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4-TOTAL MEAN CORDIAL LABELING OF ARROW GRAPHS AND SHELL GRAPHS

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Abstract. In this paper we investigate the 4-total mean cordial labeling behavior of arrow graphs, shell-Butterfly graph and graphs obtained by joining two copies of shell graphs by a path.

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1. Introduction

All graphs in this paper are finite, simple and undirected graphs only. The cordial labeling was introduced by Cahit [9]. Subsequently cordial related labeling was studied in [1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 16, 17, 29, 30]. The notion of k-total mean cordial labeling has been introduced in [20]. The 4-total mean cordial labeling behavior of several graphs like cycle, complete graph, star, bistar, comb and crown have been studied in [20, 21, 22, 23, 24, 25, 26, 27, 28]. In this paper we investigate the 4- total mean cordial labeling behavior of arrow graphs, shell-butterfly graph and graphs obtained by joining two copies of shell graphs by a path. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary[14] and Gallian[13].

2. preliminaries

Definition 2.1. The graph $P_m X P_n$ is called the grid.
Let $V(P_m X P_n) = \{u_{i,j} : 1 \leq i \leq m; 1j \leq n\}$ and
$E(P_m X P_n) = \{u_{1,j}u_{1,j+1}, u_{2,j}u_{2,j+1}, \cdots, u_{m,j}u_{m,j+1} : 1 \leq j \leq n-1\}$
$\cup \{u_{i,1}u_{i+1,1}, u_{i,2}u_{i+1,2}, \cdots, u_{i,n}u_{i+1,n} : 1 \leq i \leq m-1\}$.

Definition 2.2. [3] An arrow graph $A_{m,n}$ with width n and length m ia obtained by joining a vertex v to the vertices $u_{1,1}$, $u_{2,1}$, $\cdots$, $u_{n,1}$ of $P_m X P_n$.

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Definition 2.3. [3] A double arrow graph $DA_{m,n}$ with width n and length m ia obtained by joining a vertices u to the vertices $u_{1,1}$, $u_{2,1}$, $\cdots$, $u_{n,1}$ and a vertex v to the vertices $u_{1,m}$, $u_{2,m}$, $\cdots$, $u_{n,m}$ of $P_m X P_n$.

Definition 2.4. [15] A shell graph S_n is defined as a cycle C_n with $n-3$ chords sharing a common end point called the apex. Clearly the shell S_n is the fan graph F_{n-1}.

Definition 2.5. [15] A double shell graph DS_n is defined to be a collection of edge disjoint shells that have their apex is common.

Definition 2.6. [15] A shell-butterfly graph $SB_n$is defined as a double shell with exactly two pendent edges at the apex.

3. Main results

Theorem 3.1. The graph $A_{n,2}$ is 4-total mean cordial for all $n \geq 2$.

Proof. Let $V(A_{n,2}) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(A_{n,2}) = \{uu_1, uv_1\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$.
Clearly, $|V(A_{n,2})| + |E(A_{n,2})| = 5n + 1$.

Assign the label 0 to the vertex u.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Consider the vertices u_1, u_2, …, u_{4r}. Assign the label 0 to the r vertices u_1, u_2, …, u_r. Next assign the label 1 to the r vertices u_{r+1}, u_{r+2}, …, u_{2r}. We now assign the label 2 to the r vertices u_{2r+1}, u_{2r+2}, …, u_{3r}. Now we assign the label 3 to the r vertices u_{3r+1}, u_{3r+2}, …, u_{4r}.
Consider the vertices v_1, v_2, …, v_{4r}. Assign the label 0 to the r vertices v_1, v_2, …, v_r. Then we assign the label 1 to the r vertices v_{r+1}, v_{r+2}, …, v_{2r}. Now we assign the label 2 to the r vertices v_{2r+1}, v_{2r+2}, …, v_{3r}. Finally we assign the label 3 to the r vertices v_{3r+1}, v_{3r+2}, …, v_{4r}.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r + 1$, $r \in \mathbb{N}$. Assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r)$ as in Case 1. Finally we assign the labels 0, 2 to the vertices u_{4r+1}, v_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r + 2$, $r \in \mathbb{N}$. Label the vertices u_i, v_i $(1 \leq i \leq 4r + 1)$ as in Case 2. Next we assign the labels 1, 3 to the vertices u_{4r+2}, v_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r + 3$, $r \in \mathbb{N}$. In this case, we assign the label for the vertices u_i, v_i $(1 \leq i \leq 4r)$ as in Case 1. We now assign the labels 1, 1, 0, 3, 3, 0 to the vertices u_{4r+1}, u_{4r+2}, u_{4r+3}, v_{4r+1}, v_{4r+2}, v_{4r+3}.

This vertex labeling f is a 4-total mean cordial labeling of $A_{n,2}$ follows from the Tabel 1

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$5r+1$	$5r$	$5r$	$5r$
$n = 4r+1$	$5r+2$	$5r+1$	$5r+2$	$5r+1$
$n = 4r+2$	$5r+2$	$5r+3$	$5r+3$	$5r+3$
$n = 4r+3$	$5r+4$	$5r+4$	$5r+4$	$5r+4$

Table 1

Case 5. $n = 2, 3$.
A 4-total mean cordial labeling of $A_{n,2}$ is given in Tabel 2

Value of n	u	u_1	u_2	u_3	v_1	v_2	v_3
2	0	2	0		3	2	
3	0	1	0	0	2	3	3

Table 2

□

Theorem 3.2. The graph $A_{n,3}$ is a 4-total mean cordial for all $n \geq 2$.

Proof. Let $V(A_{n,3}) = \{u, u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(A_{n,3}) = \{uu_1, uv_1, uw_1\}$ $\cup \{u_i v_i, v_i w_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1}, w_i w_{i+1} : 1 \leq i \leq n-1\}$.
Obviously $|V(A_{n,3}))| + |E(A_{n,3})| = 8n + 1$.

Assign the label 0 to the vertex u. Now we assign the label 2 to the vertex u_1. Next we assign the label 0 to the $n-1$ vertices u_2, u_3, …, u_n. We now assign the label 3 to the n vertices v_1, v_2, …, v_n. Next we assign the label 0 to the vertex w_1. Finally we assign the label 1 to the $n-1$ vertices w_2, w_3, …, w_n. Clearly $t_{mf}(0) = t_{mf}(1) = t_{mf}(3) = 2n$; $t_{mf}(2) = 2n+1$.

□

Theorem 3.3. The graph $A_{n,4}$ is 4-total mean cordial for all $n \geq 2$.

Proof. Let $V(A_{n,4}) = \{u, u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(A_{n,4}) =$
$\{u_i u_{i+1}, v_i v_{i+1}, x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{uu_1, uv_1, ux_1, uy_1\}$
$\cup \{u_i v_i, v_i x_i, x_i y_i : 1 \leq i \leq n\}$.
Note that $|V(A_{n,4})| + |E(A_{n,4})| = 11n + 1$.

Assign the label 0 to the vertex u.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \geq 2$. Assign the label 0 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. Consider the vertices $v_1, v_2, \ldots, v_{4r}$. Then we assign the label 2 to the $3r$ vertices $v_1, v_2, \ldots, v_{3r}$. We now assign the label 0 to the r vertices $v_{3r+1}, v_{3r+2}, \ldots, v_{4r}$.
Next we assign the label 3 to the $4r-2$ vertices $x_1, x_2, \ldots, x_{4r-2}$. Now we assign the label 2 to the 2 vertices x_{4r-1}, x_{4r}. We now assign the label 1 to the $4r-2$ vertices $y_1, y_2, \ldots, y_{4r-2}$. Finally we assign the label 3 to the 2 vertices y_{4r-1}, y_{4r}.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$. Now we assign the label 0 to the $4r+1$ vertices $u_1, u_2, \ldots, u_{4r+1}$. Next we assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. Then we assign the label 2 to the $3r+1$ vertices $v_{r+1}, v_{r+2}, \ldots, v_{4r+1}$. Now we assign the label 2 to the vertex x_1. We now assign the label 3 to the $4r$ vertices $x_2, x_3, \ldots, x_{4r+1}$. Next we assign the label 3 to the vertex y_1. Finally we assign the label 1 to the $4r$ vertices $y_2, y_3, \ldots, y_{4r+1}$.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, x_i, y_i $(1 \leq i \leq 4r+1)$ as in Case 2. Next we assign the labels 0, 2, 3, 1 to the vertices $u_{4r+2}, v_{4r+2}, x_{4r+2}, y_{4r+2}$.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \geq 0$. Assign the label 0 to the $4r+3$ vertices $u_1, u_2, \ldots, u_{4r+3}$. Next we assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. We now assign the label 2 to the $3r+3$ vertices $v_{r+1}, v_{r+2}, \ldots, v_{4r+3}$. Now we assign the label 3 to the $4r+3$ vertices $x_1, x_2, \ldots, x_{4r+3}$. Next we assign the label 0 to the vertex y_1. Finally we assign the label 1 to the $4r+2$ vertices $y_2, y_3, \ldots, y_{4r+3}$.

This shows that vertex labeling f is a 4-total mean cordial labeling of $A_{n,4}$ follows from the Tabel 3

Size of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$11r$	$11r$	$11r$	$11r$
$n = 4r+1$	$11r+3$	$11r+3$	$11r+3$	$11r+3$
$n = 4r+2$	$11r+5$	$11r+6$	$11r+6$	$11r+6$
$n = 4r+3$	$11r+9$	$11r+8$	$11r+9$	$11r+8$

Table 3

Case 5. $n = 2, 3$.
A 4-total mean cordial labeling of $A_{n,4}$ is given in Tabel 4

n	u	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
2	0	0	0			1	2			3	3			1	3		
4	0	0	0	0	0	2	2	2	0	3	3	3	3	1	1	1	3

Table 4

□

Theorem 3.4. The graph $DA_{n,2}$ is 4-total mean cordial for all $n \geq 2$.

Proof. Let $V(DA_{n,2}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(DA_{n,2}) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{uu_1, uv_1, vu_n, vv_n\}$. Clearly, $|V(DA_{n,2})| + |E(DA_{n,2})| = 5n + 4$.

Assign the label 0, 1 to the vertices u, v.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \geq 2$. Consider the vertices $u_1, u_2, \ldots, u_{4r}$. Assign the label 0 to the r vertices $u_1, u_2, \ldots, u_r$. Next assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \ldots, u_{2r}$. We now assign the label 2 to the $r-1$ vertices $u_{2r+1}, u_{2r+2}, \ldots, u_{3r-1}$. Now we assign the label 3 to the $r+1$ vertices $u_{3r}, u_{3r+1}, \ldots, u_{4r}$. Consider the vertices $v_1, v_2, \ldots, v_{4r}$. Assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. Then we assign the label 1 to the r vertices $v_{r+1}, v_{r+2}, \ldots, v_{2r}$. Now we assign the label 2 to the $r+1$ vertices $v_{2r+1}, v_{2r+2}, \ldots, v_{3r+1}$. Finally we assign the label 3 to the $r-1$ vertices $v_{3r+2}, v_{3r+3}, \ldots, v_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices $u_1, u_2, \ldots, u_r$. Next assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \ldots, u_{2r}$. We now assign the label 2 to the r vertices $u_{2r+1}, u_{2r+2}, \ldots, u_{3r}$. Now we assign the label 3 to the r vertices $u_{3r+1}, u_{3r+2}, \ldots, u_{4r}$. Next we assign the label 0 to the vertex u_{4r+1}. We now assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. Then we assign the label 1 to the r vertices $v_{r+1}, v_{r+2}, \ldots, v_{2r}$. Now we assign the label 2 to the r vertices $v_{2r+1}, v_{2r+2}, \ldots, v_{3r}$. Next we assign the label 3 to the r vertices $v_{3r+1}, v_{3r+2}, \ldots, v_{4r}$. Finally we assign the label 3 to the vertex v_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \in \mathbb{N}$. Label the vertices u_i, v_i $(1 \leq i \leq 4r+1)$ as in Case 2. Now we assign the labels 0, 2 to the vertices u_{4r+2}, v_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \in \mathbb{N}$. As in case 2, we assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r)$. Finally we assign the labels 1, 0, 0, 3, 3, 2 to the vertices u_{4r+1},

u_{4r+2}, u_{4r+3}, v_{4r+1}, v_{4r+2}, v_{4r+3}.

This vertex labeling f is a 4-total mean cordial labeling of $DA_{n,2}$ follows from the Tabel 5

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$5r+1$	$5r+1$	$5r+1$	$5r+1$
$n = 4r+1$	$5r+2$	$5r+2$	$5r+3$	$5r+2$
$n = 4r+2$	$5r+4$	$5r+3$	$5r+4$	$5r+3$
$n = 4r+3$	$5r+4$	$5r+5$	$5r+5$	$5r+5$

Table 5

Case 5. $n = 2, 3$.
A 4-total mean cordial labeling of $DA_{n,2}$ is given in Tabel 6

Value of n	u	v	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4
2	0	1	0	1			3	3		
3	0	1	2	0	0		3	2	3	
4	0	1	0	1	3	2	0	1	2	3

Table 6

□

Theorem 3.5. The graph $DA_{n,3}$ is a 4-total mean cordial for all $n \geq 2$.

Proof. Let $V(DA_{n,3}) = \{u, v, u_i, v_i, w_i : 1 \leq i \leq n\}$ and Let $E(DA_{n,3}) = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i, v_iw_i : 1 \leq i \leq n\} \cup \{uu_1, uv_1, uw_1, vu_n, vv_n, vw_n\}$. Note that $|V(DA_{n,3})| + |E(DA_{n,3})| = 8n + 5$.

Assign the labels 0, 2 to the vertices u, v respectively. Next we assign the label 0 to the n vertices $u_1, u_2, \ldots, u_n$. We now assign the label 2 to the vertex v_1. Now we assign the label 3 to the $n-1$ vertices $v_2, v_3, \ldots, v_n$. Then we assign the label 1 to the $n-1$ vertices $w_1, w_2, \ldots, w_{n-1}$. Finally we assign the label 3 to the vertex w_n.
Obviously $t_{mf}(0) = t_{mf}(1) = t_{mf}(2) = 2n+1$; $t_{mf}(3) = 2n+2$.

□

Theorem 3.6. The graph $DA_{n,4}$ is 4-total mean cordial for all $n \geq 2$.

Proof. Let $V(DA_{n,4}) = \{u, v, u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(DA_{n,4}) = \{u_iu_{i+1}, v_iv_{i+1}, x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i, v_ix_i, x_iy_i : 1 \leq i \leq n\} \cup \{uu_1, uv_1, ux_1, uy_1, vu_n, vv_n, vx_n, vy_n\}$.

Clearly $|V(DA_{n,4})| + |E(DA_{n,4})| = 11n + 6$.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the labels 0 and 3 to the vertices u and v respectively. Now we assign the label 0 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. Next we assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. We now assign the label 2 to the $3r$ vertices $v_{r+1}, v_{r+2}, \ldots, v_{4r}$. Next we assign the label 3 to the $4r$ vertices $x_1, x_2, \ldots, x_{4r}$. Finally we assign the label 1 to the $4r$ vertices $y_1, y_2, \ldots, y_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r + 1$, $r \in \mathbb{N}$. Now we assign the labels 0 and 3 to the vertices u and v respectively. Next we assign the label 0 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. We now assign the label 1 to the vertex u_{4r+1}. Now we assign the label 0 to the $r + 1$ vertices $v_1, v_2, \ldots, v_{r+1}$. Next we assign the label 2 to the $3r$ vertices $v_{r+2}, v_{r+3}, \ldots, v_{4r+1}$. Then we assign the label 3 to the $4r + 1$ vertices $x_1, x_2, \ldots, x_{4r+1}$. Finally we assign the label 1 to the $4r+1$ vertices $y_1, y_2, \ldots, y_{4r+1}$.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r + 2$, $r \in \mathbb{N}$. We now assign the labels 0 and 1 to the vertices u and v respectively. Now we assign the label 0 to the $4r + 2$ vertices $u_1, u_2, \ldots, u_{4r+2}$. Next we assign the label 2 to the $3r + 1$ vertices $v_1, v_2, \ldots, v_{3r+1}$. We now assign the label 0 to the $r + 1$ vertices $v_{3r+2}, v_{3r+3}, \ldots, v_{4r+2}$. Then we assign the label 3 to the $4r$ vertices $x_1, x_2, \ldots, x_{4r}$. Now we assign the labels 3 and 2 to the vertices x_{4r+1} and x_{4r+2}. We now assign the label 1 to the $4r$ vertices $y_1, y_2, \ldots, y_{4r}$. Finally we assign the labels 2 and 3 to the vertices y_{4r+1} and y_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r + 3$, $r \geq 0$. Assign the labels 0 and 3 to the vertices u ans v respectively. Then we assign the label 0 to the $4r + 3$ vertices $u_1, u_2, \ldots, u_{4r+3}$. Now we assign the label 0 to the $r + 1$ vertices $v_1, v_2, \ldots, v_{r+1}$. We now assign the label 2 to the $3r + 2$ vertices $v_{r+2}, v_{r+3}, \ldots, v_{4r+3}$. Next we assign the label 3 to the $4r + 3$ vertices $x_1, x_2, \ldots, x_{4r+3}$. Finally we assign the label 1 to the $4r + 3$ vertices $y_1, y_2, \ldots, y_{4r+3}$.

Thus shows that this vertex labeling f is a 4-total mean cordial labeling of $DA_{n,4}$ follows from the Tabel 7

Case 5. $n = 2$.
A 4-total mean cordial labeling of $DA_{n,4}$ is given in Tabel 8

□

Size of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$11r + 1$	$11r + 1$	$11r + 2$	$11r + 2$
$n = 4r + 1$	$11r + 4$	$11r + 4$	$11r + 5$	$11r + 4$
$n = 4r + 2$	$11r + 7$	$11r + 7$	$11r + 7$	$11r + 7$
$n = 4r + 3$	$11r + 10$	$11r + 9$	$11r + 10$	$11r + 10$

Table 7

n	u	v	u_1	u_2	v_1	v_2	x_1	x_2	y_1	y_2
2	2	3	0	3	0	3	0	1	0	2

Table 8

Theorem 3.7. All shell-Butterfly graphs SB_n with shell order n $(n \geq 3)$ is 4-total mean cordial.

Proof. Let SB_n be a shell Butterfly graph. Let $V(V) = \{u, v, w, x_i, y_i : 1 \leq i \leq n\}$ and $E(V) = \{ux_i, uy_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{uv, uw\}$. Obviously $|V(V)| + |E(V)| = 6n + 3$.

Assign the labels 1, 1, 2 to the vertices u, v, w respectively.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the $3r + 1$ vertices x_1, x_2, …, x_{3r+1}. Next we assign the label 1 to the $r - 1$ vertices x_{3r+2}, x_{3r+2}, …, x_{4r}. Now we assign the label 3 to the $3r$ vertices y_1, y_2, …, y_{3r}. We now assign the label 2 to the r vertices y_{3r+1}, y_{3r+2}, …, y_{4r}.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r + 1$, $r \in \mathbb{N}$. Now we assign the label 0 to the $3r + 2$ vertices x_1, x_2, …, x_{3r+2}. Then we assign the label 1 to the $r - 1$ vertices x_{3r+3}, x_{3r+4}, …, x_{4r+1}. We now assign the label 3 to the $3r + 1$ vertices y_1, y_2, …, y_{3r+1}. Next we assign the label 2 to the r vertices y_{3r+2}, y_{3r+3}, …, y_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r + 2$, $r \in \mathbb{N}$. We now assign the label 0 to the $3r + 2$ vertices x_1, x_2, …, x_{3r+2}. Then we assign the label 1 to the $r - 1$ vertices x_{3r+3}, x_{3r+4}, …, x_{4r+1}. Next we assign the label 2 to the vertex u_{4r+2}. Now we assign the label 3 to the $3r + 2$ vertices y_1, y_2, …, y_{3r+2}. we now assign the label 0 to the vertex y_{3r+3}. Finally we assign the label 2 to the $r - 1$ vertices y_{3r+4}, y_{3r+5}, …, y_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r + 3$, $\in \mathbb{N}$. Assign the label 0 to the $3r + 3$ vertices x_1, x_2, …, x_{3r+3}. Next we assign the label 1 to the $r - 1$ vertices x_{3r+4}, x_{3r+5}, …, x_{4r+2}. We

now assign the label 2 to the vertex x_{4r+3}. Now we assign the label 3 to the $3r+3$ vertices $y_1, y_2, \ldots, y_{3r+3}$. Then we assign the label 0 to the vertex y_{3r+4}. Finally we assign the label 2 to the $r-1$ vertices $y_{3r+5}, y_{3r+6}, \cdots, y_{4r+3}$.

This shows that f is a 4-total mean cordial labeling follows from the Table 9. □

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n=4r$	$6r+1$	$6r+1$	$6r+1$	$6r$
$n=4r+1$	$6r+3$	$6r+2$	$6r+2$	$6r+2$
$n=4r+2$	$6r+4$	$6r+4$	$6r+4$	$6r+3$
$n=4r+3$	$6r+6$	$6r+5$	$6r+5$	$6r+5$

Table 9

Example 3.8. A 4 - total mean cordial labeling of SB_6 is given in figure 1.

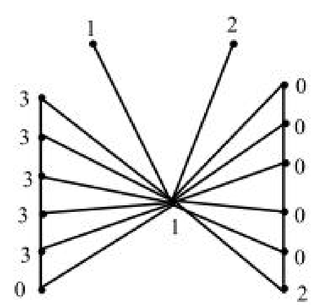

Figure 1. SB_6

Theorem 3.9. The graph obtained by joining two copies of shell graph by a path of arbitrary length is 4-total mean cordial for all $n \geq 3$.

Proof. Let G be a graph obtained by joining two copies of shell graph by a path of length. Let $V(G) = \{u, v, u_i, x_i, y_i : 1 \leq i \leq n, u = u_1; v = u_n\}$ and $E(G) = \{ux_i, vy_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1}, y_i y_{i+1}, u_i u_{i+1} : 1 \leq i \leq n-1\}$.
Obviously $|V(G)| + |E(G)| = 8n - 3$.

Assign the label 3 to the n vertices $u_1, u_2, \ldots, u_n$. We now assign the label 0 to the n vertices $x_1, x_2, \ldots, x_n$. Finally we assign the label 1 to the n vertices $y_1, y_2, \ldots, y_n$.
Clearly $t_{mf}(0) = t_{mf}(1) = t_{mf}(3) = 2n - 1$; $t_{mf}(2) = 2n$.

□

4. conclusion

Mean cordial labeling was defined in [19]. The total mean cordial labeling of graphs was introdced in [18]. Motivated on these two concepts, we have introduced k- total mean cordial labeling of graphs. In this paper we investigate the 4-total mean cordial labeling behaviour of arrow graphs, shell butterfly and two copies of shell joining by a path. Presently, it is difficult to investigate the 4-total mean cordial labeling behaviour of olive tree,parachutes and Subdivided the rim of whell graphs. The 4-total mean cordial labeling behaviour of slanting ladder, mongolial tents, friendship graph, flower snark graph are open problem for future research work.

Conflicts of interest : There is no conflicts of interest.

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