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TRIGONOMETRIC GENERATED RATE OF CONVERGENCE OF SMOOTH PICARD SINGULAR INTEGRAL OPERATORS

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Abstract. In this article we continue the study of smooth Picard singular integral operators that started in [2], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We establish the convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive. Our results are pointwise and uniform.

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1. Introduction

We are motivated by [1], [2] chapters 10-14, and [3], [4]. We use a trigonometric new Taylor formula from [3], see also [4]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their convergencnce properties quantitatively. We establish related inequalities involving the first modulus of continuity with respect to uniform norm and the estimates are pointwise and uniform. We provide a detailed proof.

2. Results

By [3], [4], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by trigonometric Taylor formula

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \tag{1}$$

$$\int_a^x \left[\left(f''(t) + f(t)\right) - \left(f''(a) + f(a)\right)\right]\sin(x-t)\,dt.$$

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For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j} \begin{pmatrix} r \\ j \end{pmatrix} j^{-n}, & j = 1, ..., r. \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \begin{pmatrix} r \\ j \end{pmatrix} j^{-n}, & j = 0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

$C_U(\mathbb{R})$ denotes the space of uniformly continuous functions on $\mathbb{R}$, and $C_B(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$.

Here we consider both $f, f'' \in C_U(\mathbb{R}) \cup C_B(\mathbb{R})$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Picard operators

$$P_{r,\xi}(f,x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-\frac{|t|}{\xi}} dt, \tag{4}$$

see [1]; $P_{r,\xi}$ are not in general positive operators, see [2].

We notice by

$$\frac{1}{2\xi} \int_{-\infty}^{\infty} e^{-\frac{|t|}{\xi}} dt = 1, \tag{5}$$

that

$$P_{r,\xi}(c,x) = c, \text{ where } c \text{ is a constant}, \tag{6}$$

and

$$P_{r,\xi}(f,x) - f(x) = \frac{1}{2\xi} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x+jt) - f(x)) e^{-\frac{|t|}{\xi}} dt \right). \tag{7}$$

Denote by

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R} \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{8}$$

the first modulus of continuity of f.

We set

$$\Delta(x) := P_{r,\xi}(f,x) - f(x) - f''(x) \left(\sum_{j=0}^{r} \frac{\alpha_j j^2}{j^2 \xi^2 + 1} \right) \xi^2; \quad \xi > 0, \ x \in \mathbb{R}. \tag{9}$$

We present our uniform approximation result.

Theorem 2.1. *It holds*

$$|\Delta(x)| \leq \|\Delta(x)\|_\infty \leq \xi^2 \omega_1(f'' + f, \xi) \left(\sum_{j=0}^{r} |\alpha_j| j^2 (j+1) \right) =: A; \quad \xi > 0, \ x \in \mathbb{R}. \tag{10}$$

And $\|\Delta(x)\|_{\infty} \to 0$*, as* $\xi \to 0$*. If* $f''(x) = 0$*, then* $|P_{r,\xi}(f,x) - f(x)| \leq A$.

Proof. By (1) we get that

$$f(x+jt) - f(x) = f'(x)\sin(jt) + 2f''(x)\sin^2\left(\frac{jt}{2}\right) +$$

$$\int_x^{x+jt} \left[(f''(s) + f(s)) - (f''(x) + f(x))\right] \sin(x + jt - s)\, ds, \tag{11}$$

or better

$$f(x+jt) - f(x) = f'(x)\sin(jt) + 2f''(x)\sin^2\left(\frac{jt}{2}\right) +$$

$$\int_0^{jt} \left[(f''(x+z) + f(x+z)) - (f''(x) + f(x))\right] \sin(jt - z)\, dz. \tag{12}$$

Furthermore, it holds

$$\sum_{j=0}^{r} \alpha_j \left[f(x+jt) - f(x)\right] =$$

$$f'(x)\sum_{j=0}^{r} \alpha_j \sin(jt) + 2f''(x)\sum_{j=0}^{r} \alpha_j \sin^2\left(\frac{jt}{2}\right) +$$

$$\sum_{j=0}^{r} \alpha_j \int_0^{jt} \left[(f''(x+z) + f(x+z)) - (f''(x) + f(x))\right] \sin(jt - z)\, dz, \tag{13}$$

or better

$$\sum_{j=0}^{r} \alpha_j \left[f(x+jt) - f(x)\right] =$$

$$f'(x)\sum_{j=0}^{r} \alpha_j \sin(jt) + 2f''(x)\sum_{j=0}^{r} \alpha_j \sin^2\left(\frac{jt}{2}\right) +$$

$$\sum_{j=0}^{r} \alpha_j j \int_0^{t} \left[(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))\right] \sin j(t - w)\, dw. \tag{14}$$

Call

$$R := R(t)$$

$$:= \sum_{j=0}^{r} \alpha_j j \int_0^{t} \left[(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))\right] \sin j(t - w)\, dw,$$

$$\forall\, t \in \mathbb{R}. \tag{15}$$

Then, for $t \geq 0$,

$$|R| \leq$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\int_0^t|(f''(x+jw)+f(x+jw))-(f''(x)+f(x))|\,|\sin j(t-w)|\,dw\le$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\int_0^t\omega_1(f''+f,jw)\,j(t-w)\,dw=$$

$(\xi>0)$

$$\sum_{j=0}^{r}|\alpha_j|\,j^2\int_0^t\omega_1\left(f''+f,\frac{\xi jw}{\xi}\right)(t-w)\,dw\le \tag{16}$$

$$\sum_{j=0}^{r}|\alpha_j|\,j^2\omega_1(f''+f,\xi)\int_0^t\left(1+\frac{jw}{\xi}\right)(t-w)\,dw=$$

$$\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\int_0^t(t-w)\,dw+\frac{j}{\xi}\int_0^t w^{2-1}(t-w)^{2-1}\,dw\right]=$$

$$\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\left.\frac{(t-w)^2}{2}\right|_t^0+\frac{j}{\xi}\frac{1}{6}t^3\right]=$$

$$\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{t^2}{2}+j\frac{t^3}{6\xi}\right].$$

Hence $(t\ge 0)$

$$|R|\le\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{t^2}{2}+j\frac{t^3}{6\xi}\right]. \tag{17}$$

Let now $t<0$, then

$$|R|\le$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\left|\int_0^t[(f''(x+jw)+f(x+jw))-(f''(x)+f(x))]\sin j(t-w)\,dw\right|\le$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\int_t^0|(f''(x+jw)+f(x+jw))-(f''(x)+f(x))|\,|\sin j(t-w)|\,dw\le$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\int_t^0\omega_1(f''+f,-jw)\,j(w-t)\,dw=$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\int_t^0\omega_1\left(f''+f,-jw\frac{\xi}{\xi}\right)j(w-t)\,dw=$$

$$\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\int_t^0\left(1-\frac{j}{\xi}w\right)(w-t)\,dw= \tag{18}$$

$$\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\int_t^0\left(w-t\right)dw+\frac{j}{\xi}\int_t^0\left(0-w\right)^{2-1}\left(w-t\right)^{2-1}dw\right]=$$

$$\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\left.\frac{\left(w-t\right)^2}{2}\right|_t^0+\frac{j}{\xi}\frac{1}{6}\left(-t\right)^3\right]=$$

$$\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{t^2}{2}+j\frac{\left(-t\right)^3}{6\xi}\right].$$

We found that $(t<0)$

$$\left|R\right|\leq\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{t^2}{2}+j\frac{\left(-t\right)^3}{6\xi}\right]. \tag{19}$$

Consequently, for $t\in\mathbb{R}$, we obtain

$$\left|R\left(t\right)\right|\leq\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{t^2}{2}+\frac{j}{6\xi}\left|t\right|^3\right],\quad \xi>0. \tag{20}$$

So, we have

$$\sum_{j=0}^{r}\alpha_j\left[f\left(x+jt\right)-f\left(x\right)\right]-f'\left(x\right)\sum_{j=0}^{r}\alpha_j\sin\left(jt\right)-2f''\left(x\right)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)=R\left(t\right). \tag{21}$$

Therefore, it holds

$$\Delta_1\left(x\right):=P_{r,\xi}\left(f,x\right)-f\left(x\right)-f'\left(x\right)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin\left(jt\right)e^{-\frac{|t|}{\xi}}dt\right)$$

$$-2f''\left(x\right)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)= \tag{22}$$

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}R\left(t\right)e^{-\frac{|t|}{\xi}}dt.$$

Hence we get

$$\left|\Delta_1\left(x\right)\right|\leq\frac{1}{2\xi}\int_{-\infty}^{\infty}\left|R\left(t\right)\right|e^{-\frac{|t|}{\xi}}dt\leq$$

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}\left[\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{t^2}{2}+\frac{j}{6\xi}\left|t\right|^3\right]\right]e^{-\frac{|t|}{\xi}}dt \tag{23}$$

(we use

$$\int_{-\infty}^{\infty}t^k e^{-\frac{|t|}{\xi}}dt=\left\{\begin{array}{l}0,\ \ k\text{ odd},\\ 2k!\xi^{k+1},\ \ k\text{ even}\end{array}\right. \tag{24}$$

)

$$= \omega_1 \left(f'' + f, \xi\right) \sum_{j=0}^{r} |\alpha_j| \, j^2 \frac{1}{2\xi} \int_{-\infty}^{\infty} \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3\right] e^{-\frac{|t|}{\xi}} dt$$

$$= \omega_1 \left(f'' + f, \xi\right) \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\frac{1}{4\xi} \int_{-\infty}^{\infty} t^2 e^{-\frac{|t|}{\xi}} dt + \frac{j}{12\xi^2} \int_{-\infty}^{\infty} |t|^3 e^{-\frac{|t|}{\xi}} dt\right] \quad (25)$$

$$= \omega_1 \left(f'' + f, \xi\right) \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\frac{1}{4\xi} 4\xi^3 + \frac{j\xi^4}{6\xi^2} \int_{0}^{\infty} \left(\frac{t}{\xi}\right)^3 e^{-\frac{|t|}{\xi}} d\frac{t}{\xi}\right]$$

$$= \omega_1 \left(f'' + f, \xi\right) \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\xi^2 + \xi^2 \frac{j}{6} \int_{0}^{\infty} z^3 e^{-z} dz\right]$$

$$= \omega_1 \left(f'' + f, \xi\right) \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\xi^2 + \xi^2 \frac{j}{6} 6\right] = \omega_1 \left(f'' + f, \xi\right) \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\xi^2 + \xi^2 j\right]$$

$$= \omega_1 \left(f'' + f, \xi\right) \xi^2 \left(\sum_{j=0}^{r} |\alpha_j| \, j^2 (1 + j)\right).$$

We have proved that

$$|\Delta_1 (x)| \leq \xi^2 \omega_1 \left(f'' + f, \xi\right) \left(\sum_{j=0}^{r} |\alpha_j| \, j^2 (j + 1)\right); \quad \xi > 0, \, x \in \mathbb{R}. \quad (26)$$

Notice that $\Delta_1 (x) \to 0, \;$ as $\xi \to 0$.

Next we simplify left hand side (22).

We observe that

$$\int_{-\infty}^{\infty} \sin (jt) \, e^{-\frac{|t|}{\xi}} dt = \int_{-\infty}^{0} \sin (jt) \, e^{-\frac{|t|}{\xi}} dt + \int_{0}^{\infty} \sin (jt) \, e^{-\frac{|t|}{\xi}} dt. \quad (27)$$

Notice $-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0$. So that

$$-\int_{-\infty}^{0} \sin (j (-(-t))) \, e^{-\frac{|t|}{\xi}} d(-t) = -\int_{-\infty}^{0} (-\sin (j (-t))) \, e^{-\frac{|t|}{\xi}} d(-t) = \quad (28)$$

$$\int_{-\infty}^{0} \sin j (-t) \, e^{-\frac{|t|}{\xi}} d(-t) = \int_{\infty}^{0} \sin j (t) \, e^{-\frac{|t|}{\xi}} dt = -\int_{0}^{\infty} \sin (jt) \, e^{-\frac{|t|}{\xi}} dt.$$

Therefore, it is

$$\int_{-\infty}^{\infty} \sin (jt) \, e^{-\frac{|t|}{\xi}} dt = 0, \quad j = 0, 1, ..., r. \quad (29)$$

Furthermore, we have that

$$\int_{-\infty}^{\infty} \sin^2 \left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = 2 \int_{0}^{\infty} \sin^2 \left(\frac{jt}{2}\right) e^{-\frac{t}{\xi}} dt, \quad j = 0, 1, ..., r. \quad (30)$$

The last follows by

$$\int_{-\infty}^{0}\sin^{2}\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=-\int_{-\infty}^{0}\left(-\sin\left(\frac{j(-t)}{2}\right)\right)^{2}e^{-\frac{|-t|}{\xi}}d(-t)\overset{(z=-t)}{=}\tag{31}$$

$$-\int_{\infty}^{0}\sin^{2}\left(\frac{jz}{2}\right)e^{-\frac{|z|}{\xi}}dz=\int_{0}^{\infty}\sin^{2}\left(\frac{jz}{2}\right)e^{-\frac{z}{\xi}}dz.$$

Next, we calculate

$$\int_{0}^{\infty}\sin^{2}\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\xi\int_{0}^{\infty}\sin^{2}\left(\left(\frac{j\xi}{2}\right)\frac{t}{\xi}\right)e^{-\frac{t}{\xi}}d\frac{t}{\xi}=$$

(call $\frac{t}{\xi}=:x$ and $\frac{j\xi}{2}=:a_1$)

$$\xi\int_{0}^{\infty}\sin^{2}(a_1x)\,e^{-x}dx=\tag{32}$$

(by Wolfram Alpha Computational Inteligence)

$$\xi\left(\frac{2a_1^2}{4a_1^2+1}\right)=\frac{j^2\xi^3}{2(j^2\xi^2+1)}.$$

Thus

$$\int_{-\infty}^{\infty}\sin^{2}\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\frac{j^2\xi^3}{j^2\xi^2+1},\quad j=0,1,...,r.\tag{33}$$

Consequently, it holds

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}\sin^{2}\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\frac{j^2\xi^2}{2(j^2\xi^2+1)}\to 0,\text{ as }\xi\to 0,\quad j=0,1,...,r.\tag{34}$$

Finally we obtain

$$-2f''(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^{2}\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)=-f''(x)\left(\sum_{j=0}^{r}\alpha_j\frac{j^2}{(j^2\xi^2+1)}\right)\xi^2.\tag{35}$$

Clearly now it is $\Delta_1(x)=\Delta(x)$.

The proof of the theorem is finally completed. □

We finish with

Corollary 2.2. *It follows* ($\xi>0$, $x\in\mathbb{R}$)

$$\|P_{r,\xi}(f,x)-f(x)\|_{\infty}\le\omega_1(f''+f.\xi)\left(\sum_{j=0}^{r}|\alpha_j|\,j^2(j+1)\right)\xi^2+\tag{36}$$

$$\|f''(x)\|_{\infty}\left(\sum_{j=0}^{r}|\alpha_j|\frac{j^2}{1+j^2\xi^2}\right)\xi^2\to 0,\quad \textit{as }\xi\to 0.$$

Proof. Easy, by (9) and (10). □

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