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AN ALGEBRAIC STRUCTURE INDUCED BY A FUZZY BI-PARTIALLY ORDERED SPACE I†

JU-MOK OH

Abstract. We introduce an algebraic structure induced by a fuzzy bi-partial order on a complete residuated lattices with the double negative law. We undertake an investigation into the properties of fuzzy bi-partial orders, including their various characteristics and features. We demonstrate that the two families of l-stable and r-stable fuzzy sets can be regarded as complete lattices, and we establish that these two families are anti-isomorphic. Furthermore, we provide two examples related to them.

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1. Introduction

The concept of a complete residuated lattice was first introduced by Ward et al. [18] as a means of extending the notion of a left-continuous t-conorm, with applications in many-valued logic [8, 9, 10]. Building on this work, Bělohl'avek [1, 2] introduced the concept of formal concepts with $R \in L^{X\times Y}$ defined on a complete residuated lattice. Let $R \in L^{X\times Y}$ be a fuzzy information system. A formal fuzzy concept is a pair $(A, B) \in L^X \times L^Y$ such that $F(A) = B$ and $G(B) = A$ where $F : L^X \to L^Y, G : L^Y \to L^X$ are defined by

$$F(A)(y) = \bigwedge_{x\in X} [A(x) \to R(x, y)] \text{ and } G(B)(x) = \bigwedge_{y\in Y} [B(y) \to R(x, y)] .$$

Many researchers developed various fuzzy concepts, information systems and decision rules on complete residuated lattices [3, 4, 11, 12, 13, 14, 15].

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Urquhart [17] defined two maps $l : P(X) \to P(X)$ and $r : P(X) \to P(X)$ on a bounded lattice $(X, \vee, \wedge, 0, 1)$ by

$$\begin{array}{ll} l(A) & = \{x \in X \mid (\forall y \in X)\ x \leq_1 y \Rightarrow y \notin A\}, \\ r(A) & = \{x \in X \mid (\forall y \in X)\ x \leq_2 y \Rightarrow y \notin A\} \end{array}$$

where $A \subseteq X$. In his work, he demonstrated that a bounded lattice's dual space can be regarded as a doubly ordered topological space. This perspective leads to the development of numerous representation theorems for various algebraic structures [5, 6, 7, 16].

In this paper, the investigation is carried out on a fuzzy bi-partially ordered space $\left(X, e_X^1, e_X^2\right)$ defined on a complete residuated lattice $(L, \leq, \vee, \wedge, \odot, \to, \perp, \top)$ with the double negative law, rather than on a doubly ordered set $\left(X, e_X^1, e_X^2\right)$ defined on a bounded lattice $(X, \vee, \wedge, \perp, \top)$. We define two maps $l : L^X \to L^X$ and $r : L^X \to L^X$ by

$$\begin{array}{ll} l(A)(x) & = \bigwedge_{y \in X} \left[e_X^1(x, y) \to n(A)(y)\right], \\ r(A)(x) & = \bigwedge_{y \in X} \left[e_X^2(x, y) \to n(A)(y)\right] \end{array}$$

where $n(A)(y) = A(y) \to \perp$.

In Theorem 3.5, we demonstrates that $L\left(L^X\right) = \{A \in L^X \mid A = l(r(A))\}$ and $R\left(L^X\right) = \{B \in L^X \mid B = r(l(A))\}$ are both complete lattices under certain suitable operations and they are anti-isomorphic. This result can be regarded as one of the important implications for the understanding of algebraic structures induced by fuzzy bi-partial orders on complete residuated lattices. Example 3.6 serves to provide a concrete instantiation of the theoretical concepts presented in this study, thereby enriching the reader's understanding of the subject matter. Example 3.7 provides a concrete illustration of these concepts in action, as we define a fuzzy bi-partially ordered space $\left(X, e_X^1, e_X^2\right)$ in the context of a fuzzy information system $L^{X \times Y}$, where X represents a set of objects and Y a set of attributes.

2. Preliminaries

Definition 2.1. [8, 9, 10, 18] An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a *complete residuated lattice* if it satisfies the following conditions:
(L1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$,
(L2) $(L, \odot, \top)$ is a commutative monoid with identity $\top$,
(L3) the residuation property, *i.e.*, $x \odot y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in L$.

Let $(L, \wedge, \vee, \odot, \to, \perp, \top)$ be a complete residuated lattice. Define $n : L \to L$ by $n(x) = x \to \perp$ for all $x \in L$.

The condition $n(n(x)) = x$ for all $x \in L$ is called the *double negative law*.

Let $\alpha \in L$ and $A \in L^X$. Define three maps $(\alpha_X \to A)$, $(\alpha_X \odot A)$, $\alpha_X : X \to L$ by $(\alpha_X \to A)(x) = \alpha \to A(x)$, $(\alpha_X \odot A)(x) = \alpha \odot A(x)$ and $\alpha_X(x) = \alpha$.

In this paper, we always assume that $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top, n)$ is a complete residuated lattice with the double negative law.

Lemma 2.2. [8, 9, 10, 18] *Let $x, y, z, w \in L$. Let $\{x_i\}_{i\in\Gamma}, \{y_i\}_{i\in\Gamma} \subseteq L$. Then the following hold.*
(1) $\top \rightarrow x = x$, $\perp \odot x = \perp$.
(2) *If* $y \leq z$, *then* $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ *and* $z \rightarrow x \leq y \rightarrow x$.
(3) $x \leq y$ *if and only if* $x \rightarrow y = \top$.
(4) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.
(5) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.
(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$.
(7) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ *and* $y \leq x \rightarrow x \odot y$.
(8) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$ *and* $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$.
(9) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$ *and* $(x \rightarrow y) \odot z \leq x \rightarrow (y \odot z)$.
(10) $\bigwedge_{i\in\Gamma}(x_i \rightarrow y_i) \leq \bigvee_{i\in\Gamma} x_i \rightarrow \bigvee_{i\in\Gamma} y_i$ *and* $\bigwedge_{i\in\Gamma}(x_i \rightarrow y_i) \leq \bigwedge_{i\in\Gamma} x_i \rightarrow \bigwedge_{i\in\Gamma} y_i$.
(11) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ *and* $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$.
(12) $n(x \odot n(y)) = x \rightarrow y$ *and* $x \rightarrow y = n(y) \rightarrow n(x)$.
(13) $x \rightarrow (y \rightarrow z) = n(z) \rightarrow (y \rightarrow n(x))$.
(14) $n\left(\bigvee_{i\in\Gamma} x_i\right) = \bigwedge_{i\in\Gamma} n(x_i)$ *and* $n\left(\bigwedge_{i\in\Gamma} x_i\right) = \bigvee_{i\in\Gamma} n(x_i)$.

Definition 2.3. [8, 9, 10, 18] Let X be a set. A map $e_X : X \times X \rightarrow L$ is called:
(E1) *reflexive* if $e_X(x, x) = \top$ for all $x \in X$,
(E2) *transitive* if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$ for all $x, y, z \in X$,
(E3) if $e_X(x, y) = e_X(y, x) = \top$ where $x, y \in X$, then $x = y$.

If e_X satisfies (E1),(E2) and (E3), e_X is called a *fuzzy partial order*. Let e^1_X and e^2_X be fuzzy partial orders on X. Then $\left(X, e^1_X, e^2_X\right)$ is called a *fuzzy bi-partially ordered space*.

Remark 2.1. [8, 9, 10, 18] Define $e_L : L \times L \rightarrow L$ by $e_L(x, y) = x \rightarrow y$. By Lemma 2.2(5) and (6), one can see that e_L is a fuzzy partial order.

Let $\tau \subseteq L^X$. Define $e_\tau : \tau \times \tau \rightarrow L$ by $e_\tau(A, B) = \bigwedge_{x\in X} [A(x) \rightarrow B(x)]$. Then e_τ is a fuzzy partial order.

Definition 2.4. [12, 14, 15] (1) Let $\tau \subseteq L^X$. τ is called an *Alexandrov topology* on X if it satisfies the following two conditions:
(A1) $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau$ for all $\{A_i\}_{i\in I} \subseteq \tau$,
(A2) $\alpha_X \rightarrow A, \alpha_X \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

The pair (X, τ) is called an *Alexandrov topological space* on X.

3. algebraic structurea induced by fuzzy bi-partially ordered spaces

Definition 3.1. Let (X, e^1_X, e^2_X) be a fuzzy bi-partially ordered space. Define two maps $l : L^X \rightarrow L^X$ and $r : L^X \rightarrow L^X$ by

$$l(A)(x) = \bigwedge_{y\in X} \left[e^1_X(x, y) \rightarrow n(A)(y)\right], \quad r(A)(x) = \bigwedge_{y\in X} \left[e^2_X(x, y) \rightarrow n(A)(y)\right]$$

where $n(A)(y) = n(A(y)) = A(y) \to \perp$.
(1) A fuzzy set $A \in L^X$ is called *l-stable* if $lr(A) = A$.
(2) A fuzzy set $A \in L^X$ is called *r-stable* if $rl(A) = A$.
(3) The family of all l-stable fuzzy sets will be denoted by $L\left(L^X\right)$.
(4) The family of all r-stable fuzzy sets will be denoted by $R\left(L^X\right)$.

Consider a fuzzy partially ordered space (X, e_X). The inverse e_X^{-1} of e_X is defined as follows:

$$e_X^{-1} : X \times X \to L \text{ by } e_X^{-1}(x,y) = e_X(y,x).$$

Theorem 3.2. *Let (X, e_X) be a fuzzy partially ordered space. Let*

$$\tau_{e_X} = \left\{A \in L^X \mid A(x) \odot e_X(x,y) \le A(y) \text{ for all } x, y \in X\right\}.$$

Then the following hold.
(1) τ_{e_X} is an Alexandrov topology.
(2) Let $A \in L^X$. Then $A \in \tau_{e_X}$ if and only if $\bigvee_{x\in X}[A(x) \odot e_X(x,y)] = A(y)$ for all $y \in X$ if and only if $\bigwedge_{y \in X}[e_X(x,y) \to A(y)] = A(x)$ for all $x \in X$.
(3) Let $\tau_{e_X^{-1}} = \left\{A \in L^X \mid A(x) \odot e_X^{-1}(x,y) \le A(y) \text{ for all } x,y \in X\right\}$. Then

$$\tau_{e_X^{-1}} = \{n(A) \mid A \in \tau_{e_X}\}.$$

(4) Let $A \in L^X$ and $z \in X$. Then $e_X(z,-)$, $[\alpha_X \to e_X(z,-)]$, $[e_X(-,z) \to \alpha_X]$, $\bigvee_{z \in X}[e_X(z,-) \odot A(z)]$, $\bigwedge_{z\in X}[e_X(-,z) \to n(A)(z)] \in \tau_{e_X}$.

Proof. (1) (A1) Let $\{A_i\}_{i\in\Gamma} \subseteq \tau_{e_X}$. Then

$$\begin{aligned}\left(\bigwedge_{i\in\Gamma} A_i\right)(x) \odot e_X(x,y) &= \bigwedge_{i\in\Gamma} A_i(x) \odot e_X(x,y)\\ &\le \bigwedge_{i\in\Gamma}[A_i(x) \odot e_X(x,y)]\\ &\le \bigwedge_{i\in\Gamma} A_i(y) \quad \text{since } A_i \in \tau_{e_X}\\ &= \left(\bigwedge_{i\in\Gamma} A_i\right)(y)\end{aligned}$$

and

$$\begin{aligned}\left(\bigvee_{i\in\Gamma} A_i\right)(x) \odot e_X(x,y) &= \bigvee_{i\in\Gamma} A_i(x) \odot e_X(x,y)\\ &= \bigvee_{i\in\Gamma}[A_i(x) \odot e_X(x,y)] \quad \text{by Lemma 2.2(6)}\\ &\le \bigvee_{i\in\Gamma} A_i(y) \quad \text{since } A_i \in \tau_{e_X}\\ &= \left(\bigvee_{i\in\Gamma} A_i\right)(y).\end{aligned}$$

Hence $\bigwedge_{i\in\Gamma} A_i, \bigvee_{i\in\Gamma} \in \tau_{e_X}$.
(A2) Let $A \in \tau_{e_X}$ and $\alpha \in L$. Then

$$\begin{aligned}[\alpha_X \to A](x) \odot e_X(x,y) &= [\alpha \to A(x)] \odot e_X(x,y)\\ &= \alpha \to [A(x) \odot e_X(x,y)] \quad \text{by Lemma 2.2(9)}\\ &\le \alpha \to A(y) \quad \text{since } A \in \tau_{e_X}\\ &= [\alpha_X \to A](y)\end{aligned}$$

and

$$\begin{aligned}[\alpha_X \odot A](x) \odot e_X(x,y) &= [\alpha \odot A(x)] \odot e_X(x,y) = \alpha \odot [A(x) \odot e_X(x,y)]\\ &\le \alpha \odot A(y) \quad \text{since } A \in \tau_{e_X}\\ &= [\alpha_X \odot A](y).\end{aligned}$$

Hence $\alpha_X \to A, \alpha_X \odot A \in \tau_{e_X}$.

Therefore τ_{e_X} is an Alexandrov topology.

(2) Assume $A \in \tau_{e_X}$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $x, y \in X$, and so $\bigvee_{x\in X} [A(x) \odot e_X(x,y)] \le A(y)$ for all $y \in X$. On the other hand, we have $\bigvee_{x\in X} [A(x) \odot e_X(x,y)] \ge A(y) \odot e_X(y,y) = A(y)$. Thus

$$\bigvee_{x\in X} [A(x) \odot e_X(x,y)] = A(y) \text{ for all } y \in X.$$

Assume that $\bigvee_{x\in X} [A(x) \odot e_X(x,y)] = A(y)$ for all $y \in X$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $y \in X$, and so $A \in \tau_{e_X}$.

Hence the first equivalent statement is proved.

Assume $A \in \tau_{e_X}$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $x, y \in X$. By residuation, $A(x) \le e_X(x,y) \to A(y)$ for all $x, y \in X$, and so

$$A(x) \le \bigwedge_{y\in X} [e_X(x,y) \to A(y)] \text{ for all } x \in X.$$

On the other hand, $\bigwedge_{y\in X} [e_X(x,y) \to A(y)] \le e_X(x,x) \to A(x) = A(x)$. Hence $\bigwedge_{y\in X} [e_X(x,y) \to A(y)] = A(x)$ for all $x \in X$.

Assume that $\bigwedge_{y\in X} [e_X(x,y) \to A(y)] = A(x)$ for all $x \in X$. Then $A(x) \le e_X(x,y) \to A(y)$ for all $x, y \in X$. By residuation, $e_X(x,y) \odot A(x) \le A(y)$ for all $x, y \in X$, and so $A \in \tau_{e_X}$.

Hence the second equivalent statement is proved.

(3) "$\supseteq$": Let $A \in \tau_{e_X}$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $x, y \in X$, and so

$$\begin{aligned} A(x) &\le e_X(x,y) \to A(y) \text{ by residuation} \\ &= n\left(e_X(x,y) \odot n(A)(y)\right) \quad \text{by Lemma 2.2(12)}. \end{aligned}$$

Now, by Lemma 2.2(2), $n\left(e_X(x,y) \odot n(A)(y)\right) \to \bot \le A(x) \to \bot$. By the double negative law, $e_X(x,y) \odot n(A)(y) \le n(A)(x)$, and so $n(A)(y) \odot e_X^{-1}(y,x) \le n(A)(x)$. Hence $n(A) \in \tau_{e_X^{-1}}$.

"$\subseteq$": Let $B \in \tau_{e_X^{-1}}$. Since $n(n(B)) = B$ by the double negative law, it is enough to show that $n(B) \in \tau_{e_X}$. Since $B \in \tau_{e_X^{-1}}$, we have $B(x) \odot e_X^{-1}(x,y) \le B(y)$, and so $B(x) \odot e_X(y,x) \le B(y)$. Then

$$\begin{aligned} B(x) &\le e_X(y,x) \to B(y) \quad \text{by residuation} \\ &= n\left(e_X(y,x) \odot n(B)(y)\right) \quad \text{by Lemma 2.2(12)}. \end{aligned}$$

By Lemma 2.2(2) and the double negative law, we have $e_X(y,x) \odot n(B)(y) \le n(B)(x)$, and so $n(B)(y) \odot e_X(y,x) \le n(B)(x)$. Hence $n(B) \in \tau_{e_X}$.

(4) Let $A \in L^X$ and $z \in X$. We show $e_X(z,-) \in \tau_{e_X}$. For all $x, y \in X$, $e_X(z,x) \odot e_X(x,y) \le e_X(z,y)$ by (E2). Hence $e_X(z,-) \in \tau_{e_X}$.

We show $[\alpha_X \to e_X(z,-)] \in \tau_{e_X}$. For all $x, y \in X$,

$$\begin{aligned} [\alpha_X \to e_X(z,-)]\,(x) \odot e_X(x,y) &= [\alpha \to e_X(z,x)] \odot e_X(x,y) \\ &\le \alpha \to [e_X(z,x) \odot e_X(x,y)] \text{ by Lemma 2.2(9)} \\ &\le \alpha \to e_X(z,y) \quad \text{by (E2)} \\ &= [\alpha_X \to e_X(z,-)]\,(y). \end{aligned}$$

Hence $[\alpha_X \to e_X(z,-)] \in \tau_{e_X}$.

We show $[e_X(-,z)\to\alpha_X]\in\tau_{e_X}$. For all $x,y\in X$,

$$\begin{aligned}[e_X(-,z)\to\alpha_X]\odot e_X(x,y)\odot e_X(y,z) &\le [e_X(x,z)\to\alpha]\odot e_X(x,z) \quad \text{by (E2)}\\ &\le \alpha.\end{aligned}$$

By residuation,

$$[e_X(-,z)\to\alpha_X](x)\odot e_X(x,y)\le e_X(y,z)\to\alpha=[e_X(-,z)\to\alpha_X](y).$$

Hence $[e_X(-,z)\to\alpha_X]\in\tau_{e_X}$.

We show $\bigvee_{z\in X}[e_X(z,-)\odot A(z)]\in\tau_{e_X}$. For all $x,y\in X$,

$$\begin{aligned}\textstyle\bigvee_{z\in X}[e_X(z,-)\odot A(z)](x)\odot e_X(x,y) &= \textstyle\bigvee_{z\in X}[e_X(z,x)\odot A(z)]\odot e_X(x,y)\\ &= \textstyle\bigvee_{z\in X}[[e_X(z,x)\odot e_X(x,y)]\odot A(z)]\\ &\le \textstyle\bigvee_{z\in X}[e_X(z,y)\odot A(z)] \quad \text{by (E2)}\\ &= \textstyle\bigvee_{z\in X}[e_X(z,-)\odot A(z)](y).\end{aligned}$$

Hence $\bigvee_{z\in X}[e_X(z,-)\odot A(z)]\in\tau_{e_X}$.

We show $\bigwedge_{z\in X}[e_X(-,z)\to n(A)(z)]\in\tau_{e_X}$. For all $x,y,w\in X$,

$$\begin{aligned}&\textstyle\bigwedge_{z\in X}[e_X(-,z)\to n(A)(z)](x)\odot e_X(x,y)\odot e_X(y,w)\\ &\le \textstyle\bigwedge_{z\in X}[e_X(x,z)\to n(A)(z)]\odot e_X(x,w) \quad \text{by (E2)}\\ &\le [e_X(x,w)\to n(A)(w)]\odot e_X(x,w)\\ &\le n(A)(w).\end{aligned}$$

By residuation, $\bigwedge_{z\in X}[e_X(-,z)\to n(A)(z)](x)\odot e_X(x,y)\le e_X(y,w)\to n(A)(w)$ for all $x,y,w\in X$, and so

$$\begin{aligned}\textstyle\bigwedge_{z\in X}[e_X(-,z)\to n(A)(z)](x)\odot e_X(x,y) &\le \textstyle\bigwedge_{w\in X}[e_X(y,w)\to n(A)(w)]\\ &= \textstyle\bigwedge_{w\in X}[e_X(-,w)\to n(A)(w)](y).\end{aligned}$$

Hence $\bigwedge_{z\in X}[e_X(-,z)\to n(A)(z)]\in\tau_{e_X}$. □

Consider a fuzzy bi-partially ordered space $\left(X,e_X^1,e_X^2\right)$. To simplify the notation, we shall denote the inverse of e_X^1 and e_X^2 by e_X^{-1} and e_X^{-2}, respectively.

Theorem 3.3. *Let $\left(X,e_X^1,e_X^2\right)$ be a fuzzy bi-partially ordered space. Then the following hold.*
(1) Let $A,B\in L^X$. Then $e_{L^X}(A,B)\le e_{L^X}(l(B),l(A))$ and $e_{L^X}(A,B)\le e_{L^X}(r(B),r(A))$. In particular, if $A\le B$, then $l(B)\le l(A)$ and $r(B)\le r(A)$.
(2) Let $A\in L^X$. Then $l(A)\in\tau_{e_X^1}$, $r(A)\in\tau_{e_X^2}$, $l(A)\le n(A)$ and $r(A)\le n(A)$.
(3) If $A\in\tau_{e_X^1}$, then $A\le l(r(A))$. Similarly, if $A\in\tau_{e_X^2}$, then $A\le r(l(A))$.
(4) $L(L^X)=\{l(r(A))\mid A\in L^X\}$ and $R(L^X)=\{r(l(A))\mid A\in L^X\}$.
(5) If $A\in L\left(L^X\right)$, then $r(A)\in R\left(L^X\right)$. Similarly, if $A\in R\left(L^X\right)$, then $l(A)\in L\left(L^X\right)$.
(6) If $\{A_i\}_{i\in\Gamma}\subseteq L\left(L^X\right)$, $A\in L\left(L^X\right)$ and $\alpha\in L$, then $\bigwedge_{i\in\Gamma}A_i,\alpha_X\to A\in L\left(L^X\right)$. Similarly, if $\{A_i\}_{i\in\Gamma}\subseteq R\left(L^X\right)$, $A\in R\left(L^X\right)$ and $\alpha\in L$, then $\bigwedge_{i\in\Gamma}A_i,\alpha_X\to A\in R\left(L^X\right)$.

Proof. (1) Let $A, B \in L^X$. Then

$$\begin{aligned}
& e_{L^X}\,(l(B), l(A)) \\
&= \bigwedge_{x\in X}\Big[\bigwedge_{y\in X}\big[e_X^1(x,y)\to n(B)(y)\big] \to \bigwedge_{y\in X}\big[e_X^1(x,y)\to n(A)(y)\big]\Big] \\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\big[\big[e_X^1(x,y)\to n(B)(y)\big]\to\big[e_X^1(x,y)\to n(A)(y)\big]\big] \\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}[n(B)(y)\to n(A)(y)] \text{ by Lemma 2.2(11)} \\
&= \bigwedge_{x\in X}\bigwedge_{y\in X}[A(y)\to B(y)] \text{ by Lemma 2.2(12)} \\
&= e_{L^X}(A,B)
\end{aligned}$$

and

$$\begin{aligned}
& e_{L^X}\,(r(B), r(A)) \\
&= \bigwedge_{x\in X}\Big[\bigwedge_{y\in X}\big[e_X^2(x,y)\to n(B)(y)\big] \to \bigwedge_{y\in X}\big[e_X^2(x,y)\to n(A)(y)\big]\Big] \\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\big[\big[e_X^2(x,y)\to n(B)(y)\big]\to\big[e_X^2(x,y)\to n(A)(y)\big]\big] \\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}[n(B)(y)\to n(A)(y)] \text{ by Lemma 2.2(11)} \\
&= \bigwedge_{x\in X}\bigwedge_{y\in X}[A(y)\to B(y)] \text{ by Lemma 2.2(12)} \\
&= e_{L^X}(A,B).
\end{aligned}$$

Hence $e_{L^X}(A,B) \leq e_{L^X}\,(l(B), l(A))$ and $e_{L^X}(A,B) \leq e_{L^X}\,(r(B), r(A))$.

Assume that $A \leq B$. Then $\top = e_{L^X}(A,B) \leq e_{L^X}(l(B), l(A))$, and so $\top = e_{L^X}(l(B), l(A))$. Hence $l(B) \leq l(A)$. Similarly, $\top = e_{L^X}(A,B) \leq e_{L^X}(r(B), r(A))$, and so $\top = e_{L^X}(r(B), r(A))$. Hence $l(B) \leq l(A)$.

(2) By Theorem 3.2(2), we have

$$\begin{aligned}
l(A) &= \bigwedge_{y\in X}\big[e_X^1(-,y)\to n(A)(y)\big] \in \tau_{e_X^1}, \\
r(A) &= \bigwedge_{y\in X}\big[e_X^2(-,y)\to n(A)(y)\big] \in \tau_{e_X^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
l(A)(x) &= \bigwedge_{y\in X}\big[e_X^1(x,y)\to n(A)(y)\big] \leq e_X^1(x,x)\to n(A)(x) = n(A)(x), \\
r(A)(x) &= \bigwedge_{y\in X}\big[e_X^2(x,y)\to n(A)(y)\big] \leq e_X^2(x,x)\to n(A)(x) = n(A)(x).
\end{aligned}$$

Hence $l(A) \leq n(A)$ and $r(A) \leq n(A)$.

(3) Let $A \in \tau_{e_X^1}$. Then

$$\begin{aligned}
lr(A)(x) &= \bigwedge_{y\in X}\big[e_X^1(x,y)\to n(r(A))(y)\big] \\
&= \bigwedge_{y\in X}\big[e_X^1(x,y)\to n\big(\bigwedge_{w\in X}\big[e_X^2(y,w)\to n(A)(w)\big]\big)\big] \\
&= \bigwedge_{y\in X}\big[e_X^1(x,y)\to n\big(\bigwedge_{w\in X} n\big(e_X^2(y,w)\odot A(w)\big)\big)\big] \\
&= \bigwedge_{y\in X}\big[e_X^1(x,y)\to \bigvee_{w\in X}\big[e_X^2(y,w)\odot A(w)\big]\big] \text{ by Lemma 2.2(14)} \\
&\geq \bigwedge_{y\in X}\big[e_X^1(x,y)\to \big[e_X^2(y,y)\odot A(y)\big]\big] \\
&= \bigwedge_{y\in X}\big[e_X^1(x,y)\to A(y)\big] \\
&= A(x) \quad \text{by Theorem 3.2(2).}
\end{aligned}$$

Hence $A \leq lr(A)$.

Let $A \in \tau_{e^2_X}$. Then

$$\begin{aligned} rl(A)(x) &= \bigwedge_{y\in X} \left[e^2_X(x,y) \to n\left(l(A)\right)(y)\right] \\ &= \bigwedge_{y\in X} \left[e^2_X(x,y) \to n\left(\bigwedge_{w\in X} \left[e^1_X(y,w) \to n(A)(w)\right]\right)\right] \\ &= \bigwedge_{y\in X} \left[e^2_X(x,y) \to n\left(\bigwedge_{w\in X} n\left(e^1_X(y,w) \odot A(w)\right)\right)\right] \\ &= \bigwedge_{y\in X} \left[e^2_X(x,y) \to \bigvee_{w\in X} \left[e^1_X(y,w) \odot A(w)\right]\right] \text{ by Lemma 2.2(14)} \\ &\geq \bigwedge_{y\in X} \left[e^2_X(x,y) \to \left[e^1_X(y,y) \odot A(y)\right]\right] \\ &= \bigwedge_{y\in X} \left[e^2_X(x,y) \to \odot A(y)\right] \\ &= A(x) \quad \text{by Theorem 3.2(2).} \end{aligned}$$

Hence $A \leq rl(A)$.

(4) Let $L_1\left(L^X\right) = \{l(r(A)) \mid A \in L^X\}$.

"$L\left(L^X\right) \supseteq L_1\left(L^X\right)$": Let $A \in L^X$. Since $r(A) \in \tau_{e^2_X}$ by (2), we have by (3) that $r(A) \leq rlr(A)$. Now, by (1), we have $lrlr(A) \leq lr(A)$. On the other hand, since $lr(A) \in \tau_{e^1_X}$ by (2), we have by (3) that $lr(A) \leq lrlr(A)$. Hence $lr(A) \in L\left(L^X\right)$.

"$L\left(L^X\right) \subseteq L_1\left(L^X\right)$": Let $A \in L\left(L^X\right)$. Then $A = lr(A)$, and so $A \in L_1\left(L^X\right)$.

Let $R_1\left(L^X\right) = \{rl(A) \mid A \in L^X\}$.

"$R\left(L^X\right) \supseteq R_1\left(L^X\right)$": Let $A \in L^X$. Since $l(A) \in \tau_{e^1_X}$ by (2), we have by (3) that $l(A) \leq lrl(A)$. Now, by (1), $rlrl(A) \leq rl(A)$. On the other hand, since $rl(A) \in \tau_{e^2_X}$ by (2), we have by (3) that $rl(A) \leq rlrl(A)$. Hence $rlrl(A) = rl(A)$, and so $rl(A) \in R\left(L^X\right)$.

"$R\left(L^X\right) \subseteq R_1\left(L^X\right)$": Let $A \in R\left(L^X\right)$. Then $A = rl(A)$, and so $A \in R_1\left(L^X\right)$.

(5) Let $A \in L\left(L^X\right)$. Since $lr(A) = A$, we have $rlr(A) = r(A)$, and so $r(A) \in R\left(L^X\right)$.

Let $A \in R\left(L^X\right)$. Since $rl(A) = A$, we have $lrl(A) = l(A)$, and so $l(A) \in L\left(L^X\right)$.

(6) Let $\{A_i\}_{i\in\Gamma} \subseteq L\left(L^X\right)$. Then $A_i = lr(A_i) \in \tau_{e^1_X}$ by (2). Since $\tau_{e^1_X}$ is an Alexandrov topology, we have $\bigwedge_{i\in\Gamma} A_i \in \tau_{e^1_X}$. Then $\bigwedge_{i\in\Gamma} A_i \leq lr\left(\bigwedge_{i\in\Gamma} A_i\right)$ by (3). On the other hand, for all $i \in \Gamma$,

$$A_i = lr(A_i) \geq lr\left(\bigwedge_{i\in\Gamma} A_i\right) \quad \text{since } lr \text{ is increasing by (1),}$$

and so $\bigwedge_{i\in\Gamma} A_i \geq lr\left(\bigwedge_{i\in\Gamma} A_i\right)$. Hence $lr\left(\bigwedge_{i\in\Gamma} A_i\right) = \bigwedge_{i\in\Gamma} A_i$ and $\bigwedge_{i\in\Gamma} A_i \in L\left(L^X\right)$.

Let $A \in L\left(L^X\right)$ and $\alpha \in L$. Then $A = lr(A) \in \tau_{e^1_X}$ by (2). Since $\tau_{e^1_X}$ is an Alexandrov topology, $\alpha_X \to A \in \tau_{e^1_X}$. Then $lr(\alpha_X \to A) \geq \alpha_X \to A$ by (3).

On the other hand, note that

$$\begin{aligned} r(\alpha_X \to A)(x) &= \bigwedge_{y\in X}\left[e_X^2(x,y) \to n(\alpha_X \to A)(y)\right] \\ &= \bigwedge_{y\in X}\left[e_X^2(x,y) \to [\alpha \odot n(A)(y)]\right] \text{ by Lemma 2.2(12)} \\ &\geq \bigwedge_{y\in X}\left[\left[e_X^2(x,y) \to n(A)(y)\right] \odot \alpha\right] \text{ by Lemma 2.2(9)} \\ &\geq \bigwedge_{y\in X}\left[e_X^2(x,y) \to n(A)(y)\right] \odot \alpha \\ &= r(A)(x) \odot \alpha. \end{aligned}$$

Then $lr(\alpha_X \to A) \leq l(r(A) \odot \alpha_X)$. Furthermore,

$$\begin{aligned} l(r(A) \odot \alpha_X)(x) &= \bigwedge_{y\in X}\left[e_X^1(x,y) \to n(r(A) \odot \alpha_X)(y)\right] \\ &= \bigwedge_{y\in X}\left[e_X^1(x,y) \to [\alpha \to n(r(A))(y)]\right] \text{ by Lemma 2.2(12)} \\ &= \bigwedge_{y\in X}\left[\alpha \to \left[e_X^1(x,y) \to n(r(A))(y)\right]\right] \text{ by Lemma 2.2(7)} \\ &= \alpha \to \bigwedge_{y\in X}\left[e_X^1(x,y) \to n(r(A))(y)\right] \text{ by Lemma 2.2(4)} \\ &= \alpha \to lr(A)(x) \\ &= \alpha \to A(x) \quad \text{since } A \in L\left(L^X\right) \\ &= (\alpha_X \to A)(x). \end{aligned}$$

Hence $lr(\alpha_X \to A) \leq l(r(A) \odot \alpha_X) = \alpha_X \to A$.

Therefore $lr(\alpha_X \to A) = \alpha_X \to A$ and $\alpha_X \to A \in L\left(L^X\right)$.

Let $\{A_i\}_{i\in\Gamma} \subseteq R\left(L^X\right)$. Then $A_i = rl(A_i) \in \tau_{e_X^2}$ by (2). Since $\tau_{e_X^2}$ is an Alexandrov topology, $\bigwedge_{i\in\Gamma} A_i \in \tau_{e_X^2}$. Then $rl\left(\bigwedge_{i\in\Gamma} A_i\right) \in \tau_{e_X^2}$ by (3).

On the other hand, for all $i \in \Gamma$,

$$A_i = rl(A_i) \geq rl\left(\bigwedge_{i\in\Gamma} A_i\right) \quad \text{since } rl \text{ is increasing by (1),}$$

and so $\bigwedge_{i\in\Gamma} A_i \geq rl\left(\bigwedge_{i\in\Gamma} A_i\right)$. Hence $rl\left(\bigwedge_{i\in\Gamma} A_i\right) = \bigwedge_{i\in\Gamma} A_i$ and $\bigwedge_{i\in\Gamma} A_i \in R\left(L^X\right)$.

Let $A \in R\left(L^X\right)$ and $\alpha \in L$. Then $A = rl(A) \in \tau_{e_X^2}$ by (2). Since $\tau_{e_X^2}$ is an Alexandrov topology, $\alpha_X \to A \in \tau_{e_X^2}$. Then $rl(\alpha_X \to A) \geq \alpha_X \to A$.

On the other hand, note that

$$\begin{aligned} l(\alpha_X \to A)(x) &= \bigwedge_{y\in X}\left[e_X^1(x,y) \to n(\alpha_X \to A)(y)\right] \\ &= \bigwedge_{y\in X}\left[e_X^1(x,y) \to [\alpha \odot n(A)(y)]\right] \text{ by Lemma 2.2(12)} \\ &\geq \bigwedge_{y\in X}\left[\left[e_X^1(x,y) \to n(A)(y)\right] \odot \alpha\right] \\ &\geq \bigwedge_{y\in X}\left[e_X^1(x,y) \to n(A)(y)\right] \odot \alpha \\ &= l(A)(x) \odot \alpha. \end{aligned}$$

Then $rl(\alpha_X \to A) \leq r(l(A) \odot \alpha_X)$. Furthermore,

$$\begin{aligned} r(l(A) \odot \alpha_X)(x) &= \bigwedge_{y\in X}\left[e_X^2(x,y) \to n(l(A) \odot \alpha_X)(y)\right] \\ &= \bigwedge_{y\in X}\left[e_X^2(x,y) \to [\alpha \to n(l(A))(y)]\right] \text{ by Lemma 2.2(12)} \\ &= \bigwedge_{y\in X}\left[\alpha \to \left[e_X^2(x,y) \to n(l(A))(y)\right]\right] \text{ by Lemma 2.2(7)} \\ &= \alpha \to \bigwedge_{y\in X}\left[e_X^2(x,y) \to n(l(A))(y)\right] \text{ by Lemma 2.2(4)} \\ &= \alpha \to rl(A)(x) = \alpha \to A(x) \quad \text{since } A \in R\left(L^X\right) \\ &= (\alpha_X \to A)(x). \end{aligned}$$

Hence $rl(\alpha_X \to A) \leq r(l(A) \odot \alpha_X) \leq \alpha_X \to A$.

Therefore $rl\left(\alpha_X \to A\right) = \alpha_X \to A$ and $\alpha_X \to A \in R\left(L^X\right)$. □

Definition 3.4. Let $(L_1, \leq, \wedge, \vee)$ and $(L_2, \leq, \wedge, \vee)$ be complete lattices. L_1 and L_2 are *anti-isomorphic* if there exists a bijective map $h : L_1 \to L_2$ such that

$$h\left(\bigvee_{i\in I} x_i\right) = \bigwedge_{i\in I} h\left(x_i\right) \text{ and } h\left(\bigwedge_{i\in I} x_i\right) = \bigvee_{i\in I} h\left(x_i\right) \text{ for all } \quad \{x_i\}_{i\in\Gamma} \subseteq L_1.$$

Theorem 3.5. *Let (X, e_X^1, e_X^2) be a fuzzy bi-partially ordered space. Let l and r be the maps defined in Definition 3.1. The the following hold.*
(1) $\left(L\left(L^X\right), \bigwedge, \sqcup^l, \bot_X, \top_X\right)$ *is a complete lattice where*

$$\left(\bigwedge_{i\in\Gamma} A_i\right)(x) = \bigwedge_{i\in\Gamma} A_i(x) \text{ and } \left(\sqcup^l_{i\in\Gamma} A_i\right)(x) = l\left(\bigwedge_{i\in\Gamma} r\left(A_i\right)\right)(x)$$

for all $\{A_i\}_{i\in\Gamma} \subseteq L\left(L^X\right)$.
(2) $\left(R\left(L^X\right), \bigwedge, \sqcup^r, \bot_X, \top_X\right)$ *is a complete lattice where*

$$\left(\bigwedge_{i\in\Gamma} A_i\right)(x) = \bigwedge_{i\in\Gamma} A_i(x) \text{ and } \left(\sqcup^r_{i\in\Gamma} A_i\right)(x) = r\left(\bigwedge_{i\in\Gamma} l\left(A_i\right)\right)(x)$$

for all $\{A_i\}_{i\in\Gamma} \subseteq R\left(L^X\right)$.
(3) $L\left(L^X\right)$ *and* $R\left(L^X\right)$ *are anti-isomorphic.*

Proof. (1) Let $\{A_i\}_{i\in\Gamma} \subseteq L\left(L^X\right)$. We first show $\sqcup_{i\in\Gamma} A_i \in L(L^X)$. Since $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \in \tau_{e_X^1}$ by Theorem 3.2(2), we have by Theorem 3.2(3) that $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \leq lrl\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$. On the other hand, since $r(A_i) \in \tau_{e_X^2}$ by Theorem 3.3(2) and $\tau_{e_X^2}$ is an Alexandrov topology, we have $\bigwedge_{i\in\Gamma} r(A_i) \in \tau_{e_X^2}$. Now, by Theorem 3.3(3), $rl\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \geq \bigwedge_{i\in\Gamma} r(A_i)$. Since l is decreasing by Theorem 3.3(1), we have $lrl\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \leq l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$. Hence $lrl\left(\bigwedge_{i\in\Gamma} r(A_i)\right) = l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$ and $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \in L\left(L^X\right)$.

We show that $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$ is an upper bound of $\{A_i\}_{i\in\Gamma}$. Since $\bigwedge_{i\in\Gamma} r(A_i) \leq r(A_i)$ for all $i \in \Gamma$ and l is decreasing by Theorem 3.3(1), we have $lr(A_i) \leq l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$. Since $A_i \in L\left(L^X\right)$, we have $lr(A_i) = A_i$. Hence we have $A_i \leq l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$.

We now show that $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$ is the supremum of $\{A_i\}_{i\in\Gamma}$. Assume that $A_i \leq B$ for all $i \in \Gamma$ where $B \in L\left(L^X\right)$. Then $r(B) \leq r(A_i)$ for all $i \in \Gamma$, and so $r(B) \leq \bigwedge_{i\in\Gamma} r(A_i)$. Since l is decreasing by Theorem 3.3(1), we have $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \leq lr(B)$. Since $B \in L\left(L^X\right)$, we have $lr(B) = B$. Hence $l\left(\bigwedge_{i\in\Gamma} r(A_i)\right) \leq B$.

Therefore $\sqcup^l_{i\in\Gamma} A_i$ is the supremum of $\{A_i\}_{i\in\Gamma}$ in $L\left(L^X\right)$.
(2) Let $\{A_i\}_{i\in\Gamma} \subseteq R\left(L^X\right)$. We first show that $\sqcup^r_{i\in\Gamma} A_i \in R\left(L^X\right)$. Since $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right) \in \tau_{e_X^2}$ by Theorem 3.3(2), we have by Theorem 3.3(3) that $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right) \leq rlr\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$. On the other hand, since $l(A_i) \in \tau_{e_X^1}$ by Theorem 3.3(1) and $\tau_{e_X^1}$ is an Alexandrov topology, we have $\bigwedge_{i\in\Gamma} l(A_i) \in \tau_{e_X^1}$. Now, by Theorem 3.3(3), we have $\bigwedge_{i\in\Gamma} l(A_i) \leq lr\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$. Since r is decreasing by Theorem 3.3(1), we have $rlr\left(\bigwedge_{i\in\Gamma} l(A_i)\right) \leq r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$. Hence $rlr\left(\bigwedge_{i\in\Gamma} l(A_i)\right) = r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$ and $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right) \in R\left(L^X\right)$.

We show that $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$ is an upper bound of $\{A_i\}_{i\in\Gamma}$. Since $\bigwedge_{i\in\Gamma} l(A_i) \leq l(A_i)$ for all $i \in \Gamma$ and r is decreasing by Theorem 3.3(1), we have $rl(A_i) \leq r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$ for all $i \in \Gamma$. Since $A_i \in R\left(L^X\right)$, we have $rl(A_i) = A_i$. Hence $A_i \leq r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$ for all $i \in \Gamma$.

We now show that $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)$ is the supremum of $\{A_i\}_{i\in\Gamma}$. Assume that $A_i \leq B$ for all $i \in \Gamma$ where $B \in R\left(L^X\right)$. Then $l(B) \leq l(A_i)$ for all $i \in \Gamma$, and so $l(B) \leq \bigwedge_{i\in\Gamma} l(A_i)$. Since r is decreasing by Theorem 3.3(1), we have $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right) \leq rl(B) = B$. Hence $r\left(\bigwedge_{i\in\Gamma} l(A_i)\right) \leq B$.

Therefore $\sqcup^r_{i\in\Gamma} A_i$ is the supremum of $\{A_i\}_{i\in\Gamma}$ in $R\left(L^X\right)$.

(3) Define $r_1 : L\left(L^X\right) \to R\left(L^X\right)$ by

$$r_1(A)(x) = \bigwedge_{y\in X} \left[e^2_X(x,y) \to n(A)(y)\right].$$

We first show that r_1 is well-defined. Let $A \in L\left(L^X\right)$. Then $lr(A) = A$. Since $rlr_1(A) = rlr(A) = r(A) = r_1(A)$, we have $r_1(A) \in R\left(L^X\right)$. Hence r_1 is well-defined.

We show that r_1 is injective. Assume that $r_1(A) = r_1(B)$ where $A, B \in L\left(L^X\right)$. Then $A = lr(A) = lr_1(A) = lr_1(B) = lr(B) = B$. Hence r_1 is injective.

We show that r_1 is surjective. Let $C \in R\left(L^X\right)$. Then $rl(C) = C$ and $l(C) \in L\left(L^X\right)$ by Theorem 3.3(5). Moreover, $C = rl(C) = r_1(l(C))$. Hence r_1 is surjective.

Let $\{A_i\}_{i\in\Gamma} \subseteq L\left(L^X\right)$. Since $r(A_i) \in R\left(L^X\right)$ by Theorem 3.3(5), we have by Theorem 3.3(6) that $\bigwedge_{i\in\Gamma} r(A_i) \in R\left(L^X\right)$. Now, note that

$$r_1\left(\sqcup^l_{i\in\Gamma} A_i\right) = rl\left(\bigwedge_{i\in\Gamma} r(A_i)\right) = \bigwedge_{i\in\Gamma} r_1(A_i)$$

and

$$r_1\left(\bigwedge_{i\in\Gamma} A_i\right) = r\left(\bigwedge_{i\in\Gamma} lr(A_i)\right) = r\left(\bigwedge_{i\in\Gamma} l(r_1(A_i))\right) = \sqcup^r_{i\in\Gamma} r_1(A_i).$$

Therefore $L\left(L^X\right)$ and $R\left(L^X\right)$ are anti-isomorphic. □

Example 3.6. Let $(X, \leq, \wedge, \vee\bot, \top)$ be a bounded lattice. Let $([0,1], \odot, \to, n, 0, 1)$ be the complete residuated lattice with the double negative law where

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\} \text{ and } n(x) = 1-x.$$

Define two maps $e^1, e^2 : X \times X \to [0,1]$ by

$$e^1(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y \end{cases} \quad \text{and } e^2(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x \not\leq y. \end{cases}$$

Let $e^{-i} : X \times X \to [0,1]$ be the map defined by $e^{-i}(x,y) = e^i(x,y)$ where $i = 1, 2$. Then one can see that e^i and e^{-i} are fuzzy partial orders where $i = 1, 2$. By

Theorem 3.2, one can see that

$$\begin{aligned}
\tau_{e^1} &= \left\{A \in [0,1]^X \mid A = \bigvee_{x\in X} \left[A(x) \odot e^1(x,-)\right]\right\} = [0,1]^X = \tau_{e^{-1}},\\
\tau_{e^2} &= \left\{A \in L^X \mid A = \bigvee_{x\in X} \left[A(x) \odot e^2(x,-)\right]\right\}\\
&= \left\{A \in L^X \mid A = \bigvee_{x\leq -} A(x)\right\}\\
&= \left\{A \in L^X \mid \text{if } x \leq y, \text{ then } A(x) \leq A(y)\right\},\\
\tau_{e^{-2}} &= \left\{A \in L^X \mid \text{if } x \leq y, \text{ then } A(y) \leq A(x)\right\}.
\end{aligned}$$

(1) Consider the fuzzy partially ordered space $\left(X, e^1, e^{-1}\right)$. Let l and r be the maps defined in Definition 3.1. Then

$$\begin{aligned}
l(A)(x) &= \bigwedge_{y\in X} \left[e^1(x,y) \to n(A)(y)\right] = e^1(x,x) \to n(A)(x) = n(A)(x),\\
r(A)(x) &= \bigwedge_{y\in X} \left[e^{-1}(x,y) \to n(A)(y)\right] = e^{-1}(x,x) \to n(A)(x) = n(A)(x).
\end{aligned}$$

Let $A \in \tau_{e^1} = [0,1]^X$ and $B \in \tau_{e^{-1}} = [0,1]^X$. Then $A = lr(A)$ and $B = rl(B)$.

Let $\{A_i\}_{i\in\Gamma} \subseteq \tau_{e^1} = [0,1]^X$ and $\{B_i\}_{i\in\Gamma} \subseteq \tau_{e^{-1}} = [0,1]^X$. Define $\sqcup^l_{i\in\Gamma} A_i = l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)$ and $\sqcup^r_{i\in\Gamma} B_i = r\left(\bigwedge_{i\in\Gamma} l(B_i)\right)$. Then

$$\begin{aligned}
\sqcup^l_{i\in\Gamma} A_i &= l\left(\bigwedge_{i\in\Gamma} r(A_i)\right) = n\left(\bigwedge_{i\in\Gamma} n(A_i)\right) = \bigvee_{i\in\Gamma} A_i,\\
\sqcup^r_{i\in\Gamma} B_i &= r\left(\bigwedge_{i\in\Gamma} l(B_i)\right) = n\left(\bigwedge_{i\in\Gamma} n(B_i)\right) = \bigvee_{i\in\Gamma} B_i.
\end{aligned}$$

Moreover,

$$\begin{aligned}
L\left([0,1]^X\right) &= \left\{A \in [0,1]^X \mid lr(A) = A\right\} = [0,1]^X,\\
R\left([0,1]^X\right) &= \left\{B \in [0,1]^X \mid rl(A) = A\right\} = [0,1]^X.
\end{aligned}$$

By Theorem 3.5, $\left([0,1]^X, \bigwedge, \sqcup^l = \bigvee, 0_X, 1_X\right)$ and $\left([0,1]^X, \bigwedge, \sqcup^r = \bigvee, 0_X, 1_X\right)$ are complete lattices which are anti-isomorphic.

(2) Consider the fuzzy partially ordered space $\left(X, e^1, e^2\right)$. Let l and r be the maps defined in Definition 3.1. Then

$$\begin{aligned}
l(A)(x) &= \bigwedge_{y\in X} \left[e^1(x,y) \to n(A)(y)\right] = n(A)(x),\\
r(A)(x) &= \bigwedge_{y\in X} \left[e^2(x,y) \to n(A)(y)\right] = \bigwedge_{\substack{x\leq y\\ y\in X}} n(A)(y).
\end{aligned}$$

Let $A \in [0,1]^X = \tau_{e^1}$. Then

$$\begin{aligned}
lr(A)(x) &= nr(A)(x) \quad \text{by (1)}\\
&= n\left(\bigwedge_{\substack{x\leq y\\ y\in X}} n(A)(y)\right)\\
&= \bigvee_{\substack{x\leq y\\ y\in X}} A(y) \quad \text{by Lemma 2.2(14)}.
\end{aligned}$$

Now, by Theorem 3.3(4), we have

$$\begin{aligned}
L\left([0,1]^X\right) &= \left\{A \in [0,1]^X \mid A = \bigvee_{\substack{-\leq y\\ y\in X}} A(y)\right\}\\
&= \left\{A \in [0,1]^X \mid \text{if } x \leq y, \text{ then } A(y) \leq A(x)\right\}.
\end{aligned}$$

Let $B \in [0,1]^X$. Then

$$\begin{aligned}
rl(B)(x) &= r\left(n(B)\right)(x) \quad \text{by (1)}\\
&= \bigwedge_{\substack{x\leq y\\ y\in X}} B(y).
\end{aligned}$$

Now, by Theorem 3.3(4), we have

$$\begin{aligned} R\left([0,1]^X\right) &= \left\{B \in [0,1]^X \mid B = \bigwedge_{\substack{-\leq y \\ y\in X}} B(y)\right\} \\ &= \left\{B \in [0,1]^X \mid \text{if } x \leq y, \text{ then } B(x) \leq B(y)\right\} \\ &= \tau_{e^2} \quad \text{by (1)}. \end{aligned}$$

For all $\{A_i\}_{i\in\Gamma} \subseteq L\left([0,1]^X\right)$, define $\sqcup^l_{i\in\Gamma} A_i(x) = l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)(x)$ (see Theorem 3.5). Then

$$\begin{aligned} \sqcup^l_{i\in\Gamma} A_i(x) &= l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)(x) = n\left(\bigwedge_{i\in\Gamma} r(A_i)\right)(x) \quad \text{by (1)} \\ &= n\left(\bigwedge_{i\in\Gamma} \bigwedge_{\substack{x\leq y \\ y\in X}} n(A_i)(y)\right) \\ &= \bigvee_{i\in\Gamma} \bigvee_{\substack{x\leq y \\ y\in X}} A_i(y) \quad \text{by Lemma 2.2(14)} \\ &= \bigvee_{i\in\Gamma} A_i(x) \quad \text{since } A_i \text{ is decreasing.} \end{aligned}$$

Now, by Theorem 3.5(1), $\left(L\left([0,1]^X\right), \bigwedge, \sqcup^l = \bigvee, 0_X, 1_X\right)$ is a complete lattice.

For all $\{B_i\}_{i\in\Gamma} \subseteq R\left([0,1]^X\right)$, define $\sqcup^r_{i\in\Gamma} B_i(x) = r\left(l(B_i)\right)(x)$ (see Theorem 3.5). Then

$$\begin{aligned} \sqcup^r_{i\in\Gamma} B_i(x) &= r\left(\bigwedge_{i\in\Gamma} l(B_i)\right)(x) = r\left(\bigwedge_{i\in\Gamma} n(B_i)\right)(x) \quad \text{by (1)} \\ &= r\left(n\left(\bigvee_{i\in\Gamma} B_i\right)\right)(x) \quad \text{by Lemma 2.2(14)} \\ &= \bigwedge_{\substack{x\leq y \\ y\in X}} \left(\bigvee_{i\in\Gamma} B_i\right)(y) \\ &= \bigvee_{i\in\Gamma} B_i(x) \quad \text{since } B_i \text{ is increasing.} \end{aligned}$$

Now, by Theorem 3.5(2), $\left(R\left([0,1]^X\right), \bigwedge, \sqcup^r = \bigvee, 0_X, 1_X\right)$ is a complete lattice.

Define $r_1 : L\left([0,1]^X\right) \to R\left([0,1]^X\right)$ by $r_1(A)(x) = \bigwedge_{y\in X} \left[e^2(x,y) \to n(A)(y)\right]$. Then

$$r_1(A)(x) = \bigwedge_{\substack{x\leq y \\ y\in X}} n(A)(y) = n(A)(x) \quad \text{since } n(A) \text{ is increasing.}$$

By Theorem 3.5, $r_1\left(\sqcup^l_{i\in\Gamma} A_i\right) = \bigwedge_{i\in\Gamma} r_1(A_i)$ and $r_1\left(\bigwedge_{i\in\Gamma} A_i\right) = \sqcup^r_{i\in\Gamma} r_1(A_i)$ for all $\{A_i\}_{i\in\Gamma} \subseteq L\left([0,1]^X\right)$. Furthermore, $L\left([0,1]^X\right)$ and $R\left([0,1]^X\right)$ are anti-isomorphic.

(3) Consider the fuzzy partially ordered space $\left(X, e^2, e^{-2}\right)$. Let l and r be the maps defined in Definition 3.1. Then

$$\begin{aligned} l(A)(x) &= \bigwedge_{y\in X} \left[e^2(x,y) \to n(A)(y)\right] = \bigwedge_{\substack{x\leq y \\ y\in X}} n(A)(y), \\ r(A)(x) &= \bigwedge_{y\in X} \left[e^{-2}(x,y) \to n(A)(y)\right] = \bigwedge_{\substack{y\leq x \\ y\in X}} n(A)(y). \end{aligned}$$

Let $A \in [0,1]^X$. Then

$$\begin{aligned} lr(A)(x) &= \bigwedge_{\substack{x\leq y \\ y\in X}} n(r(A))(y) = n\left(\bigvee_{\substack{x\leq y \\ y\in X}} r(A)(y)\right) \\ &= n\left(\bigvee_{\substack{x\leq y \\ y\in X}} \bigvee_{\substack{z\leq y \\ z\in X}} n(A)(z)\right) = \bigwedge_{\substack{x\leq y \\ y\in X}} \bigwedge_{\substack{z\leq y \\ z\in X}} A(z). \end{aligned}$$

Hence

$$L\left([0,1]^X\right) = \left\{A \in [0,1]^X \mid \text{if } x \leq y, \text{ then } A(x) \leq A(y)\right\} = \tau_{e^2}.$$

Let $B \in [0,1]^X$. Then

$$\begin{aligned} rl(B)(x) &= \bigwedge_{\substack{y\le x\\ y\in X}} nl(B)(y) = n\left(\bigvee_{\substack{y\le x\\ y\in X}} l(B)(y)\right) \\ &= n\left(\bigvee_{\substack{y\le x\\ y\in X}} \bigwedge_{\substack{y\le z\\ z\in X}} n(B)(z)\right) = \bigwedge_{\substack{y\le x\\ y\in X}} \bigvee_{\substack{y\le z\\ z\in X}} B(z). \end{aligned}$$

Hence

$$R\left([0,1]^X\right) = \left\{B \in [0,1]^X \mid \text{if } x \le y, \text{ then } B(y) \le B(x)\right\} = \tau_{e^{-2}}.$$

For all $\{A_i\}_{i\in\Gamma} \subseteq L\left([0,1]^X\right)$, define $\sqcup^l_{i\in\Gamma} A_i(x) = l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)(x)$ (see Theorem 3.5). Then

$$\sqcup^l_{i\in\Gamma} A_i(x) = l\left(\bigwedge_{i\in\Gamma} r(A_i)\right)(x) = n\left(\bigwedge_{i\in\Gamma} n(A_i)\right)(x) = \bigvee_{i\in\Gamma} A_i(x).$$

By Theorem 3.5, $\left(L\left([0,1]^X\right), \bigwedge, \sqcup^l = \bigvee, 0_X, 1_X\right)$ is a complete lattice.

For all $\{B_i\}_{i\in\Gamma} \subseteq R\left([0,1]^X\right)$, define $\sqcup^r_{i\in\Gamma} B_i(x) = r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)(x)$ (see Theorem 3.5). Then

$$\sqcup^r_{i\in\Gamma} B_i(x) = r\left(\bigwedge_{i\in\Gamma} l(A_i)\right)(x) = n\left(\bigwedge_{i\in\Gamma} n(A_i)\right)(x) = \bigvee_{i\in\Gamma} B_i(x).$$

By Theorem 3.5, $\left(R\left([0,1]^X\right), \bigwedge, \sqcup^r = \bigvee, 0_X, 1_X\right)$ is a complete lattice. Moreover, $\left(L\left([0,1]^X\right), \bigwedge, \sqcup^l = \bigvee, 0_X, 1_X\right)$ and $\left(R\left([0,1]^X\right), \bigwedge, \sqcup^r = \bigvee, 0_X, 1_X\right)$ are anti-isomorphic by Theorem 3.5.

Example 3.7. Let $X = \{h_1, h_2, h_3\}$ be the set. Let $Y = \{e, b, w, c, i\}$ be the set with $h_i =$"house", $e =$"expensive", $b =$"beautiful", $w =$"wooden", $c =$"creative" and $i =$"in the green surroundings". Let $([0,1], \odot, \to, n, 0, 1)$ be the complete residuated lattice defined in Example 3.6. Let $R : X \times Y \to [0,1]$ be the fuzzy information defined as follows:

R	e	b	w	c	i
h_1	0.1	0.5	0.6	0.3	0.7
h_2	0.8	0.1	0.9	0.8	0.5
h_3	0.4	0.9	0.4	0.5	0.3

Define $e_1, e_2 : X \times X \to [0,1]$ by

$$\begin{aligned} e_1(h_i, h_j) &= \bigwedge_{x\in\{e,b,w\}} \left[R(h_i, x) \to R(h_j, x)\right], \\ e_2(h_i, h_j) &= \bigwedge_{x\in\{c,i\}} \left[R(h_i, x) \to R(h_j, x)\right]. \end{aligned}$$

Then

e_1	h_1	h_2	h_3
h_1	1	0.6	0.8
h_2	0.3	1	0.5
h_3	0.6	0.2	1

and

e_2	h_1	h_2	h_3
h_1	1	0.8	0.6
h_2	0.5	1	0.7
h_3	0.8	1	1

One can see that (X, e_1, e_2) is a fuzzy bi-partially ordered space.

(1) Let $A = (A(h_1), A(h_2), A(h_3)) = (0.7, 0.4, 0.1) \in [0,1]^X$. Since

$$\begin{aligned} 0.5 &= A(h_1) \odot e_1(h_1, h_3) \not\le A(h_3) = 0.1, \\ 0.5 &= A(h_1) \odot e_2(h_1, h_2) \not\le A(h_2) = 0.4, \end{aligned}$$

we have $A \notin \tau_{e_1}$ and $A \notin \tau_{e_2}$. A tedious computation gives us that $l(A) = (0.3, 0.6, 0.7)$, $r(A) = (0.3, 0.6, 0.5)$, $lr(A) = (0.7, 0.4, 0.5)$, $rl(A) = (0.6, 0.4, 0.3)$, $lrl(A) = (0.4, 0.6, 0.7)$, $rlrl(A) = (0.6, 0.4, 0.3)$ and $lrlrl(A) = (0.4, 0.6, 0.7)$. Then one can see that $rlr(A) = r(A)$, $l(A) \neq lrl(A)$, $A \notin L(L(X))$ and $A \notin R(L(X))$. Moreover, $r(A), rl(A) \in R(L(X))$ and $lr(A), lrl(A) \in L(L(X))$. Hence

$$\begin{aligned} r(A) \sqcup^r rl(A) &= r\left[l(r(A)) \wedge l(rl(A))\right] = r\left[(0.7, 0.4, 0.5) \wedge (0.4, 0.6, 0.7)\right] \\ &= r(0.4, 0.4, 0.5) = (0.6, 0.6, 0.5), \\ lr(A) \sqcup^l lrl(A) &= l\left[r(lr(A)) \wedge r(lrl(A))\right] = l\left[(0.3, 0.6, 0.5)\right] \wedge (0.6, 0.4, 0.3)] \\ &= l(0.3, 0.4, 0.3) = (0.7, 0.6, 0.7). \end{aligned}$$

(2) Let $A = (0.7, 0.4, 0.1) \in [0, 1]^X$ (see (1)). Let $B = (0.3, 0.8, 0.5) \in [0, 1]^X$. Then one can see that $B \in \tau_{e_1}$ and $B \in \tau_{e_2}$.

A tedious computation gives us that $l(B) = (0.6, 0.2, 0.5)$, $r(B) = (0.4, 0.2, 0.2)$, $lr(B) = (0.6, 0.8, 0.8)$, $rl(B) = (0.4, 0.8, 0.5)$, $lrl(B) = (0.6, 0.2, 0.5)$ and $rlr(B) = (0.4, 0.2, 0.2)$. Then $lrl(B) = l(B)$, $rlr(B) = r(B)$, $l(B) \in L(L(X))$ and $r(B) \in R(L(X))$. Hence

$$\begin{aligned} r(A) \sqcup^r r(B) &= r\left[l(r(A)) \wedge l(r(B))\right] = r\left[(0.7, 0.4, 0.5) \wedge (0.6, 0.8, 0.8)\right] \\ &= r(0.6, 0.4, 0.5) = (0.4, 0.6, 0.5), \\ lr(A) \sqcup^l l(B) &= l\left[r(lr(A)) \wedge r(l(B))\right] = l\left[(0.3, 0.6, 0.5) \wedge (0.4, 0.8, 0.5)\right] \\ &= l(0.3, 0.6, 0.5) = (0.7, 0.4, 0.5), \\ lr(A) \sqcup^l lr(B) &= l\left[r(lr(A)) \wedge r(lr(B))\right] = l\left[(0.3, 0.6, 0.5) \wedge (0.4, 0.2, 0.2)\right] \\ &= l(0.3, 0.2, 0.2) = (0.7, 0.8, 0.8). \end{aligned}$$

By Theorem 3.5, $\left(L\left([0,1]^X\right), \bigwedge, \sqcap^l, 0_X, 1_X\right)$ and $\left(R\left([0,1]^X\right), \bigwedge, \sqcap^r, 0_X, 1_X\right)$ are complete lattices, which are anti-isomorphic.

4. Conclusion

The present study focused on introducing an algebraic structure that is induced by a fuzzy bi-partial order on a complete residuated lattice with the double negative law. Specifically, we demonstrated that the two families of l-stable and r-stable fuzzy sets can be regarded as complete lattices, and we established that these two families are anti-isomorphic. Furthermore, we provided examples of such orders within the context of an information system. We hope that this study represents a valuable contribution to the field of mathematics and has the potential to inform future research in this area.

In forthcoming times, there is a potential for further exploration of a variety of theoretical properties of fuzzy bi-partially ordered spaces on complete residuated lattices with the double negative law.

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Ju-mok Oh received M.Sc. and Ph.D. at Pohang University of Science and Technology. Since 2012 he has been at Kangnung-Wonju National University. His research interests include applied mathematics and enumerative combinatorics.

Department of Mathematics and Physics, Kangnung-Wonju National University, Kangnung 210-702, Korea.

e-mail: jumokoh@gwnu.ac.kr