J. Appl. Math. & Informatics Vol. **41***(2023), No. 6, pp. 1155 - 1171*
https://doi.org/10.14317/jami.2023.1155

ANALYSIS OF AN M/G/1 QUEUEING SYSTEM WITH DISGRUNTLED JOBS AND DIFFERENT TYPES OF SERVICE RATE

M. KANNAN, V. POONGOTHAI AND P. GODHANDARAMAN*

Abstract. This paper investigates a non Markovian M/G/1 queue with retrial policy, different kind of service rates as well as unsatisfied clients which is inspired by an example of a transmission medium access control in wireless communications. The server tends to work continuously until it finds at least one client in the system. The server will begin its maintenance tasks after serving all of the clients and if the system becomes empty. Provisioning periods in regular working periods and maintenance service periods should be evenly divided. Using supplementary variable technique, the amount of clients in the system as well as in the orbit were found. Further few performance measures of the system were demonstrated numerically.

AMS Mathematics Subject Classification : 60J25, 60K25, 60K30.
Key words and phrases : Retrial queue, maintenance period, performance measures, disgruntled jobs.

1. Introduction

Now-a-days, queueing systems are considered as a powerful tool in various fields like wired and wireless communication networks, logistics environment, manufacturing systems and operating systems. Data communication systems and computer networks are showing their rapid growth in recent technological development that leads to significant evolution in many areas like advances in network protocols, voice transmission, video streams, etc.

If an arriving client finds the busy server, he temporarily leaves the service channel and reattempt his request for service over a random time which is depicted as the retrial queueing system. This blocked client in between trials, joins a waiting space of disappointed clients known as orbit. For example, packet

Received June 1, 2022. Revised April 26, 2023. Accepted July 31, 2023. *Corresponding author.

switching network system, collision detection, local area networks, web access, etc.

Another important factor in communication systems are the unsatisfied clients. If a client is unsatisfied with the service provided, then he retries again and again until the successful service completion. For example, the messages with errors in multiple access telecommunication systems are re-sent may be identified as retrial queues with unsatisfied client.

The server tends to work endlessly until it finds a last existing client in the system. After completing service for all the existing clients and no more client is found in the empty system, exists the service area for a particular duration which is termed as vacation. It can be viewed in real-life scenarios like communication systems, personalized manufacturing, production lines, etc., where the server do its maintenance activities during vacation.

Kim and Kim [11] conducted a survey on retrial queues with various queueing models that focuses on queue length distributions and the waiting period of the arriving jobs are also obtained. Kulkarni and Liang [12], Templeton [18] and Artalejo [2] have gathered explicit recent reviews on queues with retrial policy.

Poongothai and Godhandaraman [14] approach a service facility with front and back room serves. The arriving customers in front room service is unsatisfied they may rejoin the service again until finished service. The main aim of the problem is constraint programing model to solve queue control problem.

Chang, Liu and Ke [4] developed a truncated classical and constant retrial policies for single server queue for which the stationary probabilities are formulated using quasi birth and death method. A recursive solver procedure is developed to obtain the stationary probabilities of the structure.

D'Arienzo, Dudin, Dudin and Manzo [5] analysed on a single server with finite capacity follows a MAP and phase type of service. This model is provided for the number of customers when attained to the pre threshold value. They also discussed with performance measures and numerical illustrations.

Fiems [9] reviewed the concept of retrial queues with general retrial times. Different queueing systems are considered with general distribution of interarrival and service times. Dutta and Choudhary [8] investigated a $M/M/1$ queueing system for its performance measures using simulation techniques. The performance of the system is effectively described with the use of substitution estimators and alternative estimators.

Dimitriou [6] has studied the asymptotic behaviour of a retrial queue with increased retrial rates and subject to varying arrival rates being dependent on events. Li and Wang [13] have discussed a single server Markovian retrial queueing system subject to catastrophes. Following a repair process, a customer may re–join or balk the system subject to observable and unobservable levels.

Arivudainambi, Godhandaraman and Rajadurai [1] analysed the performance measures of a repeated queueing model with single server and working rest where the steady state probabilities and queue size distribution were found. The steady state system size probabilities and sensitivity analysis are obtained by Som and Kumar [17] for two heterogeneous server queueing model using iterative method.

Bouchentouf, Cherfaoui and Boualem [3], followed up with the single server M/M/1/N queueing system further they obtained steady state by considering balking, reneging and vacation. The cost function and numerical illustration were analyzed.

Do [7] established a single server repeated queue and working rest period that has been illustrated using an example of a transmission medium access control in wireless communications. Jain, Dhibar and Sanga [10] investigated Markovian queueing model with working vacation, retrial concept and impatient customers.

Vishnevskii and Dudin [19] considered a model of correlated arrival of the customers with application to model communication systems for markovian batch arrival of customers. According to Sennot, Humblet and Tweedie [15], the non ergodicity of the Markov chain fulfils Kaplan's requirement explicitly.

In this paper, we look at a few exceptional situations of our model that are compatible with the existing literature. The model is reduced to the M/G/1 queue and the outcomes accord with Zhang and Hou [20]. Shortle, Thompson, Gross and Harris [16], the PGF of the amount of clients in organization, the idle probability and the average organization size in this case may be simplified to the following formulations, which are regular with the well known P-K formula.

We may notice that plenty of works have been done in existing literature tied up with disappointed clients and different types of service rates. On account of the author's perception, the present work is the foremost work that examines a one server with repeated policy, unsatisfied clients and different types of service rate.

2. Practical Application of the Suggested Framework

2.1. Semiconductor Manufacturing Process. Semiconductors widely used in all electronic devices hold a conductivity between conducting and non conducting materials. They have a major contribution to all our electronic gadgets like radio, television, computers, medical diagnostic tools, etc. With the development in semiconductor technology, the construction of electronic gadgets in a smaller size, faster performance, and reliable mode emerged, which in turn has made our lives much easier.

In Figure 1, the manufacturing of a semiconductor process is considered. Silicon wafer sheets are widely used in the manufacturing process. The silicon sheets (arrival rate) enter the processing machine (service rate), if available. If the processing machines are occupied in service, the silicon wafers await their turn in the waiting space (orbit). From the orbit, the sheets enter the processing unit upon their availability. After success under the fabrication wafer and dicing process, it comes out as a semiconductor. Some of the semiconductors are defective by Laser Fault Injection (LFI) attacks.

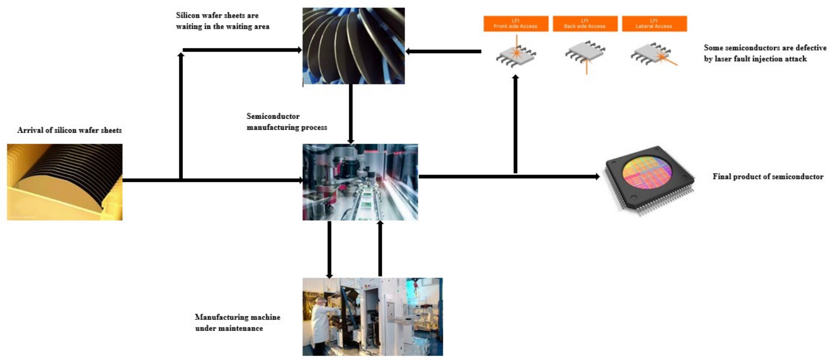

FIGURE 1. Semiconductor Manufacturing Process

The defective semiconductors are joined in the waiting space and thereby reprocessed to become a finished product. The machines are subject to regular maintenance for improving their efficiency. During this maintenance period, the fabrication wafer and dicing processing undergo a slower pace of service (lower service rate). After the completion of the maintenance period, then the processing undergo a regular pace of service (normal service rate).

2.2. E - Commerce. Nowadays, with the bloom in technology, most people prefer online purchases through e-commerce sites. E-commerce sites like Amazon, Flipkart, Myntra, etc., are selling various products such as electronic devices, clothes, home appliances, grocery and so on. E-commerce websites announce enormous discounts and attractive offers during festival periods. The launching of new products on these sites also hikes up the number of demands placed online.

During this period the demands for online service are pretty high. A customer may place orders online from the comfort of his/her own device at a convenient time. This phenomenon may be referred to as getting service. For customers arriving during a special occasion or peak hours, the e-commerce site servers are often busy and the customers requesting service have to wait in a virtual space (orbit) for the availability of the servers.

The maintenance of the e-commerce sites which includes updating offers, new arrivals, and enhancing the system efficiency takes place after providing service to the existing demands. During this period, the customer services are met at a slow service rate. Customers having booked a particular item online may replace their orders and retry service while waiting in the orbit if the customer feels dissatisfied in the sense of price variation, quality, and quantity of the product, etc.

3. Model Depiction and Ergodicity State

The client emergence to request their demand pursues a Poisson fashion with rate λ and starts the service instantly on finding idle service channel. At the arrival period, if a client finds the busy service area, then the client sent to the orbit.

Random law, along with the distribution function $B(t)$ and the Laplace Stieltjes Transform (LST) $B^*(\gamma)$, determines which client in the buffer will access the service channel. The clients remains providing service at a rate μ_b in a regular busy period pursues an arbitrary variable accompanied by a distribution function $S_b(t)$ and Laplace Stieltjes transform $S_b^*(\gamma)$. After finishing service, if any client gets disappointed by the service, the client may come back to the orbit as unsatisfied client by means of possibility r $(0 \leq r \leq 1)$ or else departs the organization using possibility s.

At once the orbit becomes vacant, the server begins the maintenance phase. The clients served at a lesser servicing rate μ_v during maintenance phase pursues an arbitrary variable using distribution function $S_v(t)$ and Laplace Stieltjes transform $S_v^*(\gamma)$. Later the fulfillment of maintenance phase, the server alter its rate of service from μ_v to μ_b, in case it finds any client in waiting area.

In Markov process, the state of system is dertermined at particular duration t by $\{L(t);t \geq 0\}$=$\{(B(t),Y(t),\tau_0(t),\tau_1(t),\tau_2(t),t \geq 0\}$, whereas $B(t)$ indicates state of the server $(0, 1$ & 2, for an idle server, busy as well as maintenance phase correspondingly) & $Y(t)$ signifies the total count of clients in orbit at period t.

If $B(t) = 0$ & $Y(t) > 0$, then $\tau_0(t)$ acts for the elapsed retrial time, if $B(t) = 1$, then $\tau_1(t)$ acts for the elapsed service phase between normal busy period at duration t, if $B(t) = 2$ & $Y(t) \geq 0$, then $\tau_2(t)$ acts for the elapsed maintenance phase at set up t. The tasks $\gamma(y)dy$, $\mu_b(y)dy$ & $\mu_v(y)dy$ are the act of restricted achievement rates for retrial, facility and maintenance phase each at set up x. i.e., $\gamma(y)dy = \frac{dB(y)}{1-B(y)}$, $\mu_b(y)dy = \frac{dS_b(y)}{1-S_b(y)}$, $\mu_v(y)dy = \frac{dS_v(y)}{1-S_v(y)}$.

3.1. Ergodicity State. During the leaving/maintenance periods, we study the ergodicity of the embedded Markov chain. Let $\{t_e;\ e \in N\}$ be a system of time periods for either service completion or maintenance expiration. The arbitrary vectors system K_e=$\{C(t_e+),\ X(t_e+)\}$ form a Markov chain, which is the queueing system's embedded Markov chain. The name of its state space is St=$\{0, 1$ and $2\} \times N$.

Theorem 3.1. *If and only if $\lambda E(S_b) < B^*(\lambda)$, the embedded Markov chain $\{K_e;\ e \in N\}$ is ergodic.*

Proof. The irreducible & aperidic Markov chain is $\{K_e;\ e \in N\}$. The sufficient state of ergodicity showed by using Foster's criterion . If there occurs a non-negative function $f(m)$, $m \in N$ and $\epsilon > 0$, so that the average drift $\chi_m = E[f(k_{e+1}) - f(k_e)|k_e = m]$ all have a limit $m \in N$ and $\chi_m \leq -\epsilon$

for everyone $m \in N$, except for a limited number of people m, then irreducible and aperiodic Markov chain is ergodic.
We study the function $f(m) = m$, then we have

$$\chi_m = \begin{cases} \lambda E(S_b) + r - B^*(\lambda);\ m = 1, 2, \cdots \\ \lambda E(S_b) - s;\ \ m = 0 \end{cases}$$

Ergodicity requires and is satisfied by the inequality $\lambda E(S_b) < B^*(\lambda)$. The required state of ergodicity proved by using Kaplan's condition. According to Sennot (1983), the non ergodicity of the Markov chain $\{K_e;\ e \geq 1\}$ fulfils Kaplan's requirement explicitly $\chi_m < \infty; m \geq 0$ & there occurs $m_0 \in N$ such that $\chi_m \geq 0; m \geq m_0$. Kaplan's condition is satisfied in our case since h exists such that $r_{lm} = 0$ for $m < l - h$ and $l > 0$, where $B = (r_{lm})$ is $\{K_e;\ e \geq 1\}$ is one-step transition matrix. The inequality $\lambda E(S_b) \geq B^*(\lambda)$ suggests that the Markov chain is non-ergodic. □

4. Steady State Distribution of the Server State

The probabilities for the procedure $\{L(t), t \geq 0\}$ are described

$$G_0(t) = P\{B(t) = 0,\ \ Y(t) = 0\}$$

$$G_e(y,t)\,dy = P\{B(t) = 0,\ \ Y(t) = e,\ \ y \leq \tau_0(t) < y + dy\};\ \ t \geq 0,\ y \geq 0,\ e \geq 1$$

$$H_{e,b}(y,t)\,dy = P\{B(t) = 1,\ \ Y(t) = e,\ \ y \leq \tau_1(t) < y + dy\};\ \ t \geq 0,\ y \geq 0,\ e \geq 0$$

$$H_{e,v}(y,t)\,dy = P\{B(t) = 2,\ \ Y(t) = e,\ \ y \leq \tau_2(t) < y + dy\};\ \ t \geq 0,\ y \geq 0,\ e \geq 0$$

The steady state condition $\lambda E(S_b) < B^*(\lambda)$ is assumed to be met, hence we can set $G_0 = \lim_{t\to\infty} G_0(t)$, $G_e(y) = \lim_{t\to\infty} G_e(t, y)$, $H_{e,b}(y) = \lim_{t\to\infty} H_{e,b}(t, y)$ and $H_{e,v}(y) = \lim_{t\to\infty} H_{e,v}(t, y)$. The following steady state balancing equations are obtained using the supplementary variables technique.

$$\lambda G_0 = \int_0^\infty H_{0,v}(y)\ \mu_v(y)dy \tag{1}$$

$$\frac{d}{dy}\ G_e(y) + [\lambda + \gamma(y)]\,G_e(y) = 0;\ \ y > 0,\ e \geq 1 \tag{2}$$

$$\frac{d}{dy}\ H_{0,\ b}(y) + [\lambda + \mu_b(y)]\,H_{0,\ b}(y) = 0;\ \ y > 0 \tag{3}$$

$$\frac{d}{dy}\ H_{e,\ b}(y) + [\lambda + \mu_b(y)]\,H_{e,\ b}(y) = \lambda H_{e-1,\ b}(y);\ \ y > 0,\ e \geq 1 \tag{4}$$

$$\frac{d}{dy}\ H_{0,\ v}(y) + [\lambda + \mu_v(y)]\,H_{0,\ v}(y) = 0;\ \ y > 0 \tag{5}$$

$$\frac{d}{dy}\ H_{e,\ v}(y) + [\lambda + \mu_v(y)]\,H_{e,\ v}(y) = \lambda H_{e-1,\ v}(y);\ \ y > 0,\ e \geq 1 \tag{6}$$

The stationary distributions of the amount of clients in the organization when the server is idle (G_0) in equation (1). Equation (2) guarantees that the amount of clients (e) in the organization when the server is busy. Equation (3, 4) certifies that the amount of clients in the organization when the server is busy with normal service rate. Equation (5, 6) shows about the amount of clients in the organization when the server is busy with lower service rate.

The steady state boundary conditions can be used to explain the aforementioned set of equations.

$$G_e(0) = \int_0^\infty H_{e,\ v}(y)\,\mu_v(y)\,dy + r\int_0^\infty H_{e-1,\ b}(y)\,\mu_b(y)\,dy$$
$$+s\int_0^\infty H_{e,\ b}(y)\,\mu_b(y)\,dy;\quad e \geq 1 \tag{7}$$

$$H_{0,\ b}(0) = \lambda G_0 + \int_0^\infty G_1(y)\ \gamma(y)dy \tag{8}$$

$$H_{e,\ b}(0) = \int_0^\infty G_{e+1}(y)\,\gamma(y)\,dy + \lambda\int_0^\infty G_e(y)\,dy;\quad e \geq 1 \tag{9}$$

$$H_{0,\ v}(0) = s\int_0^\infty H_{0,\ b}(y)\ \mu_b(y)dy \tag{10}$$

The state of normalisation is determined by

$$G_0 + \sum_{e=1}^{\infty}\int_0^\infty G_e(y)dy + \sum_{n=0}^{\infty}\int_0^\infty H_{e,b}(y)dy + \sum_{e=0}^{\infty}\int_0^\infty H_{e,\ v}(y)dy = 1 \tag{11}$$

Let's look at the probability generating function (PGF) in more detail as $G(y,k) = \sum_{e=1}^{\infty} k^e G_e(y)$, $G(0,k) = \sum_{e=1}^{\infty} k^e G_e(0)$, $H_b(y,k) = \sum_{n=0}^{\infty} k^n H_{e,b}(y)$, $H_b(0,k) = \sum_{e=0}^{\infty} k^n H_{e,b}(0)$, $H_v(y,k) = \sum_{e=0}^{\infty} k^e H_{e,v}(y)$, $H_v(0,k) = \sum_{e=0}^{\infty} k^e H_{e,v}(0)$ for $|k| \leq 1$ and $y > 0$.

Theorem 4.1. *Under the stability requirement $\lambda E(S_b) < B^*(\lambda)$, the stationary distributions of the amount of clients in the organization when the server is idle (G_0), busy $(G(k))$, busy with normal service rate $(H_b(k))$, and busy with lower service rate $(H_v(k))$ are*

$$G(k) = \frac{G_0 k[1-B^*(\lambda)]\{[1-S_v^*(\lambda(1-k))] + S_v^*(\lambda)[1-(rk+s)S_b^*(\lambda(1-k))]\}}{S_v^*(\lambda)\{[k+(1-k)B^*(\lambda)](rk+s)S_b^*(\lambda(1-k)) - k\}} \tag{12}$$

$$H_b(k) = \frac{G_0[1-S_b^*(\lambda(1-k))]\{[1-S_v^*(\lambda(1-k))][k+(1-k)B^*(\lambda)] + (1-k)B^*(\lambda)S_v^*(\lambda)\}}{S_v^*(\lambda)(1-k)\{[k+(1-k)B^*(\lambda)](rk+s)S_b^*(\lambda(1-k)) - k\}} \tag{13}$$

$$H_v(k) = \frac{G_0[1-S_v^*(\lambda(1-k))]}{S_v^*(\lambda)(1-k)} \tag{14}$$

where $G_0 = \frac{S_v^*(\lambda)[B^*(\lambda)-r-\lambda E(S_b)]}{s[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$

Proof. The partial differential equations (2) to (6) are obtained by multiplying (2) to (6) by the powers of k and summing over e.

$$\frac{\partial G(y,k)}{\partial y} + [\lambda + \gamma(y)]G(y,k) = 0 \tag{15}$$

$$\frac{\partial H_b(y,k)}{\partial y} + [\lambda(1-k) + \mu_b(y)]H_b(y,k) = 0 \tag{16}$$

$$\frac{\partial H_v(y,k)}{\partial y} + [\lambda(1-k) + \mu_v(y)]H_v(y,k) = 0 \tag{17}$$

We get the following by solving the equations from (15) to (17)

$$G(y,k) = G(0,k)[1-B(y)]e^{-\lambda y} \tag{18}$$

$$H_b(y,k) = H_b(0,k)[1-S_b(y)]e^{-\lambda(1-k)y} \tag{19}$$

$$H_v(y,k) = H_v(0,k)[1-S_v(y)]e^{-\lambda(1-k)y} \tag{20}$$

Multiplying (7) by power of k, summing over e from 1 to ∞, we obtain

$$G(0,k) = \int_0^\infty H_v(y,k)\mu_v(y)dy + (rk+s)\int_0^\infty H_b(y,k)\mu_b(y)dy - H_{0,v}(0) - \lambda G_0 \tag{21}$$

Multiplying from (8) to (10) by powers of k, summing over e from 0 to ∞, we obtain

$$H_b(0,k) = \frac{1}{k}\int_0^\infty G(y,k)\ \gamma(y)dy + \lambda\int_0^\infty G(y,k)dy + \lambda G_0 \tag{22}$$

$$H_v(0,k) = H_{0,v}(0) \tag{23}$$

Commencing equation (5),

$$H_{0,v}(y) = H_{0,v}(0)[1-S_v(y)]e^{-\lambda y} \tag{24}$$

We get equation (1) by multiplying (24) by $\mu_v(y)$ and integrating with respect to y from 0 to ∞ on both sides

$$H_{0,v}(0) = \frac{\lambda G_0}{S_v^*(\lambda)} \tag{25}$$

Substituting the equation (25) in equation (23), we get

$$H_v(0,k) = \frac{\lambda G_0}{S_v^*(\lambda)} \tag{26}$$

By using equation (18) in equation (22), we get

$$H_b(0,k) = \lambda G_0 + \left(\frac{k + (1-k)\,B^*(\lambda)}{k}\right)P(0,k) \tag{27}$$

Further using equations from (19) to (20), and (25) in equation (21), we get

$$G(0,k) = H_v(0,k)S_v^*(\lambda(1-k)) + (rk+s)H_b(0,k)S_b^*(\lambda(1-k)) - \frac{\lambda G_0}{S_v^*(\lambda)} - \lambda G_0 \tag{28}$$

Using the equations (26) and (27), we get

$$G(0,k) = \frac{\lambda k G_0}{S_v^*(\lambda)}\left\{\frac{[1-S_v^*(\lambda(1-k))] + S_v^*(\lambda)[1-(rk+s)\,S_b^*(\lambda(1-k))]}{(rk+s)\,S_b^*(\lambda(1-k))\,[k+(1-k)\,B^*(\lambda)] - k}\right\} \tag{29}$$

When we replace equation (29) with equation (27), we get

$$H_b(0,k) = \left[\frac{\lambda G_0 \left\{[1 - S_v^*\left(\lambda\left(1-k\right)\right)] + S_v^*\left(\lambda\right)\left[1 - \left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\right]\right\}}{S_v^*\left(\lambda\right)\left\{\left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\left[k + \left(1-k\right) B^*\left(\lambda\right)\right] - k\right\}}\right] \times [k + (1-k)B^*(\lambda)] + \lambda G_0 \tag{30}$$

We obtain by substituting from equations (29) to (30) and equation (26) in (18) to (20).

$$G(y,k) = \left[\frac{\lambda G_0 k \left\{[1 - S_v^*\left(\lambda\left(1-k\right)\right)] + S_v^*\left(\lambda\right)\left[1 - \left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\right]\right\}}{S_v^*\left(\lambda\right)\left\{\left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\left[k + \left(1-k\right) B^*\left(\lambda\right)\right] - k\right\}}\right] \times [1 - B(y)]\ e^{-\lambda y}$$

$$H_b(y,k) = \left\{\left[\frac{\lambda G_0 \left\{[1 - S_v^*\left(\lambda\left(1-k\right)\right)] + S_v^*\left(\lambda\right)\left[1 - \left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\right]\right\}}{S_v^*\left(\lambda\right)\left\{\left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\left[k + \left(1-k\right) B^*\left(\lambda\right)\right] - k\right\}}\right] \times \left[k + \left(1-k\right) B^*\left(\lambda\right)\right] + \lambda G_0\right\} \times [1 - S_b(y)] e^{-\lambda(1-k)y}$$

$$H_v(y,k) = \frac{\lambda G_0}{S_v^*(\lambda)}[1 - S_v(y)] e^{-\lambda(1-k)y}$$

When the above equations integrate with respect to y from 0 to ∞, the requisite final solutions are obtained from equations (12) to (14). The idle probability G is unknown and can be determined using the normalising state $G_0 + G(1) + H_b(1) + H_v(1) = 1$. Let $U(k) = G_0 + G(k) + k\left[H_b\left(k\right) + H_v\left(k\right)\right]$ and $V(k) = G_0 + G(k) + H_b(k) + H_v(k)$ denotes the PGF for the amount of clients in the system and orbit at a given point in time. □

Theorem 4.2. *The probability generating function of the system size and orbit size distribution at a stationary point of period in stability state $\lambda E(S_b) < B^*(\lambda)$ is*

$$U(k) = \left[\frac{\begin{array}{c} G_0 s \left\{[1 - S_v^*\left(\lambda\left(1-k\right)\right)]\left[k + \left(1-k\right) B^*\left(\lambda\right)\right] \\ + \left(1-k\right) B^*\left(\lambda\right) S_v^*\left(\lambda\right)\right\} S_b^*\left(\lambda\left(1-k\right)\right) \end{array}}{S_v^*\left(\lambda\right)\left\{\left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\left[k + \left(1-k\right) B^*\left(\lambda\right)\right] - k\right\}}\right] \tag{31}$$

$$V(k) = \left[\frac{\begin{array}{c} G_0[1 - rS_b^*(\lambda(1-k)]\{[1 - S_v^*(\lambda(1-k))][k + (1-k)B^*(\lambda)] \\ +(1-k)B^*(\lambda)S_v^*(\lambda)\} \end{array}}{S_v^*\left(\lambda\right)\left\{\left(rk+s\right) S_b^*\left(\lambda\left(1-k\right)\right)\left[k + \left(1-k\right) B^*\left(\lambda\right)\right] - k\right\}}\right] \tag{32}$$

where G_0 is specified in equation (15).

Proof. Let us consider the probabaility genrating function for the amount of clients in the system is $U(k) = G_0 + G(k) + k\left[H_b\left(k\right) + H_v\left(k\right)\right]$ and the PGF for the amount of clients in the orbit is $V(k) = G_0 + G(k) + H_b(k) + H_v(k)$. We get the equation (31) and (32) by using the equations from (12) to (15) for the above consideration. □

5. Performance Measures

Under steady-state conditions, we accomplish specific performance measurements for the system. Let I denote the steady state probability that the server will be idle during the retrial period, E the steady state probability that the server will be busy, F the steady state probability that the server will be on maintenance, and L the steady state probability that the server will be idle or on maintenance. B_0 is the steady state probability of the system being empty while the server is down for maintenance, M is the steady state likelihood of the system being empty, and J is the steady state probability of the orbit being empty.

$$I{=}G(1) = \frac{[1-B^*(\lambda)][\lambda E(S_v)+S_v^*(\lambda)(\lambda E(S_b)+r)]}{s[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$$

$$E{=}H_b(1) = \frac{\lambda E(S_b)}{s}$$

$$F{=}H_v(1) = \frac{\lambda E(S_v)[B^*(\lambda)-r-\lambda E(S_b)]}{s[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$$

$$L{=}G_0+G(1)+H_v(1) = 1-\frac{\lambda E(S_b)}{s}$$

$$B_0{=}H_v(0) = \frac{[1-S_v^*(\lambda)][B^*(\lambda)-r-\lambda E(S_b)]}{s[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$$

$$M{=}G_0+B_0{=}\frac{[B^*(\lambda)-r-\lambda E(S_b)]}{s[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$$

$$S{=}G_0+B_0+H_0{=}\frac{[1-rS_b^*(\lambda)][B^*(\lambda)-r-\lambda E(S_b)]}{s^2[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]S_b^*(\lambda)}$$

The averge amount of jobs in the organization is found by differentiating (32) with respect to k and evaluating at $k=1$.

$$L_s{=}\lambda E\left(S_b\right)+\frac{\lambda^2 E\left(S_b^2\right)+2\lambda E\left(S_b\right)\left[1-B^*\left(\lambda\right)\right]+2r\left[\lambda E\left(S_b\right)+1-B^*\left(\lambda\right)\right]}{2\left[B^*\left(\lambda\right)-r-\lambda E\left(S_b\right)\right]}$$

$$+\frac{\lambda^2 E(S_v^2)+2\lambda E(S_v)[1-B^*(\lambda)]}{2[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$$

The average amount of jobs in the orbit is found by differentiating (32) with respect to k and evaluating at $k=1$.

$$L_q{=}\frac{\lambda^2 E(S_b^2)+2\lambda E(S_b)(1+r-B^*(\lambda))+2r[1-B^*(\lambda)]}{2[B^*(\lambda)-r-\lambda E(S_b)]}$$

$$+\frac{s[\lambda^2 E(S_v^2)+2\lambda E(S_v)(1-B^*(\lambda))]+2r\lambda E(S_b)[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}{2s[\lambda E(S_v)+B^*(\lambda)S_v^*(\lambda)]}$$

6. Special Cases

In this paper, we look at a few exceptional situations of our model that are compatible with the existing literature.

Case 1. With a single working vacation $B^*(\lambda) \to 1$, $r=0$ and $s=1$, the model is reduced to the M/G/1 queue. The PGF of the amount of clients in the organization $U(k)$, the idle probability G_0, and the average organization size L_s can be rewritten in this case, and the outcomes accord with Zhang and Hou (2012).

$$G_0=\frac{[1-\lambda E(S)]S_v^*(\lambda)}{\lambda E(S_v)+S_v^*(\lambda)}$$

$$U(k)=\frac{[1-\lambda E(S)]\{[1-S_v^*(\lambda-\lambda k)]+(1-k)S_v^*(\lambda)\}S_b^*(\lambda-\lambda k)}{[\lambda E(S_v)+S_v^*(\lambda)][S_b^*(\lambda-\lambda k)-k]}$$

$$L_s=\lambda E(S_b)+\frac{\lambda^2 E(S_b^2)}{2[1-\lambda E(S_b)]}+\frac{\lambda^2 E(S_v^2)}{2[\lambda E(S_v)+S_v^*(\lambda)]}$$

Case 2. The M/G/1 queueing technique is used to simplify the concept $B^*(\lambda) \to 1$, r $=0$, $s=1$ and $S_v^*(\lambda)=1$. The PGF of the amount of clients in organization $U(k)$, the idle probability G_0 and the average organization size L_s in this case may be simplified to the following formulations, which are regular with the well known P-K formula [Shortle et al. (2018)].

$$G_0=1-\lambda E(S)$$

$$U(k)=\frac{[1-\lambda E(S)](1-k)S^*(\lambda-\lambda k)}{S^*(\lambda-\lambda k)-k}$$

$$L_s=\lambda E(S)+\frac{\lambda^2 E(S^2)}{2[1-\lambda E(S)]}$$

7. Numerical Illustrations

We exhibit algebraic results in Matlab to demonstrate the effect of numerous constraints on the main performance of our organization, using arbitrary values for arrival rate λ, retrial rate γ, busy with normal service rate μ_b , busy with lower service rate μ_v and unsatisfied client arrival rate r to satisfy the stability constraint. Two dimensional diagrams are drawn in Figures 2 through 6.

Figure 2 shows how the mean system size L_s increases as the unsatisfied client arrival rate r is increased while the normal service rate μ_b and maintenance service rate μ_v are varied. As displayed in Figure 3, the idle probability G_0 reduces as the unsatisfied customer arrival rate r is increased by adjusting the normal service rate μ_b and the maintenance service rate μ_v. Figure 4 shows how increasing the retrial rate lowers the idle probability G_0 while changing the normal service rate μ_b and the maintenance service rate μ_v. Figure 5 shows how the traffic intensity ρ reduces as the regular service rate μ_b is increased while the retrial rate γ is varied.

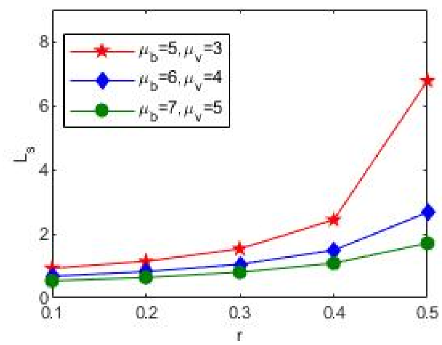

FIGURE 2. r versus L_s

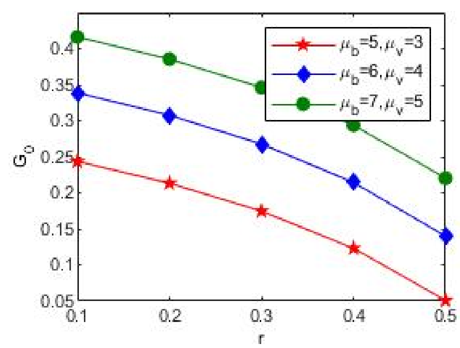

FIGURE 3. r versus G_0

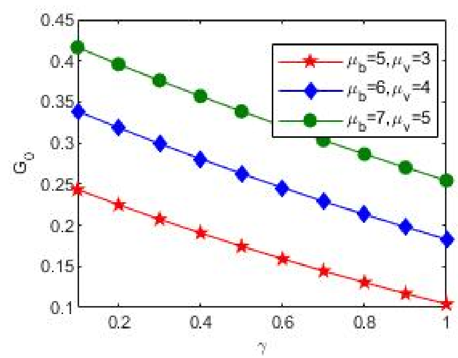

FIGURE 4. γ versus G_0

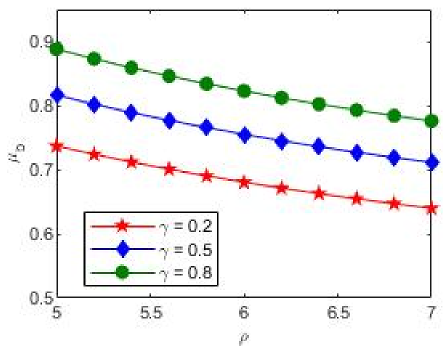

FIGURE 5. ρ versus μ_b

Figure 6 shows how the traffic intensity ρ reduces as the maintenance service rate μ_v is increased while the retrial rate is varied. Figures 7 through 10 show three-dimensional graphs. As shown in Figure 7, the surface shows an increased trend for the mean system size L_s as the regular service rate μ_b and maintenance service rate μ_v increase. Figure 8 shows a lower trend for the idle probability G_0 as the normal service rate μ_b and the maintenance service rate μ_v are increased.

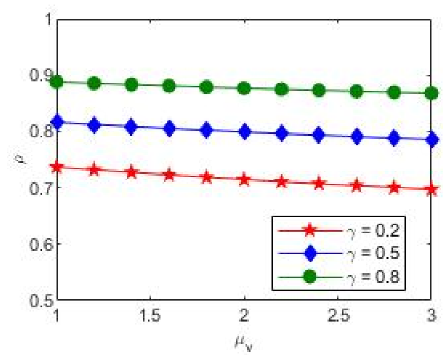

FIGURE 6. μ_v versus ρ

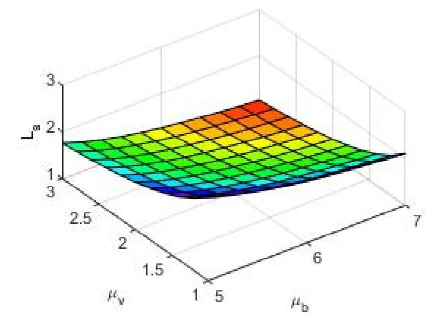

FIGURE 7. $\mu_b,\ \mu_v$ versus L_s

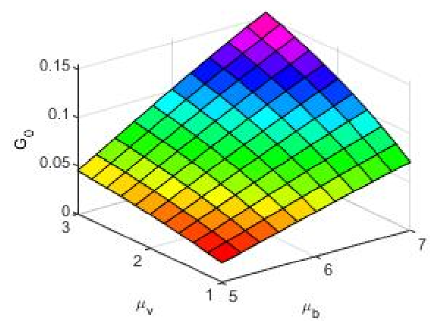

FIGURE 8. $\mu_b,\ \mu_v$ versus G_0

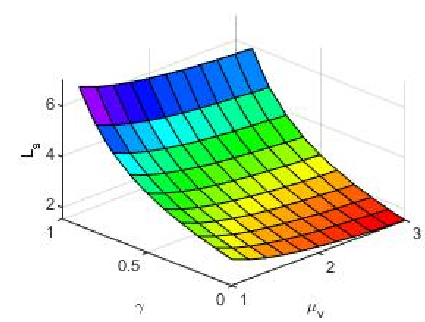

FIGURE 9. $\mu_v,\ \gamma$ versus L_s

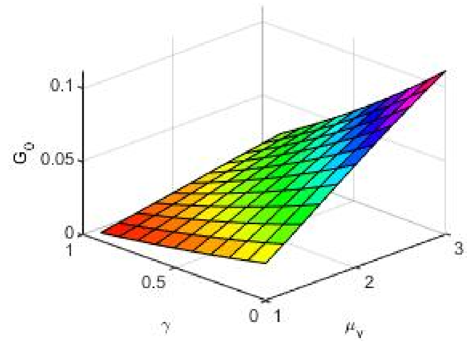

FIGURE 10. $\mu_v,\ \gamma$ versus G_0

As illustrated in Figure 9, the surface shows a declining trend for the mean system size L_s as the maintenance service rate μ_v and the retrial rate increase. The surface in Figure 10 shows an increased trend for the idle probability G_0 as the maintenance service rate μ_v and the retrial rate increase.

8. Conclusion

In the present work we have established a non Markovian $M/G/1$ queue of retrial policy, disappointed clients and variant types of service rates which is illustrated by an example of media access function in wireless networks. The explicit expressions of this model were discovered using PGF, and the total amount of clients in the organization, traffic intensity, and orbit were calculated using the supplementary variable technique. Some special cases and performance measures were also analyzed. The dissatiesfied clients parameter affect the system performance slightly. Furthermore, the numerical and graphical representations of the effects of variables utilised to portray system states are shown.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The authors would like to extend their gratitude towards the Editorial board and referees for their valuable insights and suggestions that has led to the enhancement of the quality of the original manuscript.

References

1. D. Arivudainambi, P. Godhandaraman and P. Rajadurai, *Performance analysis of a single server retrial queue with working vacation*, OPSEARCH **51** (2014), 434-462.
2. J.R. Artalejo, *Accessible bibliography on retrial queues: Progress in 2000-2009*, Top **7** (2010), 187-211.
3. A.A. Bouchentouf, M. Cherfaoui and M. Boualem, *Performance and economic analysis of a single server feedback queueing model with vacation and impatient customers*, OPSEARCH **56** (2019), 300-323.
4. F.M. Chang, T.H. Liu and J.C. Ke, *On an unreliable-server retrial queue with customer feedback & impatience*, Applied Mathematical Modelling **55** (2018), 171-182.
5. M.P. D'Arienzo, A.N. Dudin, S.A. Dudin, and R. Manzo, *Analysis of a retrial queue with group service of impatient customers*, Journal of Ambient Intelligence and Humanized Computing **11** (2020), 2591-2599.
6. I. Dimitriou, *A single server retrial queue with event-dependent arrival rates*, Annals of Operations Research (2023), 1-36.
7. T.V. Do, *M/M/1 retrial queue with working vacations*, Acta Informatica **47** (2010), 67-75.
8. K. Dutta and A. Choudhury, *Estimation of performance measures of M/M/1 queues-a simulation-based approach*, International Journal of Applied Management Science **12** (2020), 265-279.
9. D. Fiems, *Retrial queues with generally distributed retrial times*, Queueing Systems **100** (2022), 189-191.
10. M. Jain, S. Dhibar, and S.S. Sanga, *Markovian working vacation queue with imperfect service, balking and retrial*, Journal of Ambient Intelligence and Humanized Computing (2021), 1-17.
11. K. Kim and B. Kim, *A survey of retrial queueing systems*, Annals of Operations Research **247** (2016), 3-36.
12. V.G. Kulkarni and H.M. Liang, *Retrial queues revisited, In: Frontiers in queueing models and applications in science and engineering*, Ed. by J.H. Dshalalow, CRC Press, New York, 1996, 19-34.
13. K. Li and J. Wang, *Equilibrium balking strategies in the single-server retrial queue with constant retrial rate and catastrophes*, Quality Technology & Quantitative Management **18** (2021), 156-178.
14. V. Poongothai and P. Godhandaraman, *Cross trained servers with balking and feedback service facility by applying constraint programming model*, International Journal of Operational Research **35** (2019), 178-195.
15. L.T. Sennot, P.A. Humblet and R.L. Tweedie, *Mean drift and the non ergodicity of Markov chains*, Operations Research **31** (1983), 783-789.
16. J.F. Shortle, J.M. Thompson, D. Gross and C.M. Harris, *Fundamentals of queueing theory* 5th ed., Wiley, New York, 2018.
17. B.K. Som and R. Kumar, *A heterogeneous queuing system with reverse balking and reneging*, Journal of Industrial and Production Engineering **35** (2018), 1-5.
18. J.G.C. Templeton, *Retrial Queues*, Top **7** (1999), 351-353.
19. V.M. Vishnevskii and A.N. Dudin, *Queueing systems with correlated arrival flows and their applications to modeling telecommunication networks*, Automation and Remote Control **78** (2017), 1361-1403.
20. M. Zhang and Z. Hou, *M/G/1 queue with single working vacation*, Journal of Applied Mathematics and Computing **39** (2012), 221-234.

M. Kannan main research interests are: Operations Research and Queueing theory.

Department of Mathematics, SRM Institute of Science and Technology, Ramapuram, Chennai - 600089, Tamilnadu, India.

e-mail: kannanm1@srmist.edu.in

V. Poongothai has received the Ph.D. degree in Mathematics at Anna University, Chennai. She acted as a referee of number of international journals in the field of Queueing Theory and Operations Research. Her main research interests are: Operations Research, Stochastic Processes, Queueing theory and Optimization.

Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur - 603203, Tamilnadu, India.
e-mail: poongotv@srmist.edu.in

P. Godhandaraman has received the Ph.D. degree in Mathematics at Anna University, Chennai. He is referee of several international journals. His main research interests are: Operations Research, Stochastic Processes, Queueing Theory and Optimization.

Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur - 603203, Tamilnadu, India.
e-mail: godhandp@srmist.edu.in