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VECTORIAL HILFER-PRABHAKAR-HARDY TYPE FRACTIONAL INEQUALITIES

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Abstract. We present a variety of univariate and multivariate left and right side Hardy type fractional inequalities, many of them under convexity, and other also of L_p type, $p \geq 1$, in the setting of generalized Hilfer and Prabhakar fractional Calculi.

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1. Background

Let $-\infty < a < b < \infty$, the left and right Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ ($\mathcal{R}(\alpha) > 0$) are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1}$$

$x > a$; where Γ stands for the gamma function,
and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{2}$$

$x < b$.

The Riemann-Liouville left and right fractional derivatives of order $\alpha \in \mathbb{C}$ ($\mathcal{R}(\alpha) \geq 0$) are defined by

$$\left(\Delta_{a+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} y(t)\, dt \tag{3}$$

($n = \lceil \mathcal{R}(\alpha) \rceil$, $\lceil \cdot \rceil$ means ceiling of the number; $x > a$)

$$\left(\Delta_{b-}^{\alpha} y\right)(x) = (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{b-}^{n-\alpha} y\right)(x) =$$

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$$\frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_x^b (t-x)^{n-\alpha-1} y(t)\, dt \tag{4}$$

$(n = \lceil \mathcal{R}(\alpha) \rceil;\ x < b)$, respectively, where $\mathcal{R}(\alpha)$ is the real part of α.

In particular, when $\alpha = n \in \mathbb{Z}_+$, then

$$\left(\Delta_{a+}^{0} y\right)(x) = \left(\Delta_{b-}^{0} y\right)(x) = y(x);$$
$$\left(\Delta_{a+}^{n} y\right)(x) = y^{(n)}(x), \text{ and } \left(\Delta_{b-}^{n} y\right)(x) = (-1)^n y^{(n)}(x), \ \ n \in \mathbb{N}, \tag{5}$$

see [12].

Let $\alpha > 0$, $I = [a, b] \subset \mathbb{R}$, f an integrable function defined on I and $\psi \in C^1(I)$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in I$. Left fractional integrals and left Riemann-Liouville fractional derivatives of a function f with respect to another function ψ are defined as ([9], [12])

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} f(t)\, dt, \tag{6}$$

and

$$\Delta_{a+}^{\alpha,\psi} f(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{n-\alpha,\psi} f(x) = \tag{7}$$
$$\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} f(t)\, dt,$$

respectively, where $n = \lceil \alpha \rceil$.

Similarly, we define the right ones:

$$I_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} f(t)\, dt, \tag{8}$$

and

$$\Delta_{b-}^{\alpha,\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b-}^{n-\alpha,\psi} f(x) =$$
$$\frac{1}{\Gamma(n-\alpha)}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_x^b \psi'(t)(\psi(t) - \psi(x))^{n-\alpha-1} f(t)\, dt. \tag{9}$$

The following semigroup property holds; if $\alpha, \beta > 0,\ f \in C(I)$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f = I_{a+}^{\alpha+\beta,\psi} f \quad \text{and} \quad I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f = I_{b-}^{\alpha+\beta,\psi} f.$$

Next let again $\alpha > 0$, $n = \lceil \alpha \rceil$, $I = [a, b]$, $f, \psi \in C^n(I) : \psi$ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left ψ-Caputo fractional derivative of f of order α is given by ([1])

$${}^C D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x), \tag{10}$$

and the right ψ-Caputo fractional derivative ([1])

$${}^C D_{b-}^{\alpha,\psi} f(x) = I_{b-}^{n-\alpha,\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x). \tag{11}$$

We set

$$f_{\psi}^{[n]}(x) := f_{\psi}^{(n)} f(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} f(x) . \tag{12}$$

Clearly, when $\alpha = m \in \mathbb{N}$ we have

$${}^{C}D_{a+}^{\alpha,\psi} f(x) = f_{\psi}^{[m]}(x) \quad \text{and} \quad {}^{C}D_{b-}^{\alpha,\psi} f(x) = (-1)^{m} f_{\psi}^{[m]}(x) ,$$

and if $\alpha \notin \mathbb{N}$, then

$${}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t) \, dt, \tag{13}$$

and

$${}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \psi'(t) (\psi(t) - \psi(x))^{n-\alpha-1} f_{\psi}^{[n]}(t) \, dt. \tag{14}$$

If $\psi(x) = x$, then we get the usual left and right Caputo fractional derivatives

$${}^{C}D_{a+}^{m} f(x) = f^{(m)}(x) , \quad {}^{C}D_{b-}^{m} f(x) = (-1)^{m} f^{(m)}(x) ,$$

for $m \in \mathbb{N}$, and ($\alpha \notin \mathbb{N}$)

$$D_{*a}^{\alpha} f(x) = {}^{C}D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} (x-t)^{n-\alpha-1} f^{(n)}(t) \, dt, \tag{15}$$

$$D_{b-}^{\alpha}(x) = {}^{C}D_{b-}^{\alpha} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} (t-x)^{n-\alpha-1} f^{(n)}(t) \, dt. \tag{16}$$

Also we set

$${}^{C}D_{a+}^{0,\psi} f(x) = {}^{C}D_{b-}^{0,\psi} f(x) = f(x) .$$

Next we talk about the ψ-Hilfer fractional derivative.

Definition 1.1. ([14]) Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $I = [a,b] \subset \mathbb{R}$ and $f, \psi \in C^{n}([a,b])$, ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The ψ-Hilfer fractional derivative (left-sided and right-sided) ${}^{H}\mathbb{D}_{a+(b-)}^{\alpha,\beta;\psi} f$ of order α and type $0 \leq \beta \leq 1$, respectively, are defined by

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) , \tag{17}$$

and

$${}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) , \quad x \in [a,b] . \tag{18}$$

The original Hilfer fractional derivatives ([13]) come from $\psi(x) = x$, and are denoted by ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f(x)$ and ${}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f(x)$.

When $\beta = 0$, we get Riemann-Liouville fractional derivatives, while when $\beta = 1$ we have Caputo type fractional derivatives.

We define $\gamma = \alpha + \beta(n-\alpha)$. We notice that $n-1 < \alpha \leq \alpha + \beta(n-\alpha) \leq \alpha + n - \alpha = n$, hence $\lceil \gamma \rceil = n$. We can easily write that ([14])

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\gamma-\alpha;\psi} \Delta_{a+}^{\gamma;\psi} f(x) , \tag{19}$$

and

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\gamma-\alpha;\psi} \Delta_{b-}^{\gamma;\psi} f(x), \quad x \in [a,b]. \tag{20}$$

We have that ([14])

$$\Delta_{a+}^{\gamma,\psi} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x), \tag{21}$$

and

$$\Delta_{b-}^{\gamma,\psi} f(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x). \tag{22}$$

In particular, when $0 < \alpha < 1$ and $0 \leq \beta \leq 1$; $\gamma = \alpha + \beta(1-\alpha)$, we have that

$$^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\gamma-\alpha-1} \Delta_{a+}^{\gamma;\psi} f(t)\, dt, \tag{23}$$

and

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\gamma-\alpha-1} \Delta_{b-}^{\gamma;\psi} f(t)\, dt, \tag{24}$$

$x \in [a,b]$.

Remark 1.1. ([14]) Let $\mu = n(1-\beta) + \beta\alpha$, then $\lceil \mu \rceil = n$.

Assume that $g(x) = I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) \in C^n([a,b])$, we have that

$$^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{n-\mu;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n g(x). \tag{25}$$

Thus

$$^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f = {}^{C}D_{a+}^{\mu;\psi} g(x) = {}^{C}D_{a+}^{\mu;\psi} \left[I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) \right]. \tag{26}$$

Assume that $w(x) = I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \in C^n([a,b])$. Hence

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n w(x) = I_{b-}^{n-\mu;\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n w(x). \tag{27}$$

Thus

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f = {}^{C}D_{b-}^{\mu;\psi} w(x) = {}^{C}D_{b-}^{\mu;\psi} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \right). \tag{28}$$

We mention the simplified ψ-Hilfer fractional Taylor formulae:

Theorem 1.2. *(see also [14]) Let $\psi, f \in C^n([a,b])$, with ψ being increasing such that $\psi'(x) \neq 0$ over $[a,b]$, where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(n-\alpha)$, $x \in [a,b]$. Then*

$$f(x) - \sum_{k=1}^{n-1} \frac{(\psi(x) - \psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f \right)(a) =$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \, {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t)\, dt, \tag{29}$$

and

$$f(x)-\sum_{k=1}^{n-1}\frac{(-1)^k(\psi(b)-\psi(x))^{\gamma-k}}{\Gamma(\gamma-k+1)}f_{\psi}^{[n-k]}\left(I_{b-}^{(1-\beta)(n-\alpha);\psi}f\right)(b)=$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(t)\,dt. \tag{30}$$

Here notice that $\left(I_{a+}^{(1-\beta)(n-\alpha);\psi}f\right)(a)=\left(I_{b-}^{(1-\beta)(n-\alpha);\psi}f\right)(b)=0$.

We also mention the following alternative ψ-Hilfer fractional Taylor formulae:

Theorem 1.3. *([4]) Let $f,\psi\in C^n([a,b])$, with ψ being increasing, $\psi'(x)\neq 0$ over $[a,b]\subset\mathbb{R}$, $\alpha>0:\lceil\alpha\rceil=n$, $0\leq\beta\leq 1$, $\mu=n(1-\beta)+\beta\alpha$. Assume that $g(x)=I_{a+}^{(1-\beta)(n-\alpha);\psi}f(x)$, $w(x)=I_{b-}^{(1-\beta)(n-\alpha);\psi}f(x)\in C^n([a,b])$.*

Then

1)

$$I_{a+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x)=g(x)-\sum_{k=0}^{n-1}\frac{g_{\psi}^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k, \tag{31}$$

where

$$g_{\psi}^{[k]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k g(x),\quad k=0,1,...,n-1,$$

and

2)

$$I_{b-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x)=w(x)-\sum_{k=0}^{n-1}\frac{(-1)^k w_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^k, \tag{32}$$

where

$$w_{\psi}^{[k]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k w(x),\quad k=0,1,...,n-1;\ x\in[a,b].$$

Next we list two Hilfer fractional derivatives representation formulae:

Theorem 1.4. *([4]) Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $\lceil\alpha\rceil=n$, $0<\beta<1$; $f\in C^n([a,b])$, $[a,b]\subset\mathbb{R}$; and set $\gamma=\alpha+\beta(n-\alpha)$. Assume further that $\Delta_{a+}^{\gamma}f\in C([a,b])$: $\Delta_{a+}^{\gamma-j}f(a)=0$, for $j=1,...,n$. Let also $\overline{\alpha}>0:\lceil\overline{\alpha}\rceil=\overline{n}$, with $\overline{\gamma}=\overline{\alpha}+\beta(\overline{n}-\overline{\alpha})$, and assume that $\alpha>\overline{\alpha}$ and $\gamma>\overline{\gamma}$. Then*

$${}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f(x)=\frac{1}{\Gamma(\alpha-\overline{\alpha})}\int_a^x(x-t)^{\alpha-\overline{\alpha}-1}\ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta}f(t)\,dt, \tag{33}$$

$\forall\ x\in[a,b]$,

furthermore ${}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f\in AC([a,b])$ (absolutely continuous functions) if $\alpha-\overline{\alpha}\geq 1$ and ${}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f\in C([a,b])$ if $\alpha-\overline{\alpha}\in(0,1)$.

Theorem 1.5. *([4]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$, $0 < \beta < 1$; $f \in C^n([a,b])$, $[a,b] \subset \mathbb{R}$; and set $\gamma = \alpha + \beta(n-\alpha)$. Assume further that $\Delta_{b-}^{\gamma} f \in C([a,b])$: $\Delta_{b-}^{\gamma-j} f(b) = 0$, $j = 1,...,n$. Let also $\overline{\alpha} > 0 : \lceil \overline{\alpha} \rceil = \overline{n}$, with $\overline{\gamma} = \overline{\alpha} + \beta(\overline{n} - \overline{\alpha})$, and assume that $\alpha > \overline{\alpha}$ and $\gamma > \overline{\gamma}$. Then*

$$ {}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x) = \frac{1}{\Gamma(\alpha - \overline{\alpha})} \int_x^b (t-x)^{\alpha - \overline{\alpha} - 1} \ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f(t)\, dt, \tag{34} $$

$\forall\ x \in [a,b]$,

furthermore ${}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta} f \in AC([a,b])$ if $\alpha - \overline{\alpha} \geq 1$ and ${}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta} f \in C([a,b])$ if $\alpha - \overline{\alpha} \in (0,1)$.

The fractional integral operator $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$, $\alpha > 0$, are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$ \left\| I_{a+}^{\alpha} f \right\|_p \leq K \left\| f \right\|_p, \quad \left\| I_{b-}^{\alpha} f \right\|_p \leq K \left\| f \right\|_p, \tag{35} $$

where

$$ K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}. \tag{36} $$

The left inequality (35) was proved by H.G. Hardy in one of his first papers, see [8].

We continue this Background section with the following material from [5], where the author introduced the genralized ψ-Prabhakar type of fractional calculus and mixed it with the ψ-Hilfer fractional calculus.

So we consider the Prabhakar function (also known as the three parameter Mittag-Laffler function), (see [7], p. 97; [6])

$$ E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k + \beta)} z^k, \tag{37} $$

where Γ is the gamma function; $\alpha, \beta, \gamma \in \mathbb{R} : \alpha, \beta > 0$, $z \in \mathbb{R}$, and $(\gamma)_k = \gamma(\gamma+1)...(\gamma+k-1)$. It is $E_{\alpha,\beta}^{0}(z) = \frac{1}{\Gamma(\beta)}$.

Let $a, b \in \mathbb{R}$, $a < b$ and $x \in [a,b]$; $f \in C([a,b])$. Let also $\psi \in C^1([a,b])$ which is increasing. The left and right Prabhakar fractional integrals with respect to ψ are defined as follows:

$$ \left(e_{\rho,\mu,\omega,a+}^{\gamma;\psi} f \right)(x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} E_{\rho,\mu}^{\gamma} \left[\omega (\psi(x) - \psi(t))^{\rho} \right] f(t)\, dt, \tag{38} $$

and

$$ \left(e_{\rho,\mu,\omega,b-}^{\gamma;\psi} f \right)(x) = \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\mu-1} E_{\rho,\mu}^{\gamma} \left[\omega (\psi(t) - \psi(x))^{\rho} \right] f(t)\, dt, \tag{39} $$

where $\rho, \mu > 0$; $\gamma, \omega \in \mathbb{R}$.

Functions (38) and (39) are continuous ([5]).

Next, additionally, assume that $\psi'(x) \neq 0$ over $[a,b]$.

Let $\psi, f \in C^N([a,b])$, where $N = \lceil \mu \rceil$, ($\lceil \cdot \rceil$ is the ceiling of the number), $0 < \mu \notin \mathbb{N}$. We define the ψ-Prabhakar-Caputo left and right fractional derivatives of order μ as follows ($x \in [a,b]$):

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = \int_a^x \psi'(t)\left(\psi(x)-\psi(t)\right)^{N-\mu-1}$$
$$E^{-\gamma}_{\rho,N-\mu}\left[\omega\left(\psi(x)-\psi(t)\right)^{\rho}\right]\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{N} f(t)\,dt, \tag{40}$$

and

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = (-1)^N \int_x^b \psi'(t)\left(\psi(t)-\psi(x)\right)^{N-\mu-1}$$
$$E^{-\gamma}_{\rho,N-\mu}\left[\omega\left(\psi(t)-\psi(x)\right)^{\rho}\right]\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{N} f(t)\,dt. \tag{41}$$

One can write these (see (40), (41)) as

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = \left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,a+}f^{[N]}_{\psi}\right)(x), \tag{42}$$

and

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = (-1)^N\left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,b-}f^{[N]}_{\psi}\right)(x), \tag{43}$$

where

$$f^{[N]}_{\psi}(x) = f^{(N)}_{\psi}f(x) := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N} f(x), \tag{44}$$

$\forall\, x \in [a,b]$.

Functions (42) and (43) are continuous on $[a,b]$.

Next we define the ψ-Prabhakar-Riemann Liouville left and right fractional derivatives of order μ as follows ($x \in [a,b]$):

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\int_a^x \psi'(t)\left(\psi(x)-\psi(t)\right)^{N-\mu-1}$$
$$E^{-\gamma}_{\rho,N-\mu}\left[\omega\left(\psi(x)-\psi(t)\right)^{\rho}\right] f(t)\,dt, \tag{45}$$

and

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\int_x^b \psi'(t)\left(\psi(t)-\psi(x)\right)^{N-\mu-1}$$
$$E^{-\gamma}_{\rho,N-\mu}\left[\omega\left(\psi(t)-\psi(x)\right)^{\rho}\right] f(t)\,dt. \tag{46}$$

That is we have

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,a+}f\right)(x), \tag{47}$$

and

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,b-}f\right)(x), \tag{48}$$

$\forall\, x \in [a,b]$.

We define also the ψ-Hilfer-Prabhakar left and right fractional derivatives of order μ and type $0 \leq \beta \leq 1$, as follows

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,a+} f \right)(x) = e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,a+} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{N} e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,a+} f(x), \tag{49}$$

and

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,b-} f \right)(x) = e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,b-} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{N} e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,b-} f(x), \tag{50}$$

$\forall\ x \in [a,b]$.

When $\beta = 0$, we get the Riemann-Liouville version, and when $\beta = 1$, we get the Caputo version.

We call $\xi = \mu + \beta(N-\mu)$, we have that $N - 1 < \mu \leq \mu + \beta(N-\mu) \leq \mu + N - \mu = N$, hence $\lceil \xi \rceil = N$.

We can easily write that

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,a+} f \right)(x) = e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\omega,a+} \ {}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,a+} f(x), \tag{51}$$

and

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,b-} f \right)(x) = e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\omega,b-} \ {}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,b-} f(x), \tag{52}$$

$\forall\ x \in [a,b]$.

In this article we prove univariate and multivariate Hardy type inequalities based on the above mentioned fractional background and convexity of functions. Our work is inspired by [2], [3], [8], [10], [11].

2. Prerequisites

I) Here we follow [3], p. 441, see Chapter 22.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x,y)\, d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{53}$$

$i = 1, ..., m \in \mathbb{N}$. We assume that $K_i(x) > 0$ a.e. on Ω_1 and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y) f_i(y)\, d\mu_2(y), \tag{54}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., m$.

Here u stands for a weight function on Ω_1 ($u \geq 0$, which is measurable).

We will use the following general result:

Theorem 2.1. *([3], p. 442) Assume that the functions ($i = 1, 2, ..., m \in \mathbb{N}$) $x \to \left(u(x) \frac{k_i(x,y)}{K_i(x)}\right)$ are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define u_i on Ω_2 by*

$$u_i(y) := \int_{\Omega_1} u(x) \frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{55}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i(y) \Phi_i(|f_i(y)|)^{p_i} d\mu_2(y)\right)^{\frac{1}{p_i}}, \tag{56}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ ($i = 1, ..., m$) such that

(i) $f_i, \Phi_i(|f_i|)^{p_i}$, are both $k_i(x,y) d\mu_2(y)$ - integrable, μ_1-a.e. in $x \in \Omega_1$, $i = 1, ..., m$,

(ii) $u_i \Phi_i(|f_i|)^{p_i}$ is μ_2-integrable, $i = 1, ..., m$,

and for all corresponding functions g_i ($i = 1, ..., m$) given by (54).

II) Here we foolow [3], Chapter 27.

The basic setting follows:

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k(x, \cdot)$ measurable on Ω_2, and

$$K(x) = \int_{\Omega_2} k(x,y) d\mu_2(y), \text{ for any } x \in \Omega_1, \tag{57}$$

$i = 1, ..., m \in \mathbb{N}$. We assume that $K(x) > 0$ a.e. on Ω_1 and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k(x,y) f_i(y) d\mu_2(y), \tag{58}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., n$.

Denote by $\vec{x} = x := (x_1, ..., x_n) \in \mathbb{R}^n$, $\vec{g} := (g_1, ..., g_n)$ and $\vec{f} := (f_1, ..., f_n)$.

We consider here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ a convex function, which is increasing per coordinate, i.e. if $x_i \leq y_i$, $i = 1, ..., n$, then $\Phi(x_1, ..., x_n) \leq \Phi(y_1, ..., y_n)$.

Next we may write

$$\vec{g}(x) = \int_{\Omega_2} k(x,y) \vec{f}(y) d\mu_2(y), \tag{59}$$

which means

$$(g_1(x), ..., g_n(x)) = \left(\int_{\Omega_2} k(x,y) f_1(y) d\mu_2(y), ..., \int_{\Omega_2} k(x,y) f_1(y) d\mu_2(y) \right). \tag{60}$$

Similarly, we may write

$$|\vec{g}(x)| = \left| \int_{\Omega_2} k(x,y) \vec{f}(y) d\mu_2(y) \right|, \tag{61}$$

and we mean

$$(|g_1(x)|, ..., |g_n(x)|) = \left(\left| \int_{\Omega_2} k(x,y) f_1(y) d\mu_2(y) \right|, ..., \left| \int_{\Omega_2} k(x,y) f_n(y) d\mu_2(y) \right| \right). \tag{62}$$

We also can write that

$$|\vec{g}(x)| \leq \int_{\Omega_2} k(x,y) \left| \vec{f}(y) \right| d\mu_2(y), \tag{63}$$

and we mean the fact that

$$|g_i(x)| \leq \int_{\Omega_2} k(x,y) |f_i(y)| d\mu_2(y), \tag{64}$$

for all $i = 1, ..., n$, etc.

More precisely here we follow:

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_j : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_j(x, \cdot)$ measurable on Ω_2, and

$$K_j(x) = \int_{\Omega_2} k_j(x,y) d\mu_2(y), \quad x \in \Omega_1, \; j = 1, ..., m. \tag{65}$$

We suppose that $K_j(x) > 0$ a.e. on Ω_1. Let the measurable functions $g_{ji} : \Omega_1 \to \mathbb{R}$ with the representation

$$g_{ji}(x) = \int_{\Omega_2} k_j(x,y) f_{ji}(y) d\mu_2(y),$$

written also as

$$\vec{g}_j(x) = \int_{\Omega_2} k_j(x,y) \vec{f}_j(y) d\mu_2(y), \tag{66}$$

where $f_{ji} : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., n$ and $j = 1, ..., m$.

We denote above the function vectors $\vec{g}_j := (g_{j1}, g_{j2}, ..., g_{jn})$ and $\vec{f}_j := (f_{j1}, ..., f_{jn})$, $j = 1, ..., m$.

We say $\vec{f}_j$ is integrable with respect to measure μ, iff all f_{ji} are integrable with respect to μ.

We also consider here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}$, $j = 1, ..., m$, convex functions that are increasing per coordinate. Again u is a weight function on Ω_1.

We will use the following theorem

Theorem 2.2. *([3], p. 628) Assume that the functions ($j = 1, 2, ..., m \in \mathbb{N}$) $x \to \left(u(x) \frac{k_j(x,y)}{K_j(x)} \right)$ are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define u_j on Ω_2 by*

$$u_j(y) := \int_{\Omega_1} u(x) \frac{k_j(x,y)}{K_j(x)} d\mu_1(x) < \infty. \tag{67}$$

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, be convex and increasing per coordinate.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_j \left(\left| \frac{\vec{g}_j(x)}{K_j(x)} \right| \right) d\mu_1(x) \leq$$

$$\prod_{j=1}^{m} \left(\int_{\Omega_2} u_j(y) \Phi_j \left(\left| \vec{f}_j(y) \right| \right)^{p_j} d\mu_2(y) \right)^{\frac{1}{p_j}}, \tag{68}$$

under the assumptions:

(i) $\vec{f}_j, \Phi_j \left(\left| \vec{f}_j \right| \right)^{p_j}$, are both $k_j(x,y) d\mu_2(y)$ - integrable, μ_1-a.e. in $x \in \Omega_1$, $j = 1, ..., m$,

(ii) $u_j \Phi_j \left(\left| \vec{f}_j \right| \right)^{p_j}$ is μ_2-integrable, $j = 1, ..., m$.

III) We will also use from [3], Chapter 26, the following theorem:

Theorem 2.3. *([3], p. 598) Let $\rho \in \{1, ..., m\}$ be fixed. Assume that the function $x \to \left(\frac{u(x) \prod_{j=1}^{m} k_j(x,y)}{\prod_{j=1}^{m} K_j(x)} \right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m on Ω_2 by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{j=1}^{m} k_j(x,y)}{\prod_{j=1}^{m} K_j(x)} \right) d\mu_1(x) < \infty. \tag{69}$$

Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, be convex and increasing per coordinate. Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left(\left| \frac{\vec{g}_j(x)}{K_j(x)} \right| \right) d\mu_1(x) \leq \tag{70}$$

$$\left(\prod_{\substack{j=1 \\ j \neq \rho}}^{m} \int_{\Omega_2} \Phi_j \left(\left| \vec{f}_j(y) \right| \right) d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_\rho \left(\left| \vec{f}_\rho(y) \right| \right) \lambda_m(y) d\mu_2(y) \right),$$

under the assumptions:

(i) $\vec{f}_j, \Phi_j \left(\left| \vec{f}_j \right| \right)$, are $k_j(x,y) d\mu_2(y)$ - integrable, μ_1-a.e. in $x \in \Omega_1$, $j = 1, ..., m$,

(ii) $\lambda_m \Phi_\rho\left(\left|\vec{f}_\rho\right|\right)$; $\Phi_1\left(\left|\vec{f}_1\right|\right)$, $\Phi_2\left(\left|\vec{f}_2\right|\right)$, $\Phi_3\left(\left|\vec{f}_3\right|\right)$, ..., $\widehat{\Phi_\rho\left(\left|\vec{f}_\rho\right|\right)}$, ..., $\Phi_m\left(\left|\vec{f}_m\right|\right)$, *are all* μ_2*-integrable,*

and for all corresponding functions g_i *given by (54). Above* $\widehat{\Phi_\rho\left(\left|\vec{f}_\rho\right|\right)}$ *means a missing item.*

Above all symbols are as in (II).

3. Main Results

I)' Here we apply Theorem 2.1.

Let here $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$.

We present

Theorem 3.1. *Here* $i = 1, ..., m$. *Let* $\alpha_i > 0$, $\alpha_i \notin \mathbb{N}$, $\lceil \alpha_i \rceil = n_i$, $0 < \beta_i < 1$; $f_i \in C^{n_i}([a,b])$, $[a,b] \subset \mathbb{R}$; *and set* $\gamma_i = \alpha_i + \beta_i (n_i - \alpha_i)$. *Assume further that* $\Delta_{a+}^{\gamma_i} f_i \in C([a,b]) : \Delta_{a+}^{\gamma_i - j_i} f_i(a) = 0$, *for* $j_i = 1, ..., n_i$. *Let also* $\overline{\alpha}_i > 0 : \lceil \overline{\alpha}_i \rceil = \overline{n}_i$, *with* $\overline{\gamma}_i = \overline{\alpha}_i + \beta_i(\overline{n}_i - \overline{\alpha}_i)$, *and assume that* $\alpha_i > \overline{\alpha}_i$ *and* $\gamma_i > \overline{\gamma}_i$.

Let also $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing functions;* $u \geq 0$ *is a weight measurable function on* $[a,b]$. *We assume that*

$$u_i(y) = (\alpha_i - \overline{\alpha}_i) \int_y^b u(x) \frac{(x-y)^{(\alpha_i - \overline{\alpha}_i) - 1}}{(x-a)^{(\alpha_i - \overline{\alpha}_i)}} dx < \infty, \tag{71}$$

for all $a < y < b$ *and* u_i *is Lebesgue integrable.*

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i\left(\frac{\left|{}^H\mathbb{D}_{a+}^{\overline{\alpha}_i, \beta_i} f_i(x)\right|}{(x-a)^{\alpha_i - \overline{\alpha}_i}} \Gamma(\alpha_i - \overline{\alpha}_i + 1)\right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b u_i(y) \left(\Phi_i\left(\left|{}^H\mathbb{D}_{a+}^{\alpha_i, \beta_i} f_i(y)\right|\right)\right)^{p_i} dy\right)^{\frac{1}{p_i}}. \tag{72}$$

Proof. By Theorems 1.4, 2.1 and from [2], pp. 31-33, see relations there (2.40)-(2.47). □

Remark 3.1. (to Theorem 3.1) One can have Φ_i =identity map or e^x, or $\Phi_i(x) = x^{\overline{p}_i}$, $x \in \mathbb{R}_+$, $\overline{p}_i > 1$, etc.

To save space in this work we skip these interesting applications here and later.

We continue with

Theorem 3.2. *Here* $i = 1, ..., m$. *Let* $\alpha_i > 0$, $\alpha_i \notin \mathbb{N}$, $\lceil \alpha_i \rceil = n_i$, $0 < \beta_i < 1$; $f_i \in C^{n_i}([a,b])$, $[a,b] \subset \mathbb{R}$; *and set* $\gamma_i = \alpha_i + \beta_i(n_i - \alpha_i)$. *Assume further that* $\Delta_{b-}^{\gamma_i} f_i \in C([a,b]) : \Delta_{b-}^{\gamma_i - j_i} f_i(b) = 0$, $j_i = 1, ..., n_i$. *Let also* $\overline{\alpha}_i > 0 : \lceil \overline{\alpha}_i \rceil = \overline{n}_i$, *with* $\overline{\gamma}_i = \overline{\alpha}_i + \beta_i(\overline{n}_i - \overline{\alpha}_i)$, *and assume that* $\alpha_i > \overline{\alpha}_i$ *and* $\gamma_i > \overline{\gamma}_i$. *Let also*

$\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing functions; $u \geq 0$ is a weight measurable function on $[a, b]$. We assume that

$$\overline{u}_i(y) := (\alpha_i - \overline{\alpha}_i) \int_a^y u(x) \frac{(y-x)^{(\alpha_i - \overline{\alpha}_i)-1}}{(b-x)^{(\alpha_i - \overline{\alpha}_i)}} dx < \infty, \tag{73}$$

for all $a < y < b$ and $\overline{u}_i$ is Lebesgue integrable.

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| {}^H\mathbb{D}_{b-}^{\overline{\alpha}_i, \beta_i} f_i(x) \right|}{(b-x)^{\alpha_i - \overline{\alpha}_i}} \Gamma(\alpha_i - \overline{\alpha}_i + 1) \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b \overline{u}_i(y) \left(\Phi_i \left(\left| {}^H\mathbb{D}_{b-}^{\alpha_i, \beta_i} f_i(y) \right| \right) \right)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{74}$$

Proof. By Theorems 1.5, 2.1 and from [2], pp. 35-37, see relations there (2.58)-(2.67). □

We continue with

Theorem 3.3. *Here $i = 1, ..., m$. Let $f_i \in C^{n_i}([a, b])$, $\theta := \max\{n_1, ..., n_m\}$, $\psi \in C^\theta([a, b])$, with ψ being increasing: $\psi'(x) \neq 0$ over $[a, b]$, where $n_i - 1 < \alpha_i < n_i$, $0 \leq \beta_i \leq 1$, and $\gamma_i = \alpha_i + \beta_i(n_i - \alpha_i)$.*

Assume that $f_{i\psi}^{[n_i - k_i]} \left(I_{a+}^{(1-\beta_i)(n_i - \alpha_i); \psi} f_i \right)(a) = 0$, $k_i = 1, ..., n_i - 1$; $i = 1, ..., m$.

Let also $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing functions; $u \geq 0$ is a weight measurable function on $[a, b]$. We assume that

$$u_i^\psi(y) = \alpha_i \psi'(y) \int_y^b u(x) \frac{(\psi(x) - \psi(y))^{\alpha_i - 1}}{(\psi(x) - \psi(a))^{\alpha_i}} dx < \infty, \tag{75}$$

for all $a < y < b$ and u_i^ψ is Lebesgue integrable.

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{|f_i(x)|}{(\psi(x) - \psi(a))^{\alpha_i}} \Gamma(\alpha_i + 1) \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b u_i^\psi(y) \Phi_i \left(\left| \left({}^H\mathbb{D}_{a+}^{\alpha_i, \beta_i; \psi} f_i \right)(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}, \tag{76}$$

true for continuous ${}^H\mathbb{D}_{a+}^{\alpha_i, \beta_i; \psi} f_i$, $i = 1, ..., m$.

Proof. From (29) we get:

$$f_i(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha_i - 1} \; {}^H\mathbb{D}_{a+}^{\alpha_i, \beta_i; \psi} f_i(t) \, dt, \tag{77}$$

$\forall \, x \in [a, b]$; $i = 1, ..., m$.

Then we apply Theorem 2.1, along with [2], pp. 47-49, see the relations there (2.107)-(2.119). □

We also give

Theorem 3.4. *Here* $i = 1, ..., m$. *Let* $f_i \in C^{n_i}([a,b])$, $\theta := \max\{n_1, ..., n_m\}$, $\psi \in C^{\theta}([a,b])$, *with* ψ *being increasing:* $\psi'(x) \neq 0$ *over* $[a,b]$, *where* $n_i - 1 < \alpha_i < n_i$, $0 \leq \beta_i \leq 1$, *and* $\gamma_i = \alpha_i + \beta_i(n_i - \alpha_i)$.

Assume that $f_{i\psi}^{[n_i-k_i]}\left(I_{b-}^{(1-\beta_i)(n_i-\alpha_i);\psi} f_i\right)(b) = 0$, $k_i = 1, ..., n_i - 1$; $i = 1, ..., m$.

Let also $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing functions;* $u \geq 0$ *is a weight measurable function on* $[a,b]$. *We assume that*

$$\overline{u}_i^{\psi}(y) = \alpha_i \psi'(y) \int_a^y u(x) \frac{(\psi(y) - \psi(x))^{\alpha_i - 1}}{(\psi(b) - \psi(x))^{\alpha_i}} dx < \infty, \tag{78}$$

for all $a < y < b$ *and* $\overline{u}_i^{\psi}$ *is Lebesgue integrable.*

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i\left(\frac{|f_i(x)|}{(\psi(b) - \psi(x))^{\alpha_i}} \Gamma(\alpha_i + 1)\right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b \overline{u}_i^{\psi}(y) \Phi_i\left(\left|\left({}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i\right)(y)\right|\right)^{p_i} dy\right)^{\frac{1}{p_i}}, \tag{79}$$

true for continuous ${}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i$, $i = 1, ..., m$.

Proof. From (30) we get:

$$f_i(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha_i - 1} \; {}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i(t) dt, \tag{80}$$

$\forall\, x \in [a,b]$; $i = 1, ..., m$.

Then we apply Theorem 2.1, along with [2], pp. 51-53, see the relations there (2.132)-(2.142). □

We present

Theorem 3.5. *Here* $i = 1, ..., m$. *Let* $f_i \in C^{n_i}([a,b])$, $\theta := \max\{n_1, ..., n_m\}$, $\psi \in C^{\theta}([a,b])$, *with* ψ *being increasing:* $\psi'(x) \neq 0$ *over* $[a,b] \subset \mathbb{R}$, $\alpha_i > 0 : \lceil \alpha_i \rceil = n_i$, $0 \leq \beta_i \leq 1$, $\mu_i = n_i(1 - \beta_i) + \beta_i \alpha_i$. *Assume that* $g_i(x) := I_{a+}^{(1-\beta_i)(n_i-\alpha_i);\psi} f_i(x) \in C^{n_i}([a,b])$, *and* $g_{i\psi}^{[k_i]}(a) = 0$, $k_i = 0, ..., n_i - 1$, *where* $g_{i\psi}^{[k_i]}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{k_i} g_i(x)$, $k_i = 0, 1, ..., n_i - 1$.

Let also $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing functions;* $u \geq 0$ *is a weight measurable function on* $[a,b]$. *We assume that*

$$\lambda_i^{\psi}(y) = \mu_i \psi'(y) \int_y^b u(x) \frac{(\psi(x) - \psi(y))^{\mu_i - 1}}{(\psi(x) - \psi(a))^{\mu_i}} dx < \infty, \tag{81}$$

for all $a < y < b$ *and* λ_i^{ψ} *is Lebesgue integrable.*

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{|g_i(x)|}{(\psi(x)-\psi(a))^{\mu_i}} \Gamma(\mu_i+1) \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b \lambda_i^{\psi}(y) \Phi_i \left(\left| \left({}^H\mathbb{D}_{a+}^{\alpha_i,\beta_i;\psi} f_i \right)(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}, \quad (82)$$

true for continuous ${}^H\mathbb{D}_{a+}^{\alpha_i,\beta_i;\psi} f_i$, $i=1,...,m$.

Proof. From (31) we get that

$$g_i(x) = I_{a+}^{\mu_i;\psi} \, {}^H\mathbb{D}_{a+}^{\alpha_i,\beta_i;\psi} f_i(x), \quad (83)$$

$\forall \, x \in [a,b]$; $i=1,...,m$.

Then we apply Theorem 2.1, along with [2], pp. 47-49, see the relations there (2.107)-(2.119). □

We also give

Theorem 3.6. *Here* $i=1,...,m$. *Let* $f_i \in C^{n_i}([a,b])$, $\theta := \max\{n_1,...,n_m\}$, $\psi \in C^{\theta}([a,b])$, *with* ψ *being increasing,* $\psi'(x) \neq 0$ *over* $[a,b] \subset \mathbb{R}$, $\alpha_i > 0 : \lceil \alpha_i \rceil = n_i$, $0 \leq \beta_i \leq 1$, $\mu_i = n_i(1-\beta_i) + \beta_i \alpha_i$. *Assume that* $w_i(x) := I_{b-}^{(1-\beta_i)(n_i-\alpha_i);\psi} f_i(x) \in C^{n_i}([a,b])$, *and* $w_{i\psi}^{[k_i]}(b) = 0$, $k_i = 0,...,n_i-1$, *where* $w_{i\psi}^{[k_i]}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{k_i} w_i(x)$, $k_i = 0,1,...,n_i-1$.

Let also $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i=1,...,m$, *be convex and increasing functions;* $u \geq 0$ *is a weight measurable function on* $[a,b]$. *We assume that*

$$\overline{\lambda}_i^{\psi}(y) = \mu_i \psi'(y) \int_a^y u(x) \frac{(\psi(y)-\psi(x))^{\mu_i-1}}{(\psi(b)-\psi(x))^{\mu_i}} dx < \infty, \quad (84)$$

for all $a < y < b$ *and* $\overline{\lambda}_i^{\psi}$ *is Lebesgue integrable.*

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{|w_i(x)|}{(\psi(b)-\psi(x))^{\mu_i}} \Gamma(\mu_i+1) \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b \overline{\lambda}_i^{\psi}(y) \Phi_i \left(\left| \left({}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i \right)(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}, \quad (85)$$

true for continuous ${}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i$, $i=1,...,m$.

Proof. From (32) we get that

$$w_i(x) = I_{b-}^{\mu_i;\psi} \, {}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i(x), \quad (86)$$

$\forall \, x \in [a,b]$, $i=1,...,m$.

Then we apply Theorem 2.1, along with [2], pp. 51-53, see the relations there (2.132)-(2.142). □

We continue with

Theorem 3.7. *Here* $i = 1, ..., m$. *Let* $n_i - 1 < \alpha_i < n_i$, $n_i \in \mathbb{N}$, $I = [a, b] \subset \mathbb{R}$ *and* $f_i \in C^{n_i}([a, b])$, $\theta := \max\{n_1, ..., n_m\}$, $\psi \in C^{\theta}([a, b])$, ψ *is increasing and* $\psi'(x) \neq 0$, *for all* $x \in I$. *Here* $0 \leq \beta_i \leq 1$ *and* $\gamma_i = \alpha_i + \beta_i(n_i - \alpha_i)$. *Assume that* $\Delta_{a+}^{\gamma_i;\psi} f_i, \Delta_{b-}^{\gamma_i;\psi} f_i \in C([a, b])$, $i = 1, ..., m$.

Let also $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing functions;* $u \geq 0$ *is a weight measurable function on* $[a, b]$. *We assume that*

$$\lambda_{i+}^{\psi}(y) = (\gamma_i - \alpha_i)\psi'(y) \int_y^b u(x) \frac{(\psi(x) - \psi(y))^{(\gamma_i - \alpha_i)-1}}{(\psi(x) - \psi(a))^{(\gamma_i - \alpha_i)}} dx < \infty, \tag{87}$$

for all $a < y < b$ *and* λ_{i+}^{ψ} *is Lebesgue integrable; and*

$$\lambda_{i-}^{\psi}(y) = (\gamma_i - \alpha_i)\psi'(y) \int_a^y u(x) \frac{(\psi(y) - \psi(x))^{(\gamma_i - \alpha_i)-1}}{(\psi(b) - \psi(x))^{(\gamma_i - \alpha_i)}} dx < \infty, \tag{88}$$

for all $a < y < b$ *and* λ_{i-}^{ψ} *is Lebesgue integrable.*

Then

i)

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| {}^H\mathbb{D}_{a+}^{\alpha_i,\beta_i;\psi} f_i(x) \right|}{(\psi(x) - \psi(a))^{\gamma_i - \alpha_i}} \Gamma(\gamma_i - \alpha_i + 1) \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b \lambda_{i+}^{\psi}(y) \Phi_i \left(\left| \left(\Delta_{a+}^{\gamma_i;\psi} f_i \right)(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}, \tag{89}$$

and

ii)

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| {}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i(x) \right|}{(\psi(b) - \psi(x))^{\gamma_i - \alpha_i}} \Gamma(\gamma_i - \alpha_i + 1) \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b \lambda_{i-}^{\psi}(y) \Phi_i \left(\left| \left(\Delta_{b-}^{\gamma_i;\psi} f_i \right)(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{90}$$

Proof. By (19) and (20), respectively, we have that

$${}^H\mathbb{D}_{a+}^{\alpha_i,\beta_i;\psi} f_i(x) = I_{a+}^{\gamma_i - \alpha_i;\psi} \ \Delta_{a+}^{\gamma_i;\psi} f_i(x), \tag{91}$$

and

$${}^H\mathbb{D}_{b-}^{\alpha_i,\beta_i;\psi} f_i(x) = I_{b-}^{\gamma_i - \alpha_i;\psi} \ \Delta_{b-}^{\gamma_i;\psi} f_i(x), \tag{92}$$

$\forall \, x \in [a, b]$, $i = 1, ..., m$.

Then, we apply Theorem 2.1 twice, along with [2], pp. 47-49, see there (2.107)-(2.119), and [2], pp. 51-53, see there (2.132)-(2.142), respectively. ☐

We make

Remark 3.2. We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure. Here $i = 1,...,m$. Let $\rho_i, \mu_i, \gamma_i, \omega_i > 0$, and $f_i \in C([a,b])$, with $\psi \in C^1([a,b])$ which is increasing.

We have that ($x \in [a,b]$):

$$\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_i\right)(x) =$$

$$\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\mu_i - 1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x) - \psi(t))^{\rho_i}\right] f_i(t)\,dt \tag{93}$$

$$= \int_a^b \chi_{(a,x]}(t)\,\psi'(t)(\psi(x) - \psi(t))^{\mu_i - 1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x) - \psi(t))^{\rho_i}\right] f_i(t)\,dt,$$

where χ is the characteristic function.

So, we choose here

$$k_i(x,t) := \chi_{(a,x]}(t)\,\psi'(t)(\psi(x) - \psi(t))^{\mu_i - 1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x) - \psi(t))^{\rho_i}\right], \tag{94}$$

$i = 1,...,m$.

That is

$$k_i(x,y) = \begin{cases} \psi'(y)(\psi(x) - \psi(y))^{\mu_i - 1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x) - \psi(y))^{\rho_i}\right], & a < y \le x, \\ \\ 0, \quad x < y < b, \end{cases} \tag{95}$$

$i = 1,...,m$.

Therefore we obtain

$$K_i(x) = \int_a^b \chi_{(a,x]}(y)\,\psi'(y)(\psi(x) - \psi(y))^{\mu_i - 1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x) - \psi(y))^{\rho_i}\right] dy =$$

$$\int_a^x \psi'(y)(\psi(x) - \psi(y))^{\mu_i - 1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x) - \psi(y))^{\rho_i}\right] dy =$$

(by [5])

$$\sum_{k_i=0}^{\infty} \frac{(\gamma_i)_{k_i}\,\omega_i^{k_i}}{k_i!\Gamma(\rho_i k_i + \mu_i)} \int_a^x \psi'(y)(\psi(x) - \psi(y))^{(\rho_i k_i + \mu_i) - 1} dy =$$

$$\sum_{k_i=0}^{\infty} \frac{(\gamma_i)_{k_i}}{k_i!\Gamma(\rho_i k_i + \mu_i)} \frac{\omega_i^{k_i}(\psi(x) - \psi(a))^{(\rho_i k_i + \mu_i)}}{(\rho_i k_i + \mu_i)} = \tag{96}$$

$$(\psi(x) - \psi(a))^{\mu_i} \sum_{k_i=0}^{\infty} \frac{(\gamma_i)_{k_i}}{k_i!\Gamma(\rho_i k_i + \mu_i + 1)} \left(\omega_i(\psi(x) - \psi(a))^{\rho_i}\right)^{k_i} =$$

$$(\psi(x) - \psi(a))^{\mu_i} E^{\gamma_i}_{\rho_i,\mu_i+1}\left[\omega_i(\psi(x) - \psi(a))^{\rho_i}\right].$$

That is

$$K_i(x) = (\psi(x) - \psi(a))^{\mu_i} E^{\gamma_i}_{\rho_i,\mu_i+1}\left[\omega_i(\psi(x) - \psi(a))^{\rho_i}\right], \tag{97}$$

$\forall\, x \in [a,b]$, $i = 1,...,m$.

Notice that

$$\frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{(a,x]}(y)\,\psi'(y)\,(\psi(x)-\psi(y))^{\mu_i-1}\,E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x)-\psi(y))^{\rho_i}\right]}{(\psi(x)-\psi(a))^{\mu_i}\,E^{\gamma_i}_{\rho_i,\mu_i+1}\left[\omega_i(\psi(x)-\psi(a))^{\rho_i}\right]} \tag{98}$$

$$= \left(\chi_{(a,x]}(y)\,(\psi'(y))\,\frac{(\psi(x)-\psi(y))^{\mu_i-1}}{(\psi(x)-\psi(a))^{\mu_i}}\right)$$

$$\left(\frac{E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x)-\psi(y))^{\rho_i}\right]}{E^{\gamma_i}_{\rho_i,\mu_i+1}\left[\omega_i(\psi(x)-\psi(a))^{\rho_i}\right]}\right),$$

$\forall\ x,y\in[a,b]$.

Therefore for (55), we get for appropiate weight $u\geq 0$ that the Lebesgue integrable

$$u^{\psi}_{i+}(y) = \psi'(y)\int_y^b u(x)\left(\frac{(\psi(x)-\psi(y))^{\mu_i-1}}{(\psi(x)-\psi(a))^{\mu_i}}\right)$$

$$\left(\frac{E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x)-\psi(y))^{\rho_i}\right]}{E^{\gamma_i}_{\rho_i,\mu_i+1}\left[\omega_i(\psi(x)-\psi(a))^{\rho_i}\right]}\right)dx<\infty, \tag{99}$$

for all $a<y<b$.

Based on Theorem 2.1 and the above, we have established the following generalized Prabhakar left fractional Hardy type inequality:

Theorem 3.8. *Here $i=1,...,m$. Let $\rho_i,\mu_i,\gamma_i,\omega_i>0$, and $f_i\in C([a,b])$, with $\psi\in C^1([a,b])$ which is increasing. The function $u^{\psi}_{i+}(y)\in\mathbb{R}$ by assumption, $\forall$ $y\in[a,b]$, is given by (99). Here $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,...,m$, are convex and increasing functions. Then*

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x)\right|}{(\psi(x)-\psi(a))^{\mu_i}\,E^{\gamma_i}_{\rho_i,\mu_i+1}\left[\omega_i(\psi(x)-\psi(a))^{\rho_i}\right]}\right)dx\leq$$

$$\prod_{i=1}^m\left(\int_a^b u^{\psi}_{i+}(u)\,\Phi_i(|f_i(y)|)^{p_i}\,dy\right)^{\frac{1}{p_i}}. \tag{100}$$

We make

Remark 3.3. We pick $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, the Lebesgue measure. Here $i=1,...,m$. Let $\rho_i,\mu_i,\gamma_i,\omega_i>0$, and $f_i\in C([a,b])$, with $\psi\in C^1([a,b])$ which is increasing.

We have that ($x\in[a,b]$):

$$\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_i\right)(x)=$$

$$\int_x^b \psi'(t)\,(\psi(t)-\psi(x))^{\mu_i-1}\,E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(t)-\psi(x))^{\rho_i}\right]f_i(t)\,dt \tag{101}$$

$$= \int_a^b \chi_{[x,b)}(t)\, \psi'(t) (\psi(t) - \psi(x))^{\mu_i - 1} E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(t) - \psi(x))^{\rho_i}] f_i(t)\, dt,$$

where χ is the characteristic function.

So, we choose here

$$k_i(x,t) := \chi_{[x,b)}(t)\, \psi'(t) (\psi(t) - \psi(x))^{\mu_i - 1} E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(t) - \psi(x))^{\rho_i}], \quad (102)$$

$i = 1, ..., m$.

That is

$$k_i(x,y) = \begin{cases} \psi'(y) (\psi(y) - \psi(x))^{\mu_i - 1} E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(y) - \psi(x))^{\rho_i}], & x \leq y < b, \\ 0, \quad a < y < x. \end{cases} \quad (103)$$

$i = 1, ..., m$.

Therefore we obtain

$$K_i(x) = \int_a^b \chi_{[x,b)}(y)\, \psi'(y) (\psi(y) - \psi(x))^{\mu_i - 1} E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(y) - \psi(x))^{\rho_i}]\, dy =$$

$$\int_x^b \psi'(y) (\psi(y) - \psi(x))^{\mu_i - 1} E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(y) - \psi(x))^{\rho_i}]\, dy =$$

(by [5])

$$\sum_{k_i=0}^{\infty} \frac{(\gamma_i)_{k_i} \omega_i^{k_i}}{k_i! \Gamma(\rho_i k_i + \mu_i)} \int_x^b \psi'(y) (\psi(y) - \psi(x))^{(\rho_i k_i + \mu_i) - 1}\, dy =$$

$$\sum_{k_i=0}^{\infty} \frac{(\gamma_i)_{k_i}}{k_i! \Gamma(\rho_i k_i + \mu_i)} \frac{\omega_i^{k_i} (\psi(b) - \psi(x))^{(\rho_i k_i + \mu_i)}}{(\rho_i k_i + \mu_i)} = \quad (104)$$

$$(\psi(b) - \psi(x))^{\mu_i} \sum_{k_i=0}^{\infty} \frac{(\gamma_i)_{k_i}}{k_i! \Gamma(\rho_i k_i + \mu_i + 1)} (\omega_i (\psi(b) - \psi(x))^{\rho_i})^{k_i} =$$

$$(\psi(b) - \psi(x))^{\mu_i} E_{\rho_i,\mu_i+1}^{\gamma_i} [\omega_i (\psi(b) - \psi(x))^{\rho_i}].$$

That is

$$K_i(x) = (\psi(b) - \psi(x))^{\mu_i} E_{\rho_i,\mu_i+1}^{\gamma_i} [\omega_i (\psi(b) - \psi(x))^{\rho_i}], \quad (105)$$

$\forall\, x \in [a,b]$, $i = 1, ..., m$.

Notice that

$$\frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{[x,b)}(y)\, \psi'(y) (\psi(y) - \psi(x))^{\mu_i - 1} E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(y) - \psi(x))^{\rho_i}]}{(\psi(b) - \psi(x))^{\mu_i} E_{\rho_i,\mu_i+1}^{\gamma_i} [\omega_i (\psi(b) - \psi(x))^{\rho_i}]}$$

$$= \left(\chi_{[x,b)}(y) (\psi'(y)) \frac{(\psi(y) - \psi(x))^{\mu_i - 1}}{(\psi(b) - \psi(x))^{\mu_i}} \right)$$

$$\left(\frac{E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(y) - \psi(x))^{\rho_i}]}{E_{\rho_i,\mu_i+1}^{\gamma_i} [\omega_i (\psi(b) - \psi(x))^{\rho_i}]} \right), \quad (106)$$

$\forall\, x, y \in [a,b]$.

Therefore for (55), we get for appropiate weight $u \geq 0$ that the Lebesgue integrable:

$$u_{i-}^{\psi}(y) = \psi'(y) \int_a^y u(x) \left(\frac{(\psi(y) - \psi(x))^{\mu_i - 1}}{(\psi(b) - \psi(x))^{\mu_i}} \right) \left(\frac{E_{\rho_i,\mu_i}^{\gamma_i}\left[\omega_i (\psi(y) - \psi(x))^{\rho_i}\right]}{E_{\rho_i,\mu_i+1}^{\gamma_i}\left[\omega_i (\psi(b) - \psi(x))^{\rho_i}\right]} \right) dx < \infty, \tag{107}$$

for all $a < y < b$.

Based on Theorem 2.1 and the above, we have established the following generalized Prabhakar right fractional Hardy type inequality:

Theorem 3.9. *Here $i = 1, ..., m$. Let $\rho_i, \mu_i, \gamma_i, \omega_i > 0$, and $f_i \in C([a,b])$, with $\psi \in C^1([a,b])$ which is increasing. The function $u_{i-}^{\psi}(y) \in \mathbb{R}$ by assumption, $\forall$ $y \in [a,b]$, is given by (107). Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions. Then*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| \left(e_{\rho_i,\mu_i,\omega_i,b-}^{\gamma_i;\psi} f_i \right)(x) \right|}{(\psi(b) - \psi(x))^{\mu_i} E_{\rho_i,\mu_i+1}^{\gamma_i}\left[\omega_i (\psi(b) - \psi(x))^{\rho_i}\right]} \right) dx \leq \prod_{i=1}^m \left(\int_a^b u_{i-}^{\psi}(y) \Phi_i (|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{108}$$

We continue with left and right ψ-Prabhakar-Caputo Hardy fractional inequalities:

Theorem 3.10. *Here $i = 1, ..., m$. Let $\rho_i, \mu_i, \omega_i > 0$, $\gamma_i < 0$, and $f_i \in C^{N_i}([a,b])$, $N_i = \lceil \mu_i \rceil$, $\mu_i \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x) \neq 0$ over $[a,b]$. Set $f_{i\psi}^{[N_i]}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{N_i} f_i(x)$, $x \in [a,b]$. We assume that the weight function $u \geq 0$ is such that*

$$^{C}\lambda_{i+}^{\psi}(y) := \psi'(y) \int_y^b u(x) \left(\frac{(\psi(x) - \psi(y))^{(N_i - \mu_i) - 1}}{(\psi(x) - \psi(a))^{(N_i - \mu_i)}} \right) \left(\frac{E_{\rho_i,N_i-\mu_i}^{-\gamma_i}\left[\omega_i (\psi(x) - \psi(y))^{\rho_i}\right]}{E_{\rho_i,N_i-\mu_i+1}^{-\gamma_i}\left[\omega_i (\psi(x) - \psi(a))^{\rho_i}\right]} \right) dx < \infty, \tag{109}$$

for all $a < y < b$, which is a Lebesgue integrable function.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| \left(^{C}D_{\rho_i,\mu_i,\omega_i,a+}^{\gamma_i;\psi} f_i \right)(x) \right|}{(\psi(x) - \psi(a))^{N_i - \mu_i} E_{\rho_i,N_i-\mu_i+1}^{-\gamma_i}\left[\omega_i (\psi(x) - \psi(a))^{\rho_i}\right]} \right) dx \leq \prod_{i=1}^m \left(\int_a^b {^{C}\lambda_{i+}^{\psi}}(y) \Phi_i \left(\left| f_{i\psi}^{[N_i]}(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{110}$$

Proof. By (42) we have that

$$\left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x) = \left(e^{-\gamma_i;\psi}_{\rho_i,N_i-\mu_i,\omega_i,a+}f^{[N_i]}_{i\psi}\right)(x), \tag{111}$$

$\forall\ x \in [a,b]$, $i = 1, ..., m$.

We apply Theorem 3.8. □

Theorem 3.11. *Here $i = 1, ..., m$. Let $\rho_i, \mu_i, \omega_i > 0$, $\gamma_i < 0$, and $f_i \in C^{N_i}([a,b])$, $N_i = \lceil \mu_i \rceil$, $\mu_i \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x) \neq 0$ over $[a,b]$. Set $f^{[N_i]}_{i\psi}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_i} f_i(x)$, $x \in [a,b]$. We assume that the weight function $u \geq 0$ is such that*

$$^{C}\lambda^{\psi}_{i-}(y) := \psi'(y)\int_a^y u(x)\left(\frac{(\psi(y)-\psi(x))^{(N_i-\mu_i)-1}}{(\psi(b)-\psi(x))^{(N_i-\mu_i)}}\right)$$

$$\left(\frac{E^{-\gamma_i}_{\rho_i,N_i-\mu_i}\left[\omega_i(\psi(y)-\psi(x))^{\rho_i}\right]}{E^{-\gamma_i}_{\rho_i,N_i-\mu_i+1}\left[\omega_i(\psi(b)-\psi(x))^{\rho_i}\right]}\right)dx < \infty, \tag{112}$$

for all $a < y < b$, which is integrable.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions. Then

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|\left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_i\right)(x)\right|}{(\psi(b)-\psi(x))^{N_i-\mu_i}E^{-\gamma_i}_{\rho_i,N_i-\mu_i+1}\left[\omega_i(\psi(b)-\psi(x))^{\rho_i}\right]}\right)dx \leq$$

$$\prod_{i=1}^m\left(\int_a^b {}^{C}\lambda^{\psi}_{i-}(y)\,\Phi_i\left(\left|f^{[N_i]}_{i\psi}(y)\right|\right)^{p_i}dy\right)^{\frac{1}{p_i}}. \tag{113}$$

Proof. By (43) we have that

$$\left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_i\right)(x) = (-1)^{N_i}\left(e^{-\gamma_i;\psi}_{\rho_i,N_i-\mu_i,\omega_i,b-}f^{[N_i]}_{i\psi}\right)(x), \tag{114}$$

$\forall\ x \in [a,b]$, $i = 1, ..., m$.

We apply Theorem 3.9. □

Next we present left and right ψ-Hilfer-Prabhakar Hardy fractional inequalities:

Theorem 3.12. *Here $i = 1, ..., m$. Let $\rho_i, \mu_i, \omega_i > 0$, $\gamma_i < 0$, and $f_i \in C^{N_i}([a,b])$, $N_i = \lceil \mu_i \rceil$, $\mu_i \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x) \neq 0$ over $[a,b]$. Here $0 \leq \beta_i \leq 1$ and $\xi_i = \mu_i + \beta_i(N_i - \mu_i)$. We assume that ${}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+}f_i \in C([a,b])$, $i = 1, ..., m$. We assume further that the weight function $u \geq 0$ is such that*

$$^{P}\lambda^{\psi}_{i+}(y) := \psi'(y)\int_y^b u(x)\left(\frac{(\psi(x)-\psi(y))^{(\xi_i-\mu_i)-1}}{(\psi(x)-\psi(a))^{(\xi_i-\mu_i)}}\right)$$

$$\left(\frac{E^{-\gamma_i \beta_i}_{\rho_i, \xi_i - \mu_i} \left[\omega_i \left(\psi(x) - \psi(y) \right)^{\rho_i} \right]}{E^{-\gamma_i \beta_i}_{\rho_i, \xi_i - \mu_i + 1} \left[\omega_i \left(\psi(x) - \psi(a) \right)^{\rho_i} \right]} \right) dx < \infty, \tag{115}$$

for all $a < y < b$, which is integrable.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| \left({}^H \mathbb{D}^{\gamma_i, \beta_i; \psi}_{\rho_i, \mu_i, \omega_i, a+} f_i \right)(x) \right|}{\left(\psi(x) - \psi(a) \right)^{\xi_i - \mu_i} E^{-\gamma_i \beta_i}_{\rho_i, \xi_i - \mu_i + 1} \left[\omega_i \left(\psi(x) - \psi(a) \right)^{\rho_i} \right]} \right) dx \le$$

$$\prod_{i=1}^m \left(\int_a^b {}^P\lambda^{\psi}_{i+}(y) \, \Phi_i \left(\left| {}^{RL} D^{\gamma_i(1-\beta_i); \psi}_{\rho_i, \xi_i, \omega_i, a+} f_i(y) \right| \right)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{116}$$

Proof. By (51) we have that

$$\left({}^H \mathbb{D}^{\gamma_i, \beta_i; \psi}_{\rho_i, \mu_i, \omega_i, a+} f_i \right)(x) = e^{-\gamma_i \beta_i; \psi}_{\rho_i, \xi_i - \mu_i, \omega_i, a+} \; {}^{RL} D^{\gamma_i(1-\beta_i); \psi}_{\rho_i, \xi_i, \omega_i, a+} f_i(x), \tag{117}$$

$\forall \; x \in [a, b]$, $i = 1, ..., m$.

We apply Theorem 3.8. □

Theorem 3.13. *Here $i = 1, ..., m$. Let $\rho_i, \mu_i, \omega_i > 0$, $\gamma_i < 0$, and $f_i \in C^{N_i}([a, b])$, $N_i = \lceil \mu_i \rceil$, $\mu_i \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^{\theta}([a, b])$, ψ is increasing with $\psi'(x) \ne 0$ over $[a, b]$. Here $0 \le \beta_i \le 1$ and $\xi_i = \mu_i + \beta_i(N_i - \mu_i)$. We assume that ${}^{RL} D^{\gamma_i(1-\beta_i); \psi}_{\rho_i, \xi_i, \omega_i, b-} f_i \in C([a, b])$, $i = 1, ..., m$. We assume further that the weight function $u \ge 0$ is such that*

$${}^P\lambda^{\psi}_{i-}(y) := \psi'(y) \int_a^y u(x) \left(\frac{\left(\psi(y) - \psi(x) \right)^{(\xi_i - \mu_i) - 1}}{\left(\psi(b) - \psi(x) \right)^{(\xi_i - \mu_i)}} \right)$$

$$\left(\frac{E^{-\gamma_i \beta_i}_{\rho_i, \xi_i - \mu_i} \left[\omega_i \left(\psi(y) - \psi(x) \right)^{\rho_i} \right]}{E^{-\gamma_i \beta_i}_{\rho_i, \xi_i - \mu_i + 1} \left[\omega_i \left(\psi(b) - \psi(x) \right)^{\rho_i} \right]} \right) dx < \infty, \tag{118}$$

for all $a < y < b$, which is integrable.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| \left({}^H \mathbb{D}^{\gamma_i, \beta_i; \psi}_{\rho_i, \mu_i, \omega_i, b-} f_i \right)(x) \right|}{\left(\psi(b) - \psi(x) \right)^{\xi_i - \mu_i} E^{-\gamma_i \beta_i}_{\rho_i, \xi_i - \mu_i + 1} \left[\omega_i \left(\psi(b) - \psi(x) \right)^{\rho_i} \right]} \right) dx \le$$

$$\prod_{i=1}^m \left(\int_a^b {}^P\lambda^{\psi}_{i-}(y) \, \Phi_i \left(\left| {}^{RL} D^{\gamma_i(1-\beta_i); \psi}_{\rho_i, \xi_i, \omega_i, b-} f_i(y) \right|^{p_i} \right) dy \right)^{\frac{1}{p_i}}. \tag{119}$$

Proof. By (52) we have that

$$\left({}^H \mathbb{D}^{\gamma_i, \beta_i; \psi}_{\rho_i, \mu_i, \omega_i, b-} f_i \right)(x) = e^{-\gamma_i \beta_i; \psi}_{\rho_i, \xi_i - \mu_i, \omega_i, b-} \; {}^{RL} D^{\gamma_i(1-\beta_i); \psi}_{\rho_i, \xi_i, \omega_i, b-} f_i(x), \tag{120}$$

$\forall \; x \in [a, b]$, $i = 1, ..., m$.

We apply Theorem 3.9. □

II)' Next we apply Theorem 2.2.

We present the following result.

Theorem 3.14. *Here $j = 1, ..., m$. Let $\rho_j, \mu_j, \gamma_j, \omega_j > 0$, and $f_{ji} \in C([a, b])$, $i = 1, ..., n$; with $\psi \in C^1([a, b])$, which is increasing. For appropiate weight $u \geq 0$, we assume that*

$$u_{j+}^{\psi}(y) := \psi'(y) \int_y^b u(x) \left(\frac{(\psi(x) - \psi(y))^{\mu_j - 1}}{(\psi(x) - \psi(a))^{\mu_j}} \right) \left(\frac{E_{\rho_j, \mu_j}^{\gamma_j} [\omega_j (\psi(x) - \psi(y))^{\rho_j}]}{E_{\rho_j, \mu_j + 1}^{\gamma_j} [\omega_j (\psi(x) - \psi(a))^{\rho_j}]} \right) dx < \infty, \tag{121}$$

for all $a < y < b$, which is integrable.

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. Let also the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, be convex and increasing per coordinate. Then

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_j \left(\frac{\left| \left(\overrightarrow{e_{\rho_j, \mu_j, \omega_j, a+}^{\gamma_j : \psi} f_j} \right)(x) \right|}{(\psi(x) - \psi(a))^{\mu_j} E_{\rho_j, \mu_j + 1}^{\gamma_j} [\omega_j (\psi(x) - \psi(a))^{\rho_j}]} \right) dx \leq$$

$$\prod_{j=1}^{m} \left(\int_a^b u_{j+}^{\psi}(y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right)^{p_j} dy \right)^{\frac{1}{p_j}}. \tag{122}$$

Proof. By Theorem 2.2, see also Remark 3.2. □

We continue with

Theorem 3.15. *Here $j = 1, ..., m$. Let $\rho_j, \mu_j, \gamma_j, \omega_j > 0$, and $f_{ji} \in C([a, b])$, $i = 1, ..., n$; with $\psi \in C^1([a, b])$, which is increasing. For appropiate weight $u \geq 0$, we assume that*

$$u_{j-}^{\psi}(y) := \psi'(y) \int_a^y u(x) \left(\frac{(\psi(y) - \psi(x))^{\mu_j - 1}}{(\psi(b) - \psi(x))^{\mu_j}} \right) \left(\frac{E_{\rho_j, \mu_j}^{\gamma_j} [\omega_j (\psi(y) - \psi(x))^{\rho_j}]}{E_{\rho_j, \mu_j + 1}^{\gamma_j} [\omega_j (\psi(b) - \psi(x))^{\rho_j}]} \right) dx < \infty, \tag{123}$$

for all $a < y < b$, which is integrable.

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. Let also the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, be convex and increasing per coordinate. Then

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_j \left(\frac{\left| \left(\overrightarrow{e_{\rho_j, \mu_j, \omega_j, b-}^{\gamma_j : \psi} f_j} \right)(x) \right|}{(\psi(x) - \psi(a))^{\mu_j} E_{\rho_j, \mu_j + 1}^{\gamma_j} [\omega_j (\psi(b) - \psi(x))^{\rho_j}]} \right) dx \leq$$

$$\prod_{j=1}^{m}\left(\int_{a}^{b} u_{j-}^{\psi}(y)\,\Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right)^{p_j} dy\right)^{\frac{1}{p_j}}. \tag{124}$$

Proof. By Theorem 2.2, see also Remark 3.3. □

We also give

Theorem 3.16. *Here $j=1,...,m$. Let $\rho_j,\mu_j,\omega_j>0$, $\gamma_j<0$, and $f_{ji}\in C^{N_j}([a,b])$, $N_j=\lceil\mu_j\rceil$, $\mu_j\notin\mathbb{N}$; $\theta:=\max(N_1,...,N_m)$, $\psi\in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x)\neq 0$ over $[a,b]$; $i=1,...,n$. Set $f_{ji\psi}^{[N_j]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_j} f_{ji}(x)$, $x\in[a,b]$. We assume that the weight function $u\geq 0$ is such that*

$$ {}^{C}\lambda_{j+}^{\psi}(y):=\psi'(y)\int_{y}^{b} u(x)\left(\frac{(\psi(x)-\psi(y))^{(N_j-\mu_j)-1}}{(\psi(x)-\psi(a))^{(N_j-\mu_j)}}\right)$$
$$\left(\frac{E_{\rho_j,N_j-\mu_j}^{-\gamma_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right]}{E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right)dx<\infty, \tag{125}$$

for all $a<y<b$, which is integrable.

Let $p_j>1:\sum_{j=1}^{m}\frac{1}{p_j}=1$. Let also the functions $\Phi_j:\mathbb{R}_+^n\to\mathbb{R}_+$, $j=1,...,m$, be convex and increasing per coordinate. Then

$$\int_{a}^{b} u(x)\prod_{i=1}^{m}\Phi_i\left(\frac{\left|\left(\overrightarrow{{}^{C}D_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi} f_j}\right)(x)\right|}{(\psi(x)-\psi(a))^{N_j-\mu_j} E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right)dx\leq$$
$$\prod_{j=1}^{m}\left(\int_{a}^{b} {}^{C}\lambda_{j+}^{\psi}(y)\,\Phi_j\left(\left|\overrightarrow{f_{j\psi}^{[N_j]}}(y)\right|\right)^{p_j} dy\right)^{\frac{1}{p_j}}. \tag{126}$$

Proof. By Theorem 3.14 and (42), see also (111). □

We continue with

Theorem 3.17. *Here $j=1,...,m$. Let $\rho_j,\mu_j,\omega_j>0$, $\gamma_j<0$, and $f_{ji}\in C^{N_j}([a,b])$, $N_j=\lceil\mu_j\rceil$, $\mu_j\notin\mathbb{N}$; $\theta:=\max(N_1,...,N_m)$, $\psi\in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x)\neq 0$ over $[a,b]$; $i=1,...,n$. Set $f_{ji\psi}^{[N_j]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_j} f_{ji}(x)$, $x\in[a,b]$. We assume that the weight function $u\geq 0$ is such that*

$$ {}^{C}\lambda_{j-}^{\psi}(y):=\psi'(y)\int_{a}^{y} u(x)\left(\frac{(\psi(y)-\psi(x))^{(N_j-\mu_j)-1}}{(\psi(b)-\psi(x))^{(N_j-\mu_j)}}\right)$$
$$\left(\frac{E_{\rho_j,N_j-\mu_j}^{-\gamma_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]}{E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j}\left[\omega_j(\psi(b)-\psi(x))^{\rho_j}\right]}\right)dx<\infty, \tag{127}$$

for all $a<y<b$, which is integrable.

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. *Let also the functions* $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, *be convex and increasing per coordinate. Then*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\frac{\left| \left(\overrightarrow{{}^C D_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j} \right)(x) \right|}{(\psi(b) - \psi(x))^{N_j - \mu_j} E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j} \left[\omega_j (\psi(b) - \psi(x))^{\rho_j} \right]} \right) dx \leq$$

$$\prod_{j=1}^{m} \left(\int_a^b {}^C\lambda_{j-}^{\psi}(y) \Phi_j \left(\left| \overrightarrow{f_{j\psi}^{[N_j]}}(y) \right| \right)^{p_j} dy \right)^{\frac{1}{p_j}}. \tag{128}$$

Proof. By Theorem 3.15 and (43), see also (114). □

We continue with

Theorem 3.18. *Here* $j = 1, ..., m$. *Let* $\rho_j, \mu_j, \omega_j > 0$, $\gamma_j < 0$, *and* $f_{ji} \in C^{N_j}([a,b])$, $N_j = \lceil \mu_j \rceil$, $\mu_j \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^{\theta}([a,b])$, ψ *is increasing with* $\psi'(x) \neq 0$ *over* $[a,b]$; $i = 1, ..., n$. *Here* $0 \leq \beta_j \leq 1$ *and* $\xi_j = \mu_j + \beta_j (N_j - \mu_j)$. *We assume that* ${}^{RL}D_{\rho_j,\xi_j,\omega_j,a+}^{\gamma_j(1-\beta_j);\psi} f_{ji} \in C([a,b])$, $j = 1, ..., m$ *and* $i = 1, ..., n$. *We assume further that the weight function* $u \geq 0$ *is such that*

$${}^P\lambda_{j+}^{\psi}(y) := \psi'(y) \int_y^b u(x) \left(\frac{(\psi(x) - \psi(y))^{(\xi_j - \mu_j) - 1}}{(\psi(x) - \psi(a))^{(\xi_j - \mu_j)}} \right)$$

$$\left(\frac{E_{\rho_j,\xi_j-\mu_j}^{-\gamma_j\beta_j} \left[\omega_j (\psi(x) - \psi(y))^{\rho_j} \right]}{E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j\beta_j} \left[\omega_j (\psi(x) - \psi(a))^{\rho_j} \right]} \right) dx < \infty, \tag{129}$$

for all $a < y < b$, *which is integrable.*

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. *Let also the functions* $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, *be convex and increasing per coordinate. Then*

$$\int_a^b u(x) \prod_{j=1}^{m} \Phi_j \left(\frac{\left| \left(\overrightarrow{{}^H \mathbb{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_j} \right)(x) \right|}{(\psi(x) - \psi(a))^{\xi_j - \mu_j} E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j\beta_j} \left[\omega_j (\psi(x) - \psi(a))^{\rho_j} \right]} \right) dx \leq$$

$$\prod_{i=1}^{m} \left(\int_a^b {}^P\lambda_{j+}^{\psi}(y) \Phi_j \left(\left| \left(\overrightarrow{{}^{RL}D_{\rho_j,\xi_j,\omega_j,a+}^{\gamma_j(1-\beta_j);\psi} f_j} \right)(y) \right| \right)^{p_j} dy \right)^{\frac{1}{p_j}}. \tag{130}$$

Proof. By Theorem 3.14 and (51), see also (117). □

The counter part of the last theorem follows:

Theorem 3.19. *Here $j=1,...,m$. Let $\rho_j,\mu_j,\omega_j>0$, $\gamma_j<0$, and $f_{ji}\in C^{N_j}([a,b])$, $N_j=\lceil\mu_j\rceil$, $\mu_j\notin\mathbb{N}$; $\theta:=\max(N_1,...,N_m)$, $\psi\in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x)\neq 0$ over $[a,b]$; $i=1,...,n$. Here $0\leq\beta_j\leq 1$ and $\xi_j=\mu_j+\beta_j(N_j-\mu_j)$. We assume that ${}^{RL}D^{\gamma_j(1-\beta_j);\psi}_{\rho_j,\xi_j,\omega_j,b-}f_{ji}\in C([a,b])$, $j=1,...,m$ and $i=1,...,n$. We assume further that the weight function $u\geq 0$ is such that*

$$ {}^{P}\lambda^{\psi}_{j-}(y):=\psi'(y)\int_a^y u(x)\left(\frac{(\psi(y)-\psi(x))^{(\xi_j-\mu_j)-1}}{(\psi(b)-\psi(x))^{(\xi_j-\mu_j)}}\right) $$

$$ \left(\frac{E^{-\gamma_j\beta_j}_{\rho_j,\xi_j-\mu_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]}{E^{-\gamma_j\beta_j}_{\rho_j,\xi_j-\mu_j+1}\left[\omega_j(\psi(b)-\psi(x))^{\rho_j}\right]}\right)dx<\infty, \tag{131} $$

for all $a<y<b$, which is integrable.

Let $p_j>1:\sum_{j=1}^{m}\frac{1}{p_j}=1$. Let also the functions $\Phi_j:\mathbb{R}^n_+\to\mathbb{R}_+$, $j=1,...,m$, be convex and increasing per coordinate. Then

$$ \int_a^b u(x)\prod_{j=1}^m\Phi_j\left(\frac{\left|\left(\overrightarrow{{}^{H}\mathbb{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b-}f_j}\right)(x)\right|}{(\psi(b)-\psi(x))^{\xi_j-\mu_j}E^{-\gamma_j\beta_j}_{\rho_j,\xi_j-\mu_j+1}\left[\omega_j(\psi(b)-\psi(x))^{\rho_j}\right]}\right)dx\leq $$

$$ \prod_{i=1}^m\left(\int_a^b {}^{P}\lambda^{\psi}_{j-}(y)\,\Phi_j\left(\left|\left(\overrightarrow{{}^{RL}D^{\gamma_j(1-\beta_j);\psi}_{\rho_j,\xi_j,\omega_j,b-}f_j}\right)(y)\right|^{p_j}\right)dy\right)^{\frac{1}{p_j}}. \tag{132} $$

Proof. By Theorem 3.15 and (52), see also (120). □

III)' Here we apply Theorem 2.3.

Based on (69) and Remark 3.2, we get for appropiate weight $u\geq 0$ that (denote this particular λ_m by $\overline{\lambda}^{\psi}_{m+}$) the integrable function:

$$ \overline{\lambda}^{\psi}_{m+}(y)=(\psi'(y))^m\int_y^b u(x)\left(\frac{(\psi(x)-\psi(y))^{\sum_{j=1}^m\mu_j-m}}{(\psi(x)-\psi(a))^{\sum_{j=1}^m\mu_j}}\right) $$

$$ \prod_{j=1}^m\left(\frac{E^{\gamma_j}_{\rho_j,\mu_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho j}\right]}{E^{\gamma_j}_{\rho_j,\mu_j+1}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right)dx<\infty, \tag{133} $$

for all $a<y<b$.

By Theorem 2.3 and the above, we have established the following multivariate generalized Prabhakar left fractional Hardy type inequality:

Theorem 3.20. *Here* $j = 1, ..., m$. *Let* $\rho_j, \mu_j, \gamma_j, \omega_j > 0$, *and* $f_{ji} \in C([a,b])$, $i = 1, ..., n$, *with* $\psi \in C^1([a,b])$ *which is increasing. The function* $\overline{\lambda}_{m+}^{\psi}(y) \in \mathbb{R}$ *by assumption,* $\forall\, y \in [a,b]$, *is given by (133). Here* $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, *are convex and increasing per coordinate functions. Then*

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\left| \left(\overrightarrow{e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi} f_j} \right)(x) \right|}{(\psi(x) - \psi(a))^{\mu_j} E_{\rho_j,\mu_j+1}^{\gamma_j} \left[\omega_j (\psi(x) - \psi(a))^{\rho_j} \right]} \right) dx \leq$$

$$\left(\prod_{\substack{j=1 \\ j \neq \rho}}^m \int_a^b \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) dy \right) \left(\int_a^b \Phi_\rho \left(\left| \overrightarrow{f_\rho}(y) \right| \right) \overline{\lambda}_{m+}^{\psi}(y)\, dy \right). \tag{134}$$

Based on (69) and Remark 3.3, we get for appropiate weight $u \geq 0$ that (denote this particular λ_m by $\overline{\lambda}_{m-}^{\psi}$) the integrable function:

$$\overline{\lambda}_{m-}^{\psi}(y) = (\psi'(y))^m \int_a^y u(x) \left(\frac{(\psi(y) - \psi(x))^{\sum_{j=1}^m \mu_j - m}}{(\psi(b) - \psi(x))^{\sum_{j=1}^m \mu_j}} \right)$$

$$\prod_{j=1}^m \left(\frac{E_{\rho_j,\mu_j}^{\gamma_j} \left[\omega_j (\psi(y) - \psi(x))^{\rho j} \right]}{E_{\rho_j,\mu_j+1}^{\gamma_j} \left[\omega_j (\psi(b) - \psi(x))^{\rho_j} \right]} \right) dx < \infty, \tag{135}$$

for all $a < y < b$.

By Theorem 2.3 and the above, we have established the following multivariate generalized Prabhakar right fractional Hardy type inequality:

Theorem 3.21. *Here* $j = 1, ..., m$; *i=1,...,n. Let* $\rho_j, \mu_j, \gamma_j, \omega_j > 0$, *and* $f_{ji} \in C([a,b])$, $i = 1, ..., n$, *with* $\psi \in C^1([a,b])$ *which is increasing. The function* $\overline{\lambda}_{m-}^{\psi}(y) \in \mathbb{R}$ *by assumption,* $\forall\, y \in [a,b]$, *is given by (135). Here* $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, *are convex and increasing per coordinate functions. Then*

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\left| \left(\overrightarrow{e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j} \right)(x) \right|}{(\psi(b) - \psi(x))^{\mu_j} E_{\rho_j,\mu_j+1}^{\gamma_j} \left[\omega_j (\psi(b) - \psi(x))^{\rho_j} \right]} \right) dx \leq$$

$$\left(\prod_{\substack{j=1 \\ j \neq \rho}}^m \int_a^b \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) dy \right) \left(\int_a^b \Phi_\rho \left(\left| \overrightarrow{f_\rho}(y) \right| \right) \overline{\lambda}_{m-}^{\psi}(y)\, dy \right). \tag{136}$$

We continue with multivariate left and right ψ-Prabhakar-Caputo Hardy fractional inequalities:

Theorem 3.22. *Here $j=1,...,m$; $i=1,...,n$. Let $\rho_j,\mu_j,\omega_j>0$, $\gamma_j<0$, and $f_{ji}\in C^{N_j}([a,b])$, $N_j=\lceil\mu_j\rceil$, $\mu_j\notin\mathbb{N}$; $\theta:=\max(N_1,...,N_m)$, $\psi\in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x)\neq 0$ over $[a,b]$. Set $f_{ji\psi}^{[N_j]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_i}f_{ji}(x)$, $x\in[a,b]$. We assume that the weight function $u\geq 0$ is such that*

$$^{C}\lambda_{m+}^{\psi}(y):=(\psi'(y))^{m}\int_{y}^{b}u(x)\left(\frac{(\psi(x)-\psi(y))^{\sum_{j=1}^{m}(N_j-\mu_j)-m}}{(\psi(x)-\psi(a))^{\sum_{j=1}^{m}(N_j-\mu_j)}}\right)$$

$$\prod_{j=1}^{m}\left(\frac{E_{\rho_j,N_j-\mu_j}^{-\gamma_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right]}{E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right)dx<\infty, \tag{137}$$

for all $a<y<b$, which is integrable.

Here $\Phi_j:\mathbb{R}_+^n\to\mathbb{R}_+$, $j=1,...,m$, are convex and increasing per coordinate functions. Then

$$\int_a^b u(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|\left(\overrightarrow{^{C}D_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_j}\right)(x)\right|}{(\psi(x)-\psi(a))^{N_j-\mu_j}E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right)dx\leq$$

$$\left(\prod_{\substack{j=1\\ j\neq\rho}}^{m}\int_a^b\Phi_j\left(\left|\overrightarrow{f_{j\psi}^{[N_j]}}(y)\right|\right)dy\right)\left(\int_a^b\Phi_\rho\left(\left|\overrightarrow{f_{\rho\psi}^{[N_\rho]}(y)}\right|\right)\ {}^{C}\lambda_{m+}^{\psi}(y)\,dy\right). \tag{138}$$

Proof. By (42) we have that

$$\left(^{C}D_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_{ji}\right)(x)=\left(e_{\rho_j,N_j-\mu_j,\omega_j,a+}^{-\gamma_j;\psi}f_{ji\psi}^{[N_j]}\right)(x), \tag{139}$$

$\forall\ x\in[a,b]$, $j=1,...,m$, $i=1,...,n$.

We apply Theorem 3.20. ☐

Theorem 3.23. *Here $j=1,...,m$; $i=1,...,n$. Let $\rho_j,\mu_j,\omega_j>0$, $\gamma_j<0$, and $f_{ji}\in C^{N_j}([a,b])$, $N_j=\lceil\mu_j\rceil$, $\mu_j\notin\mathbb{N}$; $\theta:=\max(N_1,...,N_m)$, $\psi\in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x)\neq 0$ over $[a,b]$. Set $f_{ji\psi}^{[N_j]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_i}f_{ji}(x)$, $x\in[a,b]$. We assume that the weight function $u\geq 0$ is such that*

$$^{C}\lambda_{m-}^{\psi}(y):=(\psi'(y))^{m}\int_{a}^{y}u(x)\left(\frac{(\psi(y)-\psi(x))^{\sum_{j=1}^{m}(N_j-\mu_j)-m}}{(\psi(b)-\psi(x))^{\sum_{j=1}^{m}(N_j-\mu_j)}}\right)$$

$$\prod_{j=1}^{m}\left(\frac{E_{\rho_j,N_j-\mu_j}^{-\gamma_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]}{E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j}\left[\omega_j(\psi(b)-\psi(x))^{\rho_j}\right]}\right)dx<\infty, \tag{140}$$

for all $a<y<b$, which is integrable.

Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, *are convex and increasing per coordinate functions. Then*

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\left| \left(\overrightarrow{{}^C D_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j} \right)(x) \right|}{(\psi(b) - \psi(x))^{N_j-\mu_j} E_{\rho_j,N_j-\mu_j+1}^{-\gamma_j} \left[\omega_j (\psi(b) - \psi(x))^{\rho_j} \right]} \right) dx \leq$$

$$\left(\prod_{\substack{j=1 \\ j \neq \rho}}^m \int_a^b \Phi_j \left(\left| \overrightarrow{f_{j\psi}^{[N_j]}}(y) \right| \right) dy \right) \left(\int_a^b \Phi_\rho \left(\left| \overrightarrow{f_{\rho\psi}^{[N_\rho]}(y)} \right| \right) \ {}^C\lambda_{m-}^{\psi}(y)\, dy \right). \quad (141)$$

Proof. By (43) we have that

$$\left({}^C D_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_{ji} \right)(x) = (-1)^{N_j} \left(e_{\rho_j,N_j-\mu_j,\omega_j,b-}^{-\gamma_j;\psi} f_{ji\psi}^{[N_j]} \right)(x), \quad (142)$$

$\forall\ x \in [a,b]$, $j = 1, ..., m$, $i = 1, ..., n$..

We apply Theorem 3.21. □

Next we present multivariate left and right ψ-Hilfer-Prabhakar Hardy fractional inequalities:

Theorem 3.24. *Here* $j = 1, ..., m$, $i = 1, ..., n$. *Let* $\rho_j, \mu_j, \omega_j > 0$, $\gamma_j < 0$, *and* $f_{ji} \in C^{N_j}([a,b])$, $N_j = \lceil \mu_j \rceil$, $\mu_j \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^\theta([a,b])$, ψ *is increasing with* $\psi'(x) \neq 0$ *over* $[a,b]$. *Here* $0 \leq \beta_j \leq 1$ *and* $\xi_j = \mu_j + \beta_j (N_j - \mu_j)$. *We assume that* ${}^{RL}D_{\rho_j,\xi_j,\omega_j,a+}^{\gamma_j(1-\beta_j);\psi} f_{ji} \in C([a,b])$, $j = 1, ..., m$, $i = 1, ..., n$. *We assume further that the weight function* $u \geq 0$ *is such that*

$${}^P\lambda_{m+}^{\psi}(y) := (\psi'(y))^m \int_y^b u(x) \left(\frac{(\psi(x) - \psi(y))^{\sum_{j=1}^m (\xi_j-\mu_j)-m}}{(\psi(x) - \psi(a))^{\sum_{j=1}^m (\xi_j-\mu_j)}} \right)$$

$$\prod_{j=1}^m \left(\frac{E_{\rho_j,\xi_j-\mu_j}^{-\gamma_j\beta_j} \left[\omega_j (\psi(x) - \psi(y))^{\rho_j} \right]}{E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j\beta_j} \left[\omega_j (\psi(x) - \psi(a))^{\rho_j} \right]} \right) dx < \infty, \quad (143)$$

for all $a < y < b$, *which is integrable.*

Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, ..., m$, *are convex and increasing per coordinate functions. Then*

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\left| \overrightarrow{\left({}^H\mathbb{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_j \right)(x)} \right|}{(\psi(x) - \psi(a))^{\xi_j-\mu_j} E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j\beta_j} \left[\omega_j (\psi(x) - \psi(a))^{\rho_j} \right]} \right) dx \leq$$

$$\left(\prod_{\substack{j=1\\j\neq\overline{\rho}}}^{m}\int_{a}^{b}\Phi_{j}\left(\overrightarrow{\left|{}^{RL}D_{\rho_{j},\xi_{j},\omega_{j},a+}^{\gamma_{j}(1-\beta_{j});\psi}f_{j}\left(y\right)\right|}\right)dy\right)$$

$$\left(\int_{a}^{b}\Phi_{\overline{\rho}}\left(\left|\overrightarrow{{}^{RL}D_{\rho_{\overline{\rho}},\xi_{\overline{\rho}},\omega_{\overline{\rho}},a+}^{\gamma_{\overline{\rho}}(1-\beta_{\overline{\rho}});\psi}f_{\overline{\rho}}}\left(y\right)\right|\right)\ {}^{P}\lambda_{m+}^{\psi}\left(y\right)dy\right). \tag{144}$$

Proof. By (51) we have that

$$\left({}^{H}\mathbb{D}_{\rho_{j},\mu_{j},\omega_{j},a+}^{\gamma_{j},\beta_{j};\psi}f_{ji}\right)\left(x\right)=e_{\rho_{j},\xi_{j}-\mu_{j},\omega_{j},a+}^{-\gamma_{j}\beta_{j};\psi}\ {}^{RL}D_{\rho_{j},\xi_{j},\omega_{j},a+}^{\gamma_{j}(1-\beta_{j});\psi}f_{ji}\left(x\right), \tag{145}$$

$\forall\ x\in[a,b]$, $j=1,...,m$, $i=1,...,n$.

We apply Theorem 3.20. □

Theorem 3.25. *Here $j=1,...,m$, $i=1,...,n$. Let $\rho_j,\mu_j,\omega_j>0$, $\gamma_j<0$, and $f_{ji}\in C^{N_j}([a,b])$, $N_j=\lceil\mu_j\rceil$, $\mu_j\notin\mathbb{N}$; $\theta:=\max(N_1,...,N_m)$, $\psi\in C^{\theta}([a,b])$, ψ is increasing with $\psi'(x)\neq 0$ over $[a,b]$. Here $0\leq\beta_j\leq 1$ and $\xi_j=\mu_j+\beta_j(N_j-\mu_j)$. We assume that ${}^{RL}D_{\rho_j,\xi_j,\omega_j,b-}^{\gamma_j(1-\beta_j);\psi}f_{ji}\in C([a,b])$, $j=1,...,m$, $i=1,...,n$. We assume further that the weight function $u\geq 0$ is such that*

$$^{P}\lambda_{m-}^{\psi}\left(y\right):=\left(\psi'\left(y\right)\right)^{m}\int_{a}^{y}u\left(x\right)\left(\frac{\left(\psi\left(y\right)-\psi\left(x\right)\right)^{\sum_{j=1}^{m}(\xi_{j}-\mu_{j})-m}}{\left(\psi\left(b\right)-\psi\left(x\right)\right)^{\sum_{j=1}^{m}(\xi_{j}-\mu_{j})}}\right)$$

$$\prod_{j=1}^{m}\left(\frac{E_{\rho_{j},\xi_{j}-\mu_{j}}^{-\gamma_{j}\beta_{j}}\left[\omega_{j}\left(\psi\left(y\right)-\psi\left(x\right)\right)^{\rho_{j}}\right]}{E_{\rho_{j},\xi_{j}-\mu_{j}+1}^{-\gamma_{j}\beta_{j}}\left[\omega_{j}\left(\psi\left(b\right)-\psi\left(x\right)\right)^{\rho_{j}}\right]}\right)dx<\infty, \tag{146}$$

for all $a<y<b$, which is integrable.

Here $\Phi_j:\mathbb{R}_+^n\to\mathbb{R}_+$, $j=1,...,m$, are convex and increasing per coordinate functions. Then

$$\int_{a}^{b}u\left(x\right)\prod_{j=1}^{m}\Phi_{j}\left(\frac{\left|\left(\overrightarrow{{}^{H}\mathbb{D}_{\rho_{j},\mu_{j},\omega_{j},b-}^{\gamma_{j},\beta_{j};\psi}f_{j}}\right)\left(x\right)\right|}{\left(\psi\left(b\right)-\psi\left(x\right)\right)^{\xi_{j}-\mu_{j}}E_{\rho_{j},\xi_{j}-\mu_{j}+1}^{-\gamma_{j}\beta_{j}}\left[\omega_{j}\left(\psi\left(b\right)-\psi\left(x\right)\right)^{\rho_{j}}\right]}\right)dx\leq$$

$$\left(\prod_{\substack{j=1\\j\neq\overline{\rho}}}^{m}\int_{a}^{b}\Phi_{j}\left(\overrightarrow{\left|{}^{RL}D_{\rho_{j},\xi_{j},\omega_{j},b-}^{\gamma_{j}(1-\beta_{j});\psi}f_{j}\left(y\right)\right|}\right)dy\right)$$

$$\left(\int_{a}^{b}\Phi_{\overline{\rho}}\left(\left|\overrightarrow{{}^{RL}D_{\rho_{\overline{\rho}},\xi_{\overline{\rho}},\omega_{\overline{\rho}},b-}^{\gamma_{\overline{\rho}}(1-\beta_{\overline{\rho}});\psi}f_{\overline{\rho}}}\left(y\right)\right|\right)\ {}^{P}\lambda_{m-}^{\psi}\left(y\right)dy\right). \tag{147}$$

Proof. By (52) we have that

$$\left({}^{H}\mathbb{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_{ji} \right)(x) = e^{-\gamma_j\beta_j;\psi}_{\rho_j,\xi_j-\mu_j,\omega_j,b-} \ {}^{RL}D^{\gamma_j(1-\beta_j);\psi}_{\rho_j,\xi_j,\omega_j,b-} f_{ji}(x), \quad (148)$$

$\forall\ x \in [a,b]$, $j = 1,...,m$, $i = 1,...,n$.

We apply Theorem 3.21. □

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