JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 4, November 2023
http://dx.doi.org/10.14403/jcms.2023.36.4.267

ON ADDITIVE MAPPINGS WITH BOUNDED VALUE NEAR A POINT

Won-Gil Park, Ick-Soon Chang and Jaiok Roh*

Abstract. In this paper, we will first prove the linearity of additive mappings with bounded value near a point. And by using our theorem we will investigate the instability of the Cauchy equation in $\mathbb{R}^2$.

1. Introduction

In 1821, Cauchy [1] proved the following theorem.

Proposition 1.1. *If an additive mapping $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is linear.*

In 1946, Kestelman [3] proved the following theorem.

Proposition 1.2. *Let E_1 and E_2 be normed spaces, and let E be a subset of E_1. Suppose that there is a $\delta > 0$ satisfying*

$$\{v \in E_1 : \|v\| < \delta\} \subset \{x - y : x, y \in E\}.$$

If an additive mapping $f : E_1 \to E_2$ is bounded on E, then f is linear.

In 1991, Gajda [2] also proved the following theorem.

Proposition 1.3. *Let E_1 be a normed space and E_2 a Banach space. And let $\varepsilon > 0$ and $p \in \mathbb{R} \setminus \{1\}$. If $f : E_1 \to E_2$ is a mapping such that*

$$\|f(x + y) - f(x) - f(y)\| < \varepsilon(\|x\|^p + \|y\|^p)$$

Received October 21, 2023; Accepted November 16, 2023.
2020 Mathematics Subject Classification: 39B82, 39B72.
Key words and phrases: additive mapping, linear mapping.
This work was supported by National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C109489611).
* Correspondence should be addressed to joroh@hallym.ac.kr.

for all $x, y \in E_1$*, then there exists a unique additive mapping* $T : E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \le \frac{1-p}{|1-p|}\frac{2\varepsilon}{2-2^p}\|x\|^p$$

for all $x \in E_1$*. Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$*, then* T *is linear.*

In 2010, Reem [4] proved the following theorem.

Proposition 1.4. *If an additive mapping* $f : \mathbb{R} \to \mathbb{R}$ *is integrable on some bounded closed interval* I*, then* f *is linear.*

In this paper, we will present another version and proof of Proposition 1.2 as follows.

Theorem 1.5. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a mapping satisfying the Cauchy equation*

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$*. If* T *is bounded mapping on a small disk* $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^2 \mid |\mathbf{z} - \mathbf{x}| < r\}$ *for some* $\mathbf{x} \in \mathbb{R}^2$*, then there exists a* 2×2 *matrix* A *such that* $T(\mathbf{u}) = A\mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{R}^2$*. Therefore, the mapping* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear mapping.*

Gajda [2] investigated the instability of the Cauchy equation in $\mathbb{R}$ as follows.

Proposition 1.6. *For any* $\varepsilon > 0$*, there exists a mapping* $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$|f(x + y) - f(x) - f(y)| < \varepsilon(|x| + |y|)$$

for all $x, y \in \mathbb{R}$ *such that there is no constant* $\delta \ge 0$ *and no additive function* $T : \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$|f(x) - T(x)| \le \delta|x|$$

for all $x \in \mathbb{R}$*.*

In this paper, we will also investigate the instability of the Cauchy equation in $\mathbb{R}^2$ as follows.

Theorem 1.7. *For any* $\varepsilon > 0$*, there exists a mapping* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfying*

$$|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon(|\mathbf{x}| + |\mathbf{y}|)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ *such that there is no constant* $\delta \ge 0$ *and no additive function* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfying the condition*

$$|f(\mathbf{x}) - T(\mathbf{x})| \le \delta|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^2$. *So we can say about the instability of the Cauchy equation in* $\mathbb{R}^2$.

2. Main Results

In this section, we will prove main results.

THEOREM 2.1. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a mapping satisfying the Cauchy equation*

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. *If* T *is bounded on a small disk* $B(\mathbf{x}, r)=\{\mathbf{z} \in \mathbb{R}^2 \mid |\mathbf{z} - \mathbf{x}| < r\}$ *for some* $\mathbf{x} \in \mathbb{R}^2$, *then there exists a* 2×2 *matrix* A *such that* $T(\mathbf{u})=A\mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{R}^2$. *Therefore, the mapping* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear mapping.*

Proof : If $T = (T_1, T_2)$ is bounded on a small disk $B(\mathbf{x}, r)=\{\mathbf{z} \in \mathbb{R}^2 \mid |\mathbf{z} - \mathbf{x}| < r\}$, then T is bounded on a small disk containing $\mathbf{0} \in \mathbb{R}^{\mathbf{2}}$. Since T is bounded on a small disk containing $\mathbf{0} \in \mathbb{R}^{\mathbf{2}}$ there exists $\mathbf{z}_0 = (\pi_1(\mathbf{z}_0), \pi_2(\mathbf{z}_0))$ such that $\pi_1(\mathbf{z}_0) \neq 0$, $\pi_2(\mathbf{z}_0) \neq 0$, where $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the projection mappings given by $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$, respectively.

Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ by

(2.1)

$$G(\mathbf{z}) := T\begin{pmatrix} \pi_1(\mathbf{z}) \\ \pi_2(\mathbf{z}) \end{pmatrix} - \begin{pmatrix} \frac{1}{\pi_1(\mathbf{z}_0)} T_1\binom{\pi_1(\mathbf{z}_0)}{0} & \frac{1}{\pi_2(\mathbf{z}_0)} T_1\binom{0}{\pi_2(\mathbf{z}_0)} \\ \frac{1}{\pi_1(\mathbf{z}_0)} T_2\binom{\pi_1(\mathbf{z}_0)}{0} & \frac{1}{\pi_2(\mathbf{z}_0)} T_2\binom{0}{\pi_2(\mathbf{z}_0)} \end{pmatrix} \begin{pmatrix} \pi_1(\mathbf{z}) \\ \pi_2(\mathbf{z}) \end{pmatrix}$$

for all $\mathbf{z} \in \mathbb{R}^2$. Then, by the additive property of T, the mapping G is also additive. And, we have $G(\mathbf{z}_0) = \mathbf{0}$ and $G(\mathbf{z} + \mathbf{z}_0)=G(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^2$.

Moreover, we have

$$G\left(\mathbf{z} + \begin{pmatrix} \pi_1(\mathbf{z}_0) \\ 0 \end{pmatrix}\right) = G(\mathbf{z}) \quad \text{and} \quad G\left(\mathbf{z} + \begin{pmatrix} 0 \\ \pi_2(\mathbf{z}_0) \end{pmatrix}\right) = G(\mathbf{z})$$

for all $\mathbf{z} \in \mathbb{R}^2$. Hence the additive mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a periodic mapping which implies that G is a bounded mapping on $\mathbb{R}^2$. So, if there exists $\mathbf{w} \in \mathbb{R}^2$ such that $|G(\mathbf{w})| \neq \mathbf{0}$ then we can obtain

$$|G(n\mathbf{w})| = |nG(\mathbf{w})| \to \infty \quad \text{as} \quad n \to \infty$$

which contradicts the bounded property of G. Therefore, we have $G(\mathbf{z}) = \mathbf{0}$ for all $\mathbf{z} \in \mathbb{R}^2$. As a result we can have $T(\mathbf{z})=A\mathbf{z}$, where

$$A = \begin{pmatrix} \frac{1}{\pi_1(\mathbf{z}_0)} T_1 \binom{\pi_1(\mathbf{z}_0)}{0} & \frac{1}{\pi_2(\mathbf{z}_0)} T_1 \binom{0}{\pi_2(\mathbf{z}_0)} \\ \frac{1}{\pi_1(\mathbf{z}_0)} T_2 \binom{\pi_1(\mathbf{z}_0)}{0} & \frac{1}{\pi_2(\mathbf{z}_0)} T_2 \binom{0}{\pi_2(\mathbf{z}_0)} \end{pmatrix}.$$

Hence $T(\mathbf{z})=A\mathbf{z}$ is a linear transformation and we complete the proof. □

THEOREM 2.2. *For any $\varepsilon > 0$, there exists a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

$$|f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon(|\mathbf{x}| + |\mathbf{y}|) \tag{2.2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that there is no constant $\delta \geq 0$ and no additive function $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the condition

$$|f(\mathbf{x}) - T(\mathbf{x})| \leq \delta|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^2$. So, we can say about the instability of the Cauchy equation in $\mathbb{R}^2$.

Proof : Let $\mu := \frac{\varepsilon}{6\sqrt{2}}$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(x) := \begin{cases} \mu x & \text{for } |x| < 1 \\ \mu \frac{x}{|x|} & \text{for } |x| \geq 1. \end{cases}$$

Note that $|\phi(x)| \leq \mu$ for all $x \in \mathbb{R}$. Define $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = \big(f_1(x, y), f_2(x, y)\big) := \left(\sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n y)}{2^n} \right)$$

for all $(x, y) \in \mathbb{R}^2$, then $f(0, 0) = (0, 0)$. It is easily shown that

$$\begin{aligned} |f(x, y)|^2 = f_1(x, y)^2 + f_2(x, y)^2 &\leq \left(\sum_{n=0}^{\infty} \frac{\mu}{2^n} \right)^2 + \left(\sum_{n=0}^{\infty} \frac{\mu}{2^n} \right)^2 \\ &= 2\mu^2 \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \right)^2 = 8\mu^2 \end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$. That is, $|f(x, y)| \leq 2\sqrt{2}\mu$ for all $(x, y) \in \mathbb{R}^2$.

Now, we shall prove that there is a $\varepsilon > 0$ such that

(2.3)
$$\big|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)\big| \leq \varepsilon\big(|(x_1, x_2)| + |(y_1, y_2)|\big)$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$.

• `CASE` 1°: If $(x_1, x_2) = (y_1, y_2) = (0, 0)$, then (2.3) is fulfilled.

• `CASE` 2°: Assume that $0 < |(x_1, x_2)| + |(y_1, y_2)| < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} < |(x_1, x_2)| + |(y_1, y_2)| < \frac{1}{2^{N-1}}. \tag{2.4}$$

Hence, we see that

$$\begin{aligned}
&2^{N-1}|(x_1, x_2)| < 1, \ 2^{N-1}|(y_1, y_2)| < 1, \\
&2^{N-1}|(x_1, x_2) + (y_1, y_2)| \le 2^{N-1}(|(x_1, x_2)| + |(y_1, y_2)|) < 1.
\end{aligned}$$

Note that for $n = 0, 1, \cdots, N-1$ we have $2^n \le 2^{N-1}$. So we get the followings :

$$\begin{aligned}
&|2^n x_1|, |2^n x_2| \le |2^n(x_1, x_2)| = 2^n|(x_1, x_2)| < 2^{N-1}|(x_1, x_2)| < 1, \\
&|2^n y_1|, |2^n y_2| \le |2^n(y_1, y_2)| = 2^n|(y_1, y_2)| < 2^{N-1}|(y_1, y_2)| < 1
\end{aligned}$$

and

$$\begin{aligned}
&|2^n(x_1 + y_1)|, |2^n(x_2 + y_2)| \le |2^n(x_1 + y_1), 2^n(x_2 + y_2)| \\
&= 2^n|(x_1, x_2) + (y_1, y_2)| < 2^{N-1}|(x_1, x_2) + (y_1, y_2)| < 1.
\end{aligned}$$

Therefore, we have

$$2^n x_1, 2^n x_2, 2^n y_1, 2^n y_2, 2^n(x_1 + y_1), 2^n(x_2 + y_2) \in (-1, 1).$$

Since ϕ is linear on the interval $(-1, 1)$, we find that

$$\begin{aligned}
\phi(2^n(x_1 + y_1)) - \phi(2^n x_1) - \phi(2^n y_1) = 0, \\
\phi(2^n(x_2 + y_2)) - \phi(2^n x_2) - \phi(2^n y_2) = 0
\end{aligned} \tag{2.5}$$

for $n = 0, 1, \cdots, N-1$.

Now, we have

$$\begin{aligned}
&\frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\
&= \frac{1}{|(x_1, x_2)| + |(y_1, y_2)|} \cdot \left| \left(\sum_{n=0}^{\infty} \frac{\phi(2^n(x_1 + y_1))}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n(x_2 + y_2))}{2^n} \right) \right. \\
&\quad \left. - \left(\sum_{n=0}^{\infty} \frac{\phi(2^n x_1)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n x_2)}{2^n} \right) - \left(\sum_{n=0}^{\infty} \frac{\phi(2^n y_1)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n y_2)}{2^n} \right) \right|.
\end{aligned}$$

Thus we obtain that

$$
\begin{aligned}
&\frac{|f((x_1,x_2)+(y_1,y_2))-f(x_1,x_2)-f(y_1,y_2)|}{|(x_1,x_2)|+|(y_1,y_2)|}\\
&=\frac{1}{|(x_1,x_2)|+|(y_1,y_2)|}\\
&\quad\cdot\Bigg|\Bigg(\sum_{n=0}^{\infty}\frac{\phi(2^n(x_1+y_1))}{2^n}-\sum_{n=0}^{\infty}\frac{\phi(2^n x_1)}{2^n}-\sum_{n=0}^{\infty}\frac{\phi(2^n y_1)}{2^n},\\
&\quad\sum_{n=0}^{\infty}\frac{\phi(2^n(x_2+y_2))}{2^n}-\sum_{n=0}^{\infty}\frac{\phi(2^n x_2)}{2^n}-\sum_{n=0}^{\infty}\frac{\phi(2^n y_2)}{2^n}\Bigg)\Bigg|\\
&=\Bigg|\frac{1}{|(x_1,x_2)|+|(y_1,y_2)|}\Bigg(\sum_{n=0}^{\infty}\frac{\phi(2^n(x_1+y_1))-\phi(2^n x_1)-\phi(2^n y_1)}{2^n},\\
&\quad\sum_{n=0}^{\infty}\frac{\phi(2^n(x_2+y_2))-\phi(2^n x_2)-\phi(2^n y_2)}{2^n}\Bigg)\Bigg|\\
&=\Bigg|\Bigg(\sum_{n=0}^{\infty}\frac{\phi(2^n(x_1+y_1))-\phi(2^n x_1)-\phi(2^n y_1)}{2^n\big(|(x_1,x_2)|+|(y_1,y_2)|\big)},\\
&\quad\sum_{n=0}^{\infty}\frac{\phi(2^n(x_2+y_2))-\phi(2^n x_2)-\phi(2^n y_2)}{2^n\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg)\Bigg|.
\end{aligned}
$$

By (2.5), we see that

$$
\begin{aligned}
&\frac{|f((x_1,x_2)+(y_1,y_2))-f(x_1,x_2)-f(y_1,y_2)|}{|(x_1,x_2)|+|(y_1,y_2)|}\\
&=\Bigg|\Bigg(\sum_{n=N}^{\infty}\frac{\phi(2^n(x_1+y_1))-\phi(2^n x_1)-\phi(2^n y_1)}{2^n\big(|(x_1,x_2)|+|(y_1,y_2)|\big)},\\
&\quad\sum_{n=N}^{\infty}\frac{\phi(2^n(x_2+y_2))-\phi(2^n x_2)-\phi(2^n y_2)}{2^n\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg)\Bigg|\\
&=\Bigg|\Bigg(\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_1+y_1))-\phi(2^{k+N} x_1)-\phi(2^{k+N} y_1)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)},\\
&\quad\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_2+y_2))-\phi(2^{k+N} x_2)-\phi(2^{k+N} y_2)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg)\Bigg|.
\end{aligned}
$$

So, we obtain that

$$\frac{|f((x_1,x_2)+(y_1,y_2))-f(x_1,x_2)-f(y_1,y_2)|}{|(x_1,x_2)|+|(y_1,y_2)|}$$
$$=\Bigg(\Bigg[\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_1+y_1))-\phi(2^{k+N}x_1)-\phi(2^{k+N}y_1)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg]^2$$
$$+\Bigg[\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_2+y_2))-\phi(2^{k+N}x_2)-\phi(2^{k+N}y_2)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg]^2\Bigg)^{\frac{1}{2}}$$
$$=\Bigg(\Bigg|\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_1+y_1))-\phi(2^{k+N}x_1)-\phi(2^{k+N}y_1)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg|^2$$
$$+\Bigg|\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_2+y_2))-\phi(2^{k+N}x_2)-\phi(2^{k+N}y_2)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg|^2\Bigg)^{\frac{1}{2}}.$$

Note that

$$\Bigg|\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_1+y_1))-\phi(2^{k+N}x_1)-\phi(2^{k+N}y_1)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg|^2$$
$$\le\Bigg[\sum_{k=0}^{\infty}\Bigg|\frac{\phi(2^{k+N}(x_1+y_1))-\phi(2^{k+N}x_1)-\phi(2^{k+N}y_1)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg|\Bigg]^2$$
$$=\Bigg[\sum_{k=0}^{\infty}\frac{\big|\phi(2^{k+N}(x_1+y_1))-\phi(2^{k+N}x_1)-\phi(2^{k+N}y_1)\big|}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg]^2$$
$$\le\Bigg[\sum_{k=0}^{\infty}\frac{\big|\phi(2^{k+N}(x_1+y_1))\big|+\big|\phi(2^{k+N}x_1)\big|+\big|\phi(2^{k+N}y_1)\big|}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg]^2$$
$$\le 9\mu^2\Bigg[\sum_{k=0}^{\infty}\frac{1}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg]^2.$$

By the same argument as above, we also obtain that

$$\Bigg|\sum_{k=0}^{\infty}\frac{\phi(2^{k+N}(x_2+y_2))-\phi(2^{k+N}x_2)-\phi(2^{k+N}y_2)}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg|^2$$
$$\le 9\mu^2\Bigg[\sum_{k=0}^{\infty}\frac{1}{2^{k+N}\big(|(x_1,x_2)|+|(y_1,y_2)|\big)}\Bigg]^2.$$

Hence we obtain that

$$\begin{aligned}&\frac{|f((x_1,x_2)+(y_1,y_2))-f(x_1,x_2)-f(y_1,y_2)|}{|(x_1,x_2)|+|(y_1,y_2)|}\\&=\sum_{k=0}^{\infty}\frac{3\sqrt{2}\mu}{2^{k+N}(|(x_1,x_2)|+|(y_1,y_2)|)}<\sum_{k=0}^{\infty}\frac{3\sqrt{2}\mu}{2^k}=6\sqrt{2}\mu=\varepsilon.\end{aligned}$$

• `CASE` 3°: We finally assume that $|(x_1,x_2)|+|(y_1,y_2)|\geq 1$. Then, we get

$$\begin{aligned}&\frac{|f((x_1,x_2)+(y_1,y_2))-f(x_1,x_2)-f(y_1,y_2)|}{|(x_1,x_2)|+|(y_1,y_2)|}\\&\leq|f((x_1,x_2)+(y_1,y_2))-f(x_1,x_2)-f(y_1,y_2)|\\&\leq 6\sqrt{2}\mu=\varepsilon.\end{aligned}$$

By CASE 1°, CASE 2° and CASE 3°, we have the inequality (2.3) for all $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$.

To complete the proof we assume that there exists a $\delta\geq 0$ and an additive mapping $T:\mathbb{R}^2\to\mathbb{R}^2$ such that

$$|f(x_1,x_2)-T(x_1,x_2)|\leq\delta|(x_1,x_2)|,\quad\text{for all }(x_1,x_2)\in\mathbb{R}^2,$$

and prove that the assumption is contradictory. Since f_1 and f_2 are defined by means of a uniformly convergent series of continuous functions, they are continuous. So f is continuous. Hence the mapping $T=(T_1,T_2)$ is bounded on a neighborhood of $(0,0)$. Then, by Theorem 2.1 there exists $A\in M_2(\mathbb{R})$ such that

$$T(x_1,x_2)=A\begin{pmatrix}x_1\\x_2\end{pmatrix},\ (x_1,x_2)\in\mathbb{R}^2,$$

where

$$A=\begin{pmatrix}\frac{T_1(x_0,0)}{x_0}&\frac{T_1(y_0,0)}{y_0}\\\frac{T_2(x_0,0)}{x_0}&\frac{T_2(y_0,0)}{y_0}\end{pmatrix}$$

for some (x_0,y_0) containing in a neighborhood of $(0,0)$. Hence, we get

$$\left|f(x_1,x_2)-A\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|\leq\delta|(x_1,x_2)|,\ (x_1,x_2)\in\mathbb{R}^2.$$

So, we see that

$$|f(x_1,x_2)|\leq\delta|(x_1,x_2)|+\left|A\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|.$$

This implies that

$$\frac{|f(x_1,x_2)|}{|(x_1,x_2)|} \le \delta + \frac{\left|A\binom{x_1}{x_2}\right|}{|(x_1,x_2)|} \le \delta + \|A\| \tag{2.6}$$

for all $(x_1,x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. On the other hand, by the Archimedean property, we can choose $N \in \mathbb{N}$ such that $N\mu > \delta + \|A\|$. So

$$2^n x_1 \in (0,1) \text{ for } n = 0,1,\cdots,N-1 \text{ if } x_1 \in \Big(0, \frac{1}{2^{N-1}}\Big),$$
$$2^n x_2 \in (0,1) \text{ for } n = 0,1,\cdots,N-1 \text{ if } x_2 \in \Big(0, \frac{1}{2^{N-1}}\Big).$$

For $x_1, x_2 \in \big(0, \frac{1}{2^{N-1}}\big)$, we get

$$\begin{aligned}
&\frac{|f(x_1,x_2)|}{|(x_1,x_2)|} \\
&= \frac{1}{|(x_1,x_2)|}\left|\left(\sum_{n=0}^{\infty}\frac{\phi(2^n x_1)}{2^n}, \sum_{n=0}^{\infty}\frac{\phi(2^n x_2)}{2^n}\right)\right| \\
&= \left|\left(\sum_{n=0}^{\infty}\frac{\phi(2^n x_1)}{2^n|(x_1,x_2)|}, \sum_{n=0}^{\infty}\frac{\phi(2^n x_2)}{2^n|(x_1,x_2)|}\right)\right| \\
&= \left|\left(\sum_{n=0}^{N-1}\frac{\mu 2^n x_1}{2^n|(x_1,x_2)|} + \sum_{n=N}^{\infty}\frac{\phi(2^n x_1)}{2^n|(x_1,x_2)|},\right.\right. \\
&\qquad \left.\left.\sum_{n=0}^{N-1}\frac{\mu 2^n x_2}{2^n|(x_1,x_2)|} + \sum_{n=N}^{\infty}\frac{\phi(2^n x_2)}{2^n|(x_1,x_2)|}\right)\right|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{|f(x_1,x_2)|}{|(x_1,x_2)|} \\
&= \left(\left[\sum_{n=0}^{N-1}\frac{\mu 2^n x_1}{2^n|(x_1,x_2)|} + \sum_{n=N}^{\infty}\frac{\phi(2^n x_1)}{2^n|(x_1,x_2)|}\right]^2\right. \\
&\qquad \left. + \left[\sum_{n=0}^{N-1}\frac{\mu 2^n x_2}{2^n|(x_1,x_2)|} + \sum_{n=N}^{\infty}\frac{\phi(2^n x_2)}{2^n|(x_1,x_2)|}\right]^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{|f(x_1,x_2)|}{|(x_1,x_2)|}\\
&\geq \left(\left[\sum_{n=0}^{N-1} \frac{\mu 2^n x_1}{2^n|(x_1,x_2)|} \right]^2 + \left[\sum_{n=0}^{N-1} \frac{\mu 2^n x_2}{2^n|(x_1,x_2)|} \right]^2 \right)^{\frac{1}{2}}\\
&= \left(\left[\sum_{n=0}^{N-1} \frac{\mu x_1}{|(x_1,x_2)|} \right]^2 + \left[\sum_{n=0}^{N-1} \frac{\mu x_2}{|(x_1,x_2)|} \right]^2 \right)^{\frac{1}{2}}\\
&= \frac{N\mu}{|(x_1,x_2)|}\left(x_1^2+x_2^2\right)^{\frac{1}{2}} = N\mu > \delta + \|A\|,
\end{aligned}$$

which contradicts to (2.6). □

REMARK 2.3. *Theorem 2.1 and Theorem 2.2 can be generalized for* $\mathbb{R}^n$ $(n \in \mathbb{N})$ *as follows.*

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ *be a mapping satisfying the Cauchy equation*

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *If* T *is bounded on a small disk* $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^n \mid |\mathbf{z} - \mathbf{x}| < r\}$ *for some* $\mathbf{x} \in \mathbb{R}^n$, *then there exists an* $n \times n$ *matrix* A *such that* $T(\mathbf{u}) = A\mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{R}^n$. *Therefore, the mapping* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear mapping.*

(2) For any $\varepsilon > 0$, *there exists a mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying*

$$|f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon(|\mathbf{x}| + |\mathbf{y}|)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *such that there is no constant* $\delta \geq 0$ *and no additive function* $T : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the condition*

$$|f(\mathbf{x}) - T(\mathbf{x})| \leq \delta|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^n$. *So, we can say about the instability of the Cauchy equation in* $\mathbb{R}^n$.

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Won-Gil Park
Department of Mathematics Education,
Mokwon University
Daejeon 35349, Korea.
E-mail: `wgpark@mokwon.ac.kr`

Ick-Soon Chang
Department of Mathematics,
Chungnam National University
Daejeon 34134, Korea.
E-mail: `ischang@cnu.ac.kr`

Jaiok Roh*
Ilsong Liberal Art Schools (Mathematics), Hallym University,
Chuncheon 24252, Korea.
E-mail: `joroh@hallym.ac.kr`