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STUDY OF ENTIRE AND MEROMORPHIC FUNCTION FOR LINEAR DIFFERENCE-DIFFERENTIAL POLYNOMIALS

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ABSTRACT. We investigate the value distribution of difference-differential polynomials of entire and meromorphic functions, which can be gazed as the Hayman's Conjecture. And also we study the uniqueness and existence for sharing common value of difference-differential polynomials.

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1. Introduction

A meromorphic function always means a non constant function analytic in the whole complex plane. Nevanlinna theory is a part of the theory of meromorphic function which describes the asymptotic distributions of solutions of the equation $f(z) = a$, as a values, we adopt fundamental results and standard notations of the Nevanlinna theory of meromorphic functions as explained in [4],[9],[10],[14], [15] such as the characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$ and so on.

In addition, $S(r, f)$ denotes any quantities that satisfies the condition that $S(r, f) = O(T(r, f))$ as r tends to infinity outside of possible exceptional set of finite logarithmic measure. In this sequel, a meromorphic function $a(z)$ is called a small functions with respect to f if and only if $T[r, a(z)] = O(T(r, f))$ as r tends to infinity outside of a possible exceptional set of finite logarithmic measure. We denotes by $S_f(r)$, the family of all such small meromorphic functions.

We say f and g be two meromorphic functions, which share the value a CM (IM)(in the extended complex plane) provided that

$$f(z) \equiv a$$

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if and only if

$$g(z) \equiv a,$$

CM(IM).

Definition 1.1 ([2]). Let $f(z)$ be meromorphic and c be a non-zero complex constant, we define its shift by $f(z+c)$ and its difference operator by

$$\Delta_c f(z) = f(z+c) - f(z)$$
$$\Delta_{lc} f(z) = f(z+lc) - f(z)$$

where l is a positive integer.

$$\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in N, \ n \geq 2$$

$$\Delta_c^n f(z) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} f(z + \overline{n-k}, \ c)$$

In particular,

$$\Delta_c^n f(z) = \Delta^n f(z).$$

For $c = 1$. We **define Differential - Difference monomial** as

$$M[f] = \prod_{i=0}^{k} \prod_{j=0}^{l} [f^{(j)}(z + c_{ij})]^{n_{ij}}$$

Where c_{ij} are complex constants and n_{ij} are natural numbers, $i = 0, 1, 2, 3, ...k$ and $j = 0, 1, 2, 3, ...l$. Then the degree of $M[f]$ will be the sum of all the powers in the product on the right hand side.

Definition 1.2. Let

$$M_1[f], M_2[f], ...$$

denote the monomials in f and $a_1(z), a_2(z), ...$ be the small meromorphic function including complex numbers then

$$P[f] = P[z, f] = \sum_{j \in \Delta} a_j(z) M_j[f]$$

Where Δ is a finite set of multi-indices, $a_j(z)$ are small functions of $f, M_j[f]$ are differential-difference monomials, will be called a **differential-difference polynomial in** f**,** which is a finite sum of product of f, derivatives of f, their shift and derivatives of its shifts. We define the total degree d of $P[z, f]$ in f as

$$d = \underbrace{Max.}_{j \in \Delta} d_{M_j}.$$

If all the terms in the summation of $P[f]$ have same degrees, then $P[f]$ is known as homogeneous differential-difference polynomial. Usually, we take $P[f]$ such that $T(r, P) \neq S(r, f)$

The Difference Polynomial of degree one is called Linear Difference Polynomial.
For Example:

$$\Delta_c^n f(z)$$

One of the important part of Nevanlinna Theory is Uniqueness Theory of Meromorphic function. Nevanlinna Theory with respect to difference operators have been focused in number of papers recently. Many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts or difference operators.

Five point theorem is one of the most classical results due to Nevanlinna Theory of meromorphic functions. i.e., If two non-constant meromorphic functions f and g share five distinct values ignoring multiplicities (IM) then

$$f(z) = g(z).$$

The best possible number is five. If the number of shared values is decreased, then an additional assumptions on value distribution needs to be introduced in order to obtain uniqueness.

Definition 1.3. Let k be a positive integer and a complex number a. We denoted by $N_k(r, \frac{1}{(f-a)})$, the counting function of a point of f with multiplicity $\leq k$, by $N_k(r, \frac{1}{(f-a)})$, the counting function of a point of f with multiplicity $\geq k$. Set

$$N_k(r, \frac{1}{(f-a)}) = \overline{N}_1(r, \frac{1}{(f-a)}) + \overline{N}_2(r, \frac{1}{(f-a)}) + ... + \overline{N}_k(r, \frac{1}{(f-a)})$$

A finite value 'a' is called the Picard Exceptional Value of f, if $f - a$ has no zeros. The Picard theorem shows that a transcendental entire function has at most one Picard exceptional values, a transcendental meromorphic functions has at most two Picard exceptional values. The Hayman conjecture [4] is that if f is a transcendental meromorphic function and $n \in N$ then $f^n f'$ takes every finite non-zero values infinitely often which means that the Picard exceptional value of $f^n f'$ may only be 0. Laine and Yang [5] has proved this conjecture for shifts and difference operators as follows:

Theorem 1.4 ([5])**.** *Let f be a transcendental entire function with finite order and c be a non-zero complex constant. Then for $n \geq 2, f(z)^n f(z+c)$ assume every non-zero complex value a infinitely often.*
Liu K.et.al. [6] *proved the above result for the meromorphic functions and obtained the following result:*

Theorem 1.5 ([6])**.** *Let f be a transcendental meromorphic function with finite order and c be a non-zero complex constant. Then for $n \geq 6, f(z)^n f(z+c) - a(z)$ has infinitely many zeros.*

Theorem 1.6 ([6]). *Let f be a transcendental meromorphic function with finite order and c be non-zero complex constant. Then for $n \geq 7$, then the difference polynomial $f(z)^n[f(z+c)-f(z)]a(z)$ has infinitely many zeros.*

Theorem 1.7. *Let f be transcendental entire function with finite order and as in definition 1.2, P[f] be a linear difference polynomial defined as*

$$P[f] = c_0 f(z)+c_1 f(z+c)+c_2(f(z+2c))+.......+c_n f(z+nc);\ T(r,[p(f)]) \neq S(r,f),$$

where $c \neq 0$ and c_j, $j = 0,1,2,3......n$, are complex constants then

$$f^l P[f] - a[z],\ \ a(z) \neq 0, \infty$$

has infinitely many zeros provided $l > 2n+1$.

Theorem 1.8. *Let f be transcendental meromorphic function with finite order and as in definition 1.2, P[f] be a linear differences polynomial defined as*

$$P[f] = c_0 f(z)+c_1 f(z+c)+c_2(f(z+2c))+.......+c_n f(z+nc);\ T(r,[p(f^l)]) \neq S(r,f),$$

where $c \neq 0$ and $c_j, j = 0,1,2,3......n$, are complex constants then $f^l P[f] - a[z],\ a(z) \neq 0, \infty$ has infinitely many zeros provided $l > 4n+3$.

2. Main results

For the proof of the main results, we need the following Lemmas:

Lemma 2.1 ([2][3]). *Let f be a non-constant meromorphic function of finite order and c be a non-zero complex constant, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f)$$

for all r outside a possible exceptional set of finite logarithmic measure.

Lemma 2.2 ([1]). *Let c be a non-constant complex constant, and let f be a meromorphic function of finite order then*

$$T(r, f(z+c)) = T(r,f) + S(r,f)$$

$$N(r, f(z+c)) = N(r,f) + S(r,f)$$

$$N(r, 0, f(z+c)) = N(r,0,f) + S(r,f)$$

Lemma 2.3 ([8]). *Let F and G be two non-constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*
i. max $(T(r,F), T(r,G)) \leq N_2(r,0,F) + N_2(r,0,G) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G)$.
ii. $F \equiv G$
iii. $F.G \equiv 1$

Our main results are here

Theorem 2.4. *Let f be transcendental meromorphic function with finite order and as in definition* 1.2*, P[f] be a linear differences polynomial defined as*

$$P[f] = c_0 f(z)+c_1 f(z+c)+c_2(f(z+2c))+.......+c_n f(z+nc) : T(r,[p(f^l)]) \neq S(r,f),$$

where $c \neq 0$ *and* $c_j, j = 0,1,2,3......n,$ *are complex constants then* $f^l P[f] - a[z],\ a(z) \neq 0, \infty$ *has infinitely many zeros provided* $l > 1$.

Proof. Let $G(z) = f^l\, P[f]$ where f is an entire function and suppose $G(z) - a(z), a(z) \neq 0, \infty$ has infinitely many zeros. Then we get by using Lemma 2.1 and Lemma 2.2.

$$\begin{aligned}
T(r,G(z)) &= T(r, f^l P[f]) \\
&= T(r, f^l[c_0 f(z) + c_1 f(z+c) + ... + c_n f(z+nc)]) \\
&\leq lT(r,f) + T(r,f(z))[c_0 + c_1\frac{f(z+c)}{f(z)} ... + c_n\frac{f(z+nc)}{f(z)}] \\
&\leq lT(r,f) + T(r,f(z))[c_0 + c_1\frac{f(z+c)}{f(z)} ... + c_n\frac{f(z+nc)}{f(z)}] \\
&\leq lT(r,f) + (n+1)T(r,f) \\
&\leq (l+n+1)T(r,f)
\end{aligned} \tag{1}$$

Since f is entire, by using Nevanlinna's Second Main theorem and Lemma, we get

$$\begin{aligned}
(l+n+1)T(r,f)+S(r,f) &\leq \overline{N}\left(r, G(z)) + \overline{N}(r, \frac{1}{G(z)})\right) + \overline{N}\left(r, \frac{1}{G(z)-c}\right) + S(r,G) \\
&\leq (n+2)T(r.f) + S(r,f).
\end{aligned}$$

Which is a contradiction as $l > 1$. Then our assumption is wrong and hence, $f^l P[f] - a(z), a(z) \neq 0, \infty$ has infinitely many zeros. □

Theorem 2.5. *Let f be transcendental meromorphic function with finite order and as in definition* 1.2*, P[f] be a linear differences polynomial defined as*

$$P[f] = c_0 f(z)+c_1 f(z+c)+c_2(f(z+2c))+.......+c_n f(z+nc) : T(r,[p(f^l)]) \neq S(r,f),$$

where $c \neq 0$ *and* $c_j, j = 0,1,2,3......n,$ *are complex constants then* $f^l P[f] - a[z],\ a(z) \neq 0, \infty$ *has infinitely many zeros provided* $l > 3$.

Proof. Let $G(z) = f^l P[f]$ where f is meromorphic function and suppose $G(z) - a(z),\ a(z) \neq 0, \infty$ has finitely many zeros. Then we get by using Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
T(r,G(z)) &= T(r, f^l P[f]) \\
&= T(r, f^l[c_0 f(z) + c_1 f(z+c) + ... + c_n f(z+nc)]) \\
&\leq lT(r,f) + (2n+1)T(r,f) + S(r,f) \\
&\leq (2n+l+1)T(r,f) + S(r,f)
\end{aligned}$$

Since f is meromorphic, therefore by using Nevanlinna's Second main theorem and Lemma, we get

$$(l+2n+1)T(r,f)+S(r,f) \leq \overline{N}(r,G(z))+\overline{N}\left(r,\frac{1}{G(z)}\right)+\overline{N}\left(r,\frac{1}{G(z)-c}\right)+S(r,f)$$

$$= \overline{N}\left(r,\frac{1}{G(z)}\right)+\overline{N}(r,G(z))+S(r,f)$$

$$\leq (2n+4)T(r,f)+S(r,f)$$

So, we get

$$lT(r,f) \leq 3T(r,f)+S(r.f)$$

Which is a contradiction as $l > 3$. Thus our supposition is wrong and hence, $f^l P[f]-a(z), a(z) \neq 0, \infty$ has infinitely many zeros. □

Remark 2.1. Let $P[f] = f(z+c)$, then the above results are improvement and generalizations of Theorems 1.4 and 1.5.

Remark 2.2. Similar can be obtained for $f^l[\Delta_c^n f(z)]$ for all n.

3. Examples

Example 3.1. Let $f(z) = e^{zi}+1$. $C \neq \pi$ then $f(z), f(z+c) \neq 1$ identically. Therefore, Theorem 2.4 does not holds for $l = 1$.

Example 3.2. Let $f(z) = tanz, c = \frac{\pi}{2}, f^3, f(z+c) = -tan^2 z \neq 1$. identically, so Theorem 2.5 does not holds for $l = 3$.

4. Implementation

To implement the above main results, we present the following outcomes.

Theorem 4.1. *Let f and g be transcendental entire functions with finite order and as in definition* 1.2*, $P[f]$ and $P[g]$ be two linear difference polynomials defined as*

$$P[f] = c_0 f(z)+c_1 f(z+c)+c_2 f(z+2c)+....+c_n f(z+nc): \ T(r,P(f)) \neq S(r,f)$$

where $c \neq 0$ and $c_j, j = 0,1,2,3,.....n$, are complex constants, and $[f^l P[f]]^{(k)}$ and $[g^l f(g)]^{(n)}$ share $a(z), a(z) \neq 0, \infty$, then

$$[f^l P[f]]^{(k)} = [g^l P[g]]^{(n)}$$

or

$$[f^l P[f]]^{(k)}[g^l P(g)]^{(n)} = (a(z))^{(n)},$$

provided $l > 4k+n+3$

Proof. Let $F(z) = \frac{[f^l P[f]]^{(k)}}{a(z)}$ and $G(z) = \frac{[g^l P(g)]^{(k)}}{a(z)}$, then $F(z)$ and $G(z)$ share 1 CM except the zeros or poles of $a(z)$, we have by using Lemma 2.2.

$$\begin{aligned} N_2(r,0;f) &= N_2(r,0;f^l P[f]^{(k)}) + N_{(2+k)}(r.\frac{1}{f^l P(f)}) \\ &+ k\overline{N}(r, f^l P[f]) + S(r,f) \\ &\leq (k+2)\overline{N}(r, \frac{1}{f^l}) + N(r + \frac{r}{P(f)} + k\overline{N}(r,f) + S(r,f) \\ &\leq (2k+n+3)T(r,f) + S(r,f) \end{aligned}$$

Similarly,

$$N_2(r,0;G) \leq (2k+n+3)T(r,g) + S(r,g).$$

By Lemma 2.3, suppose (i) holds, then since f, g are entire functions,

$$\begin{aligned} max(T(r,f), T(r,G)) &\leq N_2(r,0;F) + N_2(r,0;G) \\ &+ N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G) \\ &\leq 2(2k+n+3)T(r,f) + T(r,g) + S(r,f) + S(r,g) \end{aligned}$$

Using equation (1), For $k = 1$, we have

$$(l+n+1)[T(r,f) + T(r,g)] \leq 2(2k+n+2)[T(r,f) + T(r,g)]$$

$$(l+n+1)[T(r,f) + T(r,g)] \leq (4k+2n+4)[T(r,f) + T(r,g)]$$

Which contradicts the given condition that $l > 4k + n + 3$.
Hence by Lemma 2.3, results holds. □

Remark 4.1. Similar results can be proved whole f, g are meromorphic functions.

Important Results:

1. Let f and g be transcendental entire function with finite order, c be non-zero complex constant and if $F = [f^n f(z+c)]^{(k)}$ and $G = [g^n g(z+c)]^{(n)}$ share $1CM$, then for $k = 1, n > 4k+3, F \equiv G$ or $F.G \equiv 1$.

2. Let f and g be transcendental entire function with finite order, c be non-zero complex constants and if $F = [f^n[f(z+c) - f(z)]]^{(k)}$ and $G = [g^n[g(z+c) - g(z)]]^{(k)}$ share $1CM$, then for $k = 1, n > 4k+6,\ F \equiv G$ or $F.G \equiv 1$
3. Similar results can be obtained for $[f^l[\Delta_c f(z)]]^{(k)}$ and $[g^l[\Delta_c^n g(z)]]^{(k)}$ for all n.

Example 4.2. Let $f(z) = sin\, z$ and $g(z) = cos\, z$, $l = 1$, $c = \pi$, $k = 1$ then $f^l P[f] = f^l f(z+c) = -sin^2 z$ and $g^l P[g] = g^l f(z+c) = -cos^2 z$. Here $-sin^2 z$ and $-cos^2 z$ share $\frac{-1}{2}CM$ which prove that the Theorem 4.1 may not be true when $l = 1$.

Example 4.3. In case of meromorphic functions,let $f(z) = cotz, g(z) = tanz, l = 1, c = \pi, k = 1, f^l p[f] = f^l f(z+c) = cot^2 z$ and $g^l p(g) = g^l g(z+c) = cot^2 z$. Here $tan^2 z$ and $cot^2 z$ share $1CM$ and $f^l p[f].g^l P[g] = 1$.

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