \ldots, \frac{r+4}{2}$ according to $\nu_1, \nu_3, \ldots, \nu_{\frac{r+2}{2}}$ and $\frac{-r-6}{2}, \frac{-r-10}{2}, \ldots, -r+1$ to $\nu_{\frac{r+6}{2}}, \nu_{\frac{r+10}{2}}, \ldots, \nu_{r-1}$ correspondingly. Subsequently, put labels $-2, -4, \ldots, -r$ on $\nu_2, \nu_4, \ldots, \nu_r$.

**Case** $B$: $r \equiv 1 \pmod 4$

Now allocate the labels $-1, -3, \ldots, \frac{-r-1}{2}$ corresponding to $\mu_1, \mu_3, \ldots, \mu_{\frac{r+1}{2}}$ and $\frac{r+7}{2}, \frac{r+11}{2}, \ldots, r-1$ to $\mu_{\frac{r+5}{2}}, \mu_{\frac{r+9}{2}}, \ldots, \mu_{r-2}$ correspondingly. Allocate the labels $3, 5, \ldots, r$ according to $\mu_2, \mu_4, \ldots, \mu_{r-1}$ and fix 1 with $\mu_r$. Additionally, designate the labels $2, 4, \ldots, \frac{r+3}{2}$ corresponding to $\nu_1, \nu_3, \ldots, \nu_{\frac{r+1}{2}}$ and $\frac{-r-5}{2}, \frac{-r-9}{2}, \ldots, -r$ to $\nu_{\frac{r+5}{2}}, \nu_{\frac{r+9}{2}}, \ldots, \nu_r$ correspondingly. Put labels $-2, -4, \ldots, -r+1$ on $\nu_2, \nu_4, \ldots, \nu_{r-1}$.

**Case** $C$: $r \equiv 2 \pmod 4$

Allocate the labels $-1, -3, \ldots, -r+1$ corresponding to $\mu_1, \mu_3, \ldots, \mu_{r-1}$ and $3, 5, \ldots, \frac{r+4}{2}$ to $\mu_2, \mu_4, \ldots, \mu_{\frac{r+2}{2}}$ accordingly. Put labels $\frac{-r-6}{2}, \frac{-r-10}{2}, \ldots, -n$ on $\mu_{\frac{r+6}{2}}, \mu_{\frac{r+10}{2}}, \ldots, \mu_r$. We designate the labels $2, 4, \ldots, r$ corresponding to $\nu_1, \nu_3, \ldots, \nu_{r-1}$ and $-2, -4, \ldots, \frac{-n-2}{2}$ to $\nu_2, \nu_4, \ldots, \nu_{\frac{n+2}{2}}$ accordingly. Subsequently, allocate the labels $\frac{r+8}{2}, \frac{r+12}{2}, \ldots, r-1$ according to $\nu_{\frac{r+6}{2}}, \nu_{\frac{r+10}{2}}, \ldots, \nu_{r-2}$ and fix 1 with $\nu_r$.

**Case** $D$: $r \equiv 3 \pmod 4$

**Subcase** $a$: $r = 3$

Suppose that $CL_3$ is PMC-graph. Therefore, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu) + \varphi(\nu) = 1$ and $\varphi(\mu) + \varphi(\nu) = 2$ can be utilized. The maximum number of edges labeled with 1 will be always 3. That's $\bar{\mathbb{S}}_{\varphi_1} \leq 3$ and $\bar{\mathbb{S}}_{\varphi_1^c} \geq 6$. Subsequently, $\bar{\mathbb{S}}_{\varphi_1^c} - \bar{\mathbb{S}}_{\varphi_1} \geq 6 - 3 = 3 > 1$, a contradiction.

**Subcase** $b$: $r > 3$

Put labels $-1, -3, \ldots, -r$ on $\mu_1, \mu_3, \ldots, \mu_r$. Designate the labels $3, 5, \ldots, \frac{r+3}{2}$ corresponding to $\mu_2, \mu_4, \ldots, \mu_{\frac{r+1}{2}}$ and $\frac{-r-5}{2}, \frac{-r-9}{2}, \ldots, -r+1$ to $\mu_{\frac{r+5}{2}}, \mu_{\frac{r+9}{2}}, \ldots, \mu_{r-1}$ accordingly. Allocate the labels $2, 4, \ldots, r-1$ according to $\nu_1, \nu_3, \ldots, \nu_{r-2}$ and $-2, -4, \ldots, \frac{-r-1}{2}$ to $\nu_2, \nu_4, \ldots, \nu_{\frac{r+1}{2}}$ correspondingly. Subsequently, allocate the labels $\frac{r+7}{2}, \frac{r+11}{2}, \ldots, r$ corresponding to $\nu_{\frac{r+5}{2}}, \nu_{\frac{r+9}{2}}, \ldots, \nu_{r-1}$ and fix 1 with $\nu_r$. Moreover, explain the results in Table 3.

□

**Example 3.3.** *Figure* 3 *illustrates the PMC labeling of the circular ladder graph* $CL_6$.

| $r$ | $\mathbb{S}_{\varphi_1}$ | $\mathbb{S}_{\varphi_1^c}$ |
|---|---|---|
| $r \equiv 0 \pmod 4$ | $\frac{3r}{2}$ | $\frac{3r}{2}$ |
| $r \equiv 1 \pmod 4$ | $\frac{3r-1}{2}$ | $\frac{3r+1}{2}$ |
| $r \equiv 2 \pmod 4$ | $\frac{3r}{2}$ | $\frac{3r}{2}$ |
| $r \equiv 3 \pmod 4$ | $\frac{3r-1}{2}$ | $\frac{3r+1}{2}$ |

**Table 3**

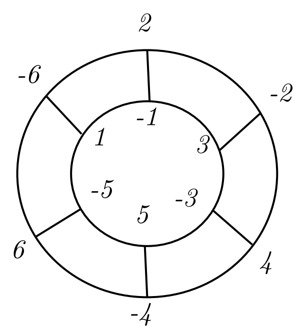


*Figure* 3

**Theorem 3.10.** *The Mobius ladder $M_r$ is PMC-graph for every $r \geq 5$.*

*Proof.* Let $E(M_r) = \{\mu_s\nu_s, \mu_1\nu_2, \mu_2\nu_1, \mu_r\mu_1, \nu_r\nu_1 \backslash 1 \leq s \leq r\} \cup \{\mu_s\mu_{s+1}, \nu_s\nu_{s+1} \backslash 2 \leq s \leq r-1\}$ and $V(M_r) = \{u_s, \nu_s \backslash 1 \leq s \leq r\}$. Evidently, the diamond ladder graph $Dl_r$ consists of $3r$ edges and $2r$ vertices.

**Case** $A$: $r = 3$

This Proof is consistent with Theorem 3.11.

**Case** $B$: $r = 4$

This Proof is consistent with Theorem 3.11.

**Case** $C$: $r > 4$

Let $\varphi(\mu_1) = -1$ and $\varphi(\nu_1) = 2$. The remaining vertices, $\mu_s, \nu_s$, $2 \leq s \leq r$, are then given the labels, following Theorem 3.11.

□

**Theorem 3.11.** *The diamond ladder graph $Dl_r$ is not PMC-graph for every $r \geq 2$.*

*Proof.* Suppose the diamond ladder graph $Dl_r$ is PMC-graph. Then, two possible outcomes exist if 1 is given to $\mu\nu$: both $\varphi(\mu)+\varphi(\nu) = 1$ and $\varphi(\mu)+\varphi(\nu) = 2$ can be utilized. The maximum number of edges labeled with 1 will be always $4r-3$. That's $\bar{\mathbb{S}}_{\varphi_1} \leq 4r-3$ and $\bar{\mathbb{S}}_{\varphi_1^c} \geq 8r-3-(4r-3) = 4r$. Subsequently $\bar{\mathbb{S}}_{\varphi_1^c} - \bar{\mathbb{S}}_{\varphi_1} \geq 4r-(4r-3) = 3 > 1$, a contradiction.

□

**Theorem 3.12.** *The subdivision of ladder graph $S(L_r)$ is PMC-graph for every $r \geq 2$*

*Proof.* Let $V(S(L_r) = \{\mu_s, \nu_s, \omega_s, \kappa_t, \eta_t : 1 \leq s \leq r \quad \text{and} \quad 1 \leq t \leq r-1\}$ and $E(S(L_r)) = \{\mu_s\omega_s, \nu_s\omega_s \backslash 1 \leq s \leq r\} \cup \{\mu_s\eta_s, \mu_{s+1}\eta_s, \nu_s\kappa_s, \nu_{s+1}\kappa_s \backslash 1 \leq s \leq r-1\}$. Evidently, the subdivision of ladder graph $S(L_r)$ consists of $6r-4$ edges and $5r-2$ vertices. We consider two cases:

**Case** $A$: For even $r$

Now allocate the labels $3, 8, \ldots, \frac{5r-4}{2}$ corresponding to $\mu_1, \mu_3, \ldots, \mu_{r-1}$ and $-3$, $-8, \ldots, \frac{-5r+14}{2}$ to $\mu_2, \mu_4, \ldots, \mu_{r-2}$ correspondingly. Designate label $\frac{-5r+2}{2}$ with $\mu_r$. Allocate the labels $-2, -7, \ldots, \frac{-5r+6}{2}$ according to $\eta_1, \eta_3, \ldots, \eta_{r-1}$ and $-5, -10, \ldots, \frac{-5r+10}{2}$ to $\eta_2, \eta_4, \ldots, \eta_{r-2}$. Additionally, designate the labels $-1, -6$, $\ldots, \frac{-5r+8}{2}$ corresponding to $\omega_1, \omega_3, \ldots, \omega_{r-1}$ and $5, 10, \ldots, \frac{5r-10}{2}$ to $\omega_2, \omega_4, \ldots, \omega_{r-2}$ correspondingly. Put label 1 on $\omega_r$. Subsequently designate the labels $2, 7, \ldots, \frac{5r-6}{2}$ corresponding to $\nu_1, \nu_3, \ldots, \nu_{r-1}$ and $-4, -9, \ldots, \frac{-5r+12}{2}$ to $\nu_2, \nu_4, \ldots, \nu_{r-2}$ correspondingly. Assign label $\frac{-5r+4}{2}$ with $\nu_r$. Allocate the labels $4, 9, \ldots, \frac{5r-2}{2}$ according to $\kappa_1, \kappa_3, \ldots, \kappa_{r-1}$ and $6, 11, \ldots, \frac{5r-8}{2}$ to $\kappa_2, \kappa_4, \ldots, \kappa_{r-2}$.

**Case** $B$: For odd $r$

Here designate the labels $3, 8, \ldots, \frac{5r-9}{2}$ corresponding to $\mu_1, \mu_3, \ldots, \mu_{r-2}$ and $-3, -8, \ldots, \frac{-5r+9}{2}$ to $\mu_2, \mu_4, \ldots, \mu_{r-1}$ correspondingly. Designate label $\frac{5r-3}{2}$ with $\mu_r$. Allocate the labels $-2, -7, \ldots, \frac{-5r+11}{2}$ according to $\eta_1, \eta_3, \ldots, \eta_{r-2}$ and $-5, -10, \ldots, \frac{-5r+5}{2}$ to $\eta_2, \eta_4, \ldots, \eta_{r-1}$. Designate the labels $-1, -6, \ldots, \frac{-5r+3}{2}$ corresponding to $\omega_1, \omega_3, \ldots, \omega_r$ and $5, 10, \ldots, \frac{5r-5}{2}$ to $\omega_2, \omega_4, \ldots, \omega_{r-1}$ correspondingly. Subsequently designate the labels $2, 7, \ldots, \frac{5r-11}{2}$ corresponding to $\nu_1, \nu_3, \ldots, \nu_{r-2}$ and $-4, -9, \ldots, \frac{-5r+7}{2}$ to $\nu_2, \nu_4, \ldots, \nu_{r-2}$ correspondingly. Fix 1 with $\nu_r$. Allocate the labels $4, 9, \ldots, \frac{5r-7}{2}$ according to $\kappa_1, \kappa_3, \ldots, \kappa_{r-2}$ and $6, 11, \ldots, \frac{5r-3}{2}$ to $\kappa_2, \kappa_4, \ldots, \kappa_{r-1}$. In both cases, $\bar{\mathbb{S}}_{\varphi_1^c} = 3r-2 = \bar{\mathbb{S}}_{\varphi_1}$ □

**Example 3.4.** *Figure* 4 *illustrates the PMC labeling of the subdivision of ladder graph* $S(L_4)$.

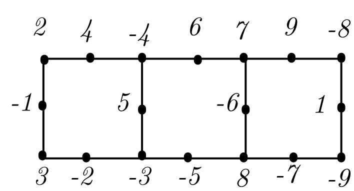


*Figure* 4

## 4. Discussion

The idea of cordial labeling was introduced by Cahit [3] and For more about the cordial and ladder related graphs in [1,2,4,7,9-19]. Inspired by the several authors who have written about graph labeling, we developed a brand-new labeling technique known as pair mean cordial labeling [8]. In this study, we have investigated the PMC-labeling behavior of several ladder-related graphs, including as open ladder, triangular ladder, diagonal ladder, slanting ladder, circular ladder and diamond ladder. We can discuss more similar results for various graphs.

## 5. Limitation of Research

It is now difficult to analyze the PMC-labeling behavior of the spectrum graph, cocktail party graph, lobster graph, clematis flower graph, bamboo tree, pumpkin graph, caterpillar graph, kayak paddle graph, $n-$ polygonal snake and layered graph.

## 6. Future Research

The rocket graph, polar grid graph, generalized web graph, generalized prism graph, generalized theta graph, lollipop graph and broom graph can also be analysed for PMC-labeling in future directions of research papers.

## 7. Conclusion

It is very interesting to find whether the graph admits PMC-labeling or not. The PMC-labeling behavior of some ladder related graphs like open ladder, triangular ladder, diagonal ladder, slanting ladder, circular ladder and diamond ladder are studied in this paper. It is possible to examine the PMC-labeling technique for some other family of graphs.

**Conflicts of interest** : The authors declare no conflict of interest.

**Data availability** : Not applicable

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