JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 4, November 2023
http://dx.doi.org/10.14403/jcms.2023.36.4.255

CONSTRUCTION OF AN EIGHT DIMENSIONAL NONALTERNATIVE, NONCOMMUTATIVE ALGEBRA

Youngkwon Song

Abstract. The purpose of this article is to construct a unital 8 dimensional hypercomplex number system H_8^* that is neither alternative nor commutative unlike the octonions by means of the unital 4 dimensional, commutative, and nonassociative hypercomplex number system H^*. We also establish some algebraic properties related to H_8^* and compare to those of octonions.

1. Introduction

Number systems $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \cdots$ can be extended from the previous number system. But, $\mathbb{H} \subset \mathbb{O} \subset \cdots$ doesn't guarantee the commutative property. In [5], we introduced the commutative number systems.

Properties of quaternions and matrices of quaternions was published [6], but the noncommutativity was a big obstacle to expand the theory. In 1892, Segre introduced commutative quaternions [3] and various properties about commutative quaternions have been established [1,2,3]. Also, we introduced modified 4 dimensional commutative quaternions H^* which is not associative [4].

As is well known, octonions $\mathbb{O}$ was constructed by means of Cayley-Dickson construction with quaternions $\mathbb{H}$. Note that $\mathbb{O}$ is a unital 8 dimensional, nonalternative, noncommutative, division algebra.

Recall the 4 dimensional, nonassociative, commutative quaternions H^*.

Received August 16, 2023; Accepted October 30, 2023.
2020 Mathematics Subject Classification: 15A18, 15B33 .
Key words and phrases: commutative quaternions, octonions.
The present research has been conducted by the Research Grant of Kwangwoon University in 2023.

Definition 1.1. ([4]) Let
$H^* = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R},\, i, j, k \notin \mathbb{R}\}$, where

$$i^2 = k^2 = -1,\ j^2 = 1,\ ij = ji = k,\ jk = kj = -i,\ ki = ik = j.$$

Then, H^* is a unital 4 dimensional, nonassociative, commutative $\mathbb{Z}_2$-graded algebra. Moreover, H^* is neither an alternative algebra nor a division algebra.

In this paper, we will construct a unital 8 dimensional, nonalternative, noncommutative hypercomplex number system H_8^* by means of the algebra H^*.

2. Eight dimensional nonalternative, noncommutative algebra

In this section, we will construct a unital 8 dimensional nonalternative, noncommutative, nondivision algebra by using the 4 dimensional nonalternative, commutative, nondivision algebra H^*.

Let $\alpha = a_0 + a_1 i + a_2 j + a_3 k \in H^*$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Instead of defining the conjugate of $\alpha \in H^*$ by $a_0 - a_1 i - a_2 j - a_3 k$, we simply define the conjugate $\alpha^{(1)}$ of α as follows:

Definition 2.1. Let $\alpha = a_0 + a_1 i + a_2 j + a_3 k \in H^*$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Then, we define $\alpha^{(1)} = a_0 - a_1 i - a_2 j + a_3 k$ as the conjugate of α.

Note that $\alpha = \alpha_1 + \alpha_2 k$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$ and

$$\alpha^{(1)} = (a_0 - a_1 i) + (a_3 - a_2 i)k = \overline{\alpha_1} + \overline{\alpha_2} k,$$

where $\overline{\alpha_t}$ is the complex conjugate of α_t, $t = 1, 2$.

The conjugate is an involution satisfying the following facts.

Proposition 2.2. *Let $\alpha, \beta \in H^*$. Then,*

(1) $\alpha^{(1)} = \alpha$ *if and only if* $\alpha \in \mathbb{R}$.

(2) $(\alpha^{(1)})^{(1)} = \alpha$.

(3) $(\alpha + \beta)^{(1)} = \alpha^{(1)} + \beta^{(1)}$.

(4) $(a\alpha)^{(1)} = a\alpha^{(1)}$ *for all* $a \in \mathbb{R}$.

(5) $(\alpha\beta)^{(1)} = \alpha^{(1)}\beta^{(1)}$.

Starting with the unital 4 dimensional, nonassociative, commutative, nondivision $\mathbb{Z}_2$-graded algebra H^*, a unital 8 dimensional nonalternative, noncommutative, nondivision $\mathbb{Z}_2$-graded algebra H_8^* is constructed as follows:

DEFINITION 2.3. Let $H_8^* = \{(\alpha, \beta) | \alpha, \beta \in H^*\}$ and define the operations on the set H_8^* as follows:

$$(\alpha, \beta) + (\gamma, \eta) = (\alpha + \gamma, \beta + \eta)$$
$$a(\alpha, \beta) = (a\alpha, a\beta)$$
$$(\alpha, \beta)(\gamma, \eta) = (\alpha\gamma - \beta\eta^{(1)}, \alpha\eta + \beta\gamma^{(1)})$$

for all elements $(\alpha, \beta), (\gamma, \eta) \in H_8^*$ and $a \in \mathbb{R}$.

PROPOSITION 2.4. *H_8^* is a unital 8 dimensional nonalternative, noncommutative, and nondivision algebra.*

Proof. Obviously, the set $\mathcal{B} = \{(1,0), (i,0), (j,0), (k,0), (0,1), (0,i), (0,j), (0,k)\}$ is a basis of H_8^*. Note that

$$\{(i,j)(i,j)\}\,(i,i) = (ii - j(-j), ij + j(-i))(i,i) = (0,0)(i,i) = (0,0)$$

$$\begin{aligned} (i,j)\,\{(i,j)(i,i)\} &= (i,j)(ii - j(-i), ii + j(-i)) = (i,j)(-1+k, -1-k) \\ &= (i(-1+k) - j(-1-k), i(-1-k) + j(-1+k)) \\ &= 2(-i+j, -i-j) \end{aligned}$$

Thus, H_8^* is not alternative. Also, H_8^* is not commutative since

$$(0,1)(0,i) = (i,0) \neq (-i,0) = (0,i)(0,1)$$

To show H_8^* is not a division algebra, consider the element $(1+j, 0) \in H_8^*$. Then,

$$(1+j,0)(\alpha,\beta) = ((1+j)\alpha, (1+j)\beta^{(1)}) = (1,0)$$

implies that

$$(1+j)\alpha = 1, \quad (1+j)\beta^{(1)} = 0$$

If we let $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ and $\beta = b_0 + b_1 i + b_2 j + b_3 k$, then

$$a_0 + a_2 = 1, \;\; a_1 - a_3 = 0, \;\; a_0 + a_2 = 0, \;\; a_1 + a_3 = 0,$$

which is impossible. Thus, the element $(1+j, 0)$ has no multiplicative inverse. Consequently, H_8^* is a unital 8 dimensional, nonalternative, noncommutative, and nondivision algebra. □

The multiplication table of basis members of the algebra H_8^* is as follows:

	$(1,0)$	$(i,0)$	$(j,0)$	$(k,0)$	$(0,1)$	$(0,i)$	$(0,j)$	$(0,k)$
$(1,0)$	$(1,0)$	$(i,0)$	$(j,0)$	$(k,0)$	$(0,1)$	$(0,i)$	$(0,j)$	$(0,k)$
$(i,0)$	$(i,0)$	$(-1,0)$	$(k,0)$	$(j,0)$	$(0,i)$	$(0,-1)$	$(0,k)$	$(0,j)$
$(j,0)$	$(j,0)$	$(k,0)$	$(1,0)$	$(-i,0)$	$(0,j)$	$(0,k)$	$(0,1)$	$(0,-i)$
$(k,0)$	$(k,0)$	$(j,0)$	$(-i,0)$	$(-1,0)$	$(0,k)$	$(0,j)$	$(0,-i)$	$(0,-1)$
$(0,1)$	$(0,1)$	$(0,-i)$	$(0,-j)$	$(0,k)$	$(-1,0)$	$(i,0)$	$(j,0)$	$(-k,0)$
$(0,i)$	$(0,i)$	$(0,1)$	$(0,-k)$	$(0,j)$	$(-i,0)$	$(-1,0)$	$(k,0)$	$(-j,0)$
$(0,j)$	$(0,j)$	$(0,-k)$	$(0,-1)$	$(0,-i)$	$(-j,0)$	$(k,0)$	$(1,0)$	$(i,0)$
$(0,k)$	$(0,k)$	$(0,-j)$	$(0,i)$	$(0,-1)$	$(-k,0)$	$(j,0)$	$(-i,0)$	$(1,0)$

Let $e_1 = (1,0)$, $e_2 = (i,0)$, $e_3 = (j,0)$, $e_4 = (k,0)$, $e_5 = (0,1)$, $e_6 = (0,i)$, $e_7 = (0,j)$, $e_8 = (0,k)$. Then we have

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	e_3	e_6	$-e_5$	e_8	e_7
e_3	e_3	e_4	e_1	$-e_2$	e_7	e_8	e_5	$-e_6$
e_4	e_4	e_3	$-e_2$	$-e_1$	e_8	e_7	$-e_6$	$-e_5$
e_5	e_5	$-e_6$	$-e_7$	e_8	$-e_1$	e_2	e_3	$-e_4$
e_6	e_6	e_5	$-e_8$	e_7	$-e_2$	$-e_1$	e_4	$-e_3$
e_7	e_7	$-e_8$	$-e_5$	$-e_6$	$-e_3$	e_4	e_1	e_2
e_8	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	$-e_2$	e_1

Theorem 2.5. *Let R_0 and R_1 be two sets defined by*

$$R_0 = \left\{ \sum_{p,q,r,s=1}^{4} a e_p e_q e_r e_s \mid a \in \mathbb{R} \right\}, \qquad R_1 = \left\{ \sum_{p,q,r,s=5}^{8} a e_p e_q e_r e_s \mid a \in \mathbb{R} \right\}.$$

Then, $R_0 \cong H^*$ and $R_1 = H_8^*$. Moreover, H_8^* is a $\mathbb{Z}_2$-graded algebra.

The $\mathbb{Z}_2$-graded algebra H_8^* is not alternative, but the subset

$$V = \left\{ \sum_{p=5}^{8} a_p e_p | a_p \in \mathbb{R}, p = 5,6,7,8 \right\}$$

satisfies similar property.

THEOREM 2.6. *Let* $V = \left\{\sum_{p=5}^{8} a_p e_p | a_p \in \mathbb{R}, p = 5,6,7,8\right\}$. *Then,*

$$(e_p e_p)e_q = e_p(e_p e_q) \quad \text{or} \quad (e_p e_p)e_q = -e_p(e_p e_q)$$

for all $e_p, e_q \in R_1$, $5 \leq p, q \leq 8$.

Proof. The proof is straightforward.

$$(e_5e_5)e_6 = -e_6 = e_5(e_5e_6),\ \ (e_5e_5)e_7 = -e_7 = e_5(e_5e_7),\ \ (e_5e_5)e_8 = -e_8 = e_5(e_5e_8)$$

$$(e_6e_6)e_5 = -e_5 = e_6(e_6e_5),\ \ (e_6e_6)e_7 = -e_7,\ \ e_6(e_6e_7) = e_7,\ \ (e_6e_6)e_8 = -e_8,\ \ e_6(e_6e_8) = e_8$$

$$(e_7e_7)e_5 = e_5 = e_7(e_7e_5),\ \ (e_7e_7)e_6 = e_6,\ \ e_7(e_7e_6) = -e_6,\ \ (e_7e_7)e_8 = e_8,\ \ e_7(e_7e_8) = -e_8,$$

$$(e_8e_8)e_5 = e_5 = e_8(e_8e_5),\ \ (e_8e_8)e_6 = e_6 = e_8(e_8e_6),\ \ (e_8e_8)e_7 = e_7 = e_8(e_8e_7)$$

□

Now, we will construct a real matrix representation of elements in H_8^*.

LEMMA 2.7. *Let* $\alpha, \beta \in H^*$ *and let* $\alpha = a_0 + a_1 i + a_2 j + a_3 k$, $\beta = b_0 + b_1 i + b_2 j + b_3 k$. *Then,*

$$\begin{aligned}
(\alpha,\beta)(1,0) &= (\alpha,\beta) = (a_0 + a_1 i + a_2 j + a_3 k,\ b_0 + b_1 i + b_2 j + b_3 k)\\
(\alpha,\beta)(i,0) &= (\alpha i, \beta i^{(1)}) = (\alpha i, -\beta i)\\
&= (-a_1 + a_0 i + a_3 j + a_2 k,\ b_1 - b_0 i - b_3 j - b_2 k)\\
(\alpha,\beta)(j,0) &= (\alpha j, \beta j^{(1)}) = (\alpha j, -\beta j)\\
&= (a_2 - a_3 i + a_0 j + a_1 k,\ -b_2 + b_3 i - b_0 j - b_1 k)\\
(\alpha,\beta)(k,0) &= (\alpha k, \beta k^{(1)}) = (\alpha k, \beta k)\\
&= (-a_3 - a_2 i + a_1 j + a_0 k,\ -b_3 - b_2 i + b_1 j + b_0 k)\\
(\alpha,\beta)(0,1) &= (-\beta, \alpha) = (-b_0 - b_1 i - b_2 j - b_3 k,\ a_0 + a_1 i + a_2 j + a_3 k)\\
(\alpha,\beta)(0,i) &= (-\beta i^{(1)}, \alpha i) = (\beta i, \alpha i)\\
&= (-b_1 + b_0 i + b_3 j + b_2 k,\ -a_1 + a_0 i + a_3 j + a_2 k)\\
(\alpha,\beta)(0,j) &= (-\beta j^{(1)}, \alpha j) = (\beta j, \alpha j)\\
&= (b_2 - b_3 i + b_0 j + b_1 k,\ a_2 - a_3 i + a_0 j + a_1 k)\\
(\alpha,\beta)(0,k) &= (-\beta k^{(1)}, \alpha k) = (-\beta k, \alpha k)\\
&= (b_3 + b_2 i - b_1 j - b_0 k,\ -a_3 - a_2 i + a_1 j + a_0 k)
\end{aligned}$$

Define the map $\phi^{(1)} : H_8^* \longrightarrow M_{8\times 8}(\mathbb{R})$ by

$$\phi^{(1)}(\alpha,\beta)=\begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & -b_0 & -b_1 & b_2 & b_3 \\ a_1 & a_0 & -a_3 & -a_2 & -b_1 & b_0 & -b_3 & b_2 \\ a_2 & a_3 & a_0 & a_1 & -b_2 & b_3 & b_0 & -b_1 \\ a_3 & a_2 & a_1 & a_0 & -b_3 & b_2 & b_1 & -b_0 \\ b_0 & b_1 & -b_2 & -b_3 & a_0 & -a_1 & a_2 & -a_3 \\ b_1 & -b_0 & b_3 & -b_2 & a_1 & a_0 & -a_3 & -a_2 \\ b_2 & -b_3 & -b_0 & b_1 & a_2 & a_3 & a_0 & a_1 \\ b_3 & -b_2 & -b_1 & b_0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

for all elements $(\alpha,\beta)\in H_8^*$. Then,

$$\phi^{(1)}(\alpha,\beta)=\begin{pmatrix} \phi_1(\alpha) & -\phi_1(\beta)G_1 \\ \phi_1(\beta)G_1 & \phi_1(\alpha) \end{pmatrix},$$

where

$$\phi_1(\alpha)=\begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 \\ a_1 & a_0 & -a_3 & -a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \quad \text{and} \quad G_1=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2.8. *Let $\alpha_t,\beta_t\in H^*$, $t=1,2$ and $a\in\mathbb{R}$. Then,*

(1) $\phi^{(1)}(a(\alpha_1,\beta_1))=a\phi^{(1)}(\alpha_1,\beta_1)$.

(2) $\phi^{(1)}(\alpha_1+\alpha_2,\beta_1+\beta_2)=\phi^{(1)}(\alpha_1,\beta_1)+\phi^{(1)}(\alpha_2,\beta_2)$.

(3) $\phi^{(1)}((\alpha_1,\beta_1)(\alpha_2\beta_2))\neq\phi^{(1)}(\alpha_1,\beta_1)\phi^{(1)}(\alpha_2,\beta_2)$ in general.

Proof. (1) and (2) are obvious.
For (3), $\phi^{(1)}((0,1)(0,1))=\phi^{(1)}((-1,0))=-I_8\neq\phi^{(1)}(0,1)\phi^{(1)}(0,1)$. □

Theorem 2.9. *Let $\alpha,\beta\in H^*$ and let $\alpha=a_0+a_1i+a_2j+a_3k$, $\beta=b_0+b_1i+b_2j+b_3k$. Then,*

(1) If $\phi_1(\alpha)$ is invertible if and only if $\phi^{(1)}(\alpha,0)$ is invertible.

(2) If $\phi_1(\beta)$ is invertible if and only if $\phi^{(1)}(0,\beta)$ is invertible.

(3) If $\phi^{(1)}(\alpha,\beta)$ is invertible and $\phi_1(\alpha)=0$, then $\phi_1(\beta)$ is invertible.

(4) If $\phi^{(1)}(\alpha,\beta)$ is invertible and $\phi_1(\beta)=0$, then $\phi_1(\alpha)$ is invertible.

(5) $tr(\phi^{(1)}(\alpha,\beta))=8a_0=8Re(\alpha)=2tr(\phi_1(\alpha))$.

Proof. (1) If $\phi_1(\alpha)$ is invertible, then

$$\phi^{(1)}(\alpha,0)\begin{pmatrix} \phi_1(\alpha)^{-1} & O \\ O & \phi_1(\alpha)^{-1} \end{pmatrix} = I_8 = \begin{pmatrix} \phi_1(\alpha)^{-1} & O \\ O & \phi_1(\alpha)^{-1} \end{pmatrix}\phi^{(1)}(\alpha,0)$$

and thus $\phi^{(1)}(\alpha,0)$ is invertible.

Conversely, if $\phi^{(1)}(\alpha,0)$ is invertible, then

$$\begin{pmatrix} \phi_1(\alpha) & O \\ O & \phi_1(\alpha) \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = I_8 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} \phi_1(\alpha) & O \\ O & \phi_1(\alpha) \end{pmatrix}$$

for some $C_{11}, C_{12}, C_{21}, C_{22} \in M_{4\times 4}(\mathbb{R})$. Thus, $\phi_1(\alpha)C_{11} = I_4$ and $\phi_1(\alpha)$ is invertible.

(2) If $\phi_1(\beta)$ is invertible, then

$$\phi^{(1)}(0,\beta)\begin{pmatrix} O & G_1^{-1}\phi_1(\beta)^{-1} \\ -G_1^{-1}\phi_1(\beta)^{-1} & O \end{pmatrix} = I_8$$

and thus $\phi^{(1)}(0,\beta)$ is invertible.

Conversely, if $\phi^{(1)}(0,\beta)$ is invertible, then

$$\begin{pmatrix} O & -\phi_1(\beta)G_1 \\ \phi_1(\beta)G_1 & O \end{pmatrix}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = I_8$$

for some $D_{11}, D_{12}, D_{21}, D_{22} \in M_{4\times 4}(\mathbb{R})$. Thus, $\phi_1(\beta)G_1D_{12} = I_4$ and $\phi_1(\beta)$ is invertible.

(3) If $\phi^{(1)}(\alpha,\beta)$ is invertible and $\phi_1(\alpha) = 0$, then $\phi^{(1)}(0,\beta)$ is invertible. Thus, by (2), $\phi_1(\beta)$ is invertible.

(4) If $\phi^{(1)}(\alpha,\beta)$ is invertible and $\phi_1(\beta) = 0$, then $\phi^{(1)}(\alpha,0)$ is invertible. Thus, by (1), $\phi_1(\alpha)$ is invertible.

(5) is obvious by the definition of $\phi^{(1)}(\alpha,\beta)$. □

3. Matrices in $M_{n\times n}(H_8^*)$

Let $\psi^{(1)} : M_{n\times n}(H_8^*) \longrightarrow M_{8n\times 8n}(\mathbb{R})$ be the map defined by $\psi^{(1)}(A) = C$, where $A = (A_{st})_{n\times n}$, $A_{st} = (\alpha_{st}, \beta_{st})$, $C = (C_{st})_{8n\times 8n}$, and

$$C_{st} = \phi^{(1)}(A_{st}) = \begin{pmatrix} \phi_1(\alpha_{st}) & -\phi_1(\beta_{st})G_1 \\ \phi_1(\beta_{st})G_1 & \phi_1(\alpha_{st}) \end{pmatrix}$$

the $(s,t)-th$ 8×8 block matrix.

DEFINITION 3.1. Let $A, B \in M_{n\times n}(H_8^*)$. Then, the $8n \times 8n$ matrix $\psi^{(1)}(A)$ is called the adjoint matrix of A.

DEFINITION 3.2. Let $A, B \in M_{n\times n}(H_8^*)$. Then, we define the determinant of the matrix A by the determinant of $\psi^{(1)}(A)$.

THEOREM 3.3. *Let $A, B \in M_{n\times n}(H_8^*)$. Then,*

(1) $\psi^{(1)}(aA) = a\psi^{(1)}(A)$ *for all* $a \in \mathbb{R}$.

(2) $\psi^{(1)}(A+B) = \psi^{(1)}(A) + \psi^{(1)}(B)$.

(3) $\psi^{(1)}(AB) \neq \psi^{(1)}(A)\psi^{(1)}(B)$ *in general.*

(4) $\det(aA) = a^{8n}\det(A)$.

(5) $\det(AB) \neq \det(A)\det(B)$ *for* $n \geq 2$ *in general.*

Proof. Let $A = (A_{st})_{n\times n}$, $A_{st} = (\alpha_{st}, \beta_{st})$, $B = (B_{st})_{n\times n}$, $B_{st} = (\gamma_{st}, \delta_{st})$ for some $\alpha_{st}, \beta_{st}, \gamma_{st}, \delta_{st} \in H_8^*$. Then,

(1) The $(s,t) - th$ 8×8 block matrix of $\psi^{(1)}(aA)$ is

$$\begin{aligned}\phi^{(1)}(aA_{st}) &= \phi^{(1)}(a(\alpha_{st}, \beta_{st})) = \phi^{(1)}(a\alpha_{st}, a\beta_{st}) = a\phi^{(1)}(\alpha_{st}, \beta_{st})\\ &= a\phi^{(1)}(A_{st})\end{aligned}$$

Since the $(s,t) - th$ 8×8 block matrix of $a\psi^{(1)}(A)$ is $a\phi^{(1)}(A_{st})$, we have $\psi^{(1)}(aA) = a\psi^{(1)}(A)$.

(2) The $(s,t) - th$ 8×8 block matrix of $\psi^{(1)}(A+B)$ is

$$\begin{aligned}\phi^{(1)}(A_{st} + B_{st}) &= \phi^{(1)}(\alpha_{st} + \gamma_{st}, \beta_{st} + \delta_{st})\\ &= \begin{pmatrix} \phi_1(\alpha_{st} + \gamma_{st}) & -\phi_1(\beta_{st} + \delta_{st})G_1 \\ \phi_1(\beta_{st} + \delta_{st})G_1 & \phi_1(\alpha_{st} + \gamma_{st}) \end{pmatrix}\\ &= \begin{pmatrix} \phi_1(\alpha_{st}) & -\phi_1(\beta_{st})G_1 \\ \phi_1(\beta_{st})G_1 & \phi_1(\alpha_{st}) \end{pmatrix} + \begin{pmatrix} \phi_1(\gamma_{st}) & -\phi_1(\delta_{st})G_1 \\ \phi_1(\delta_{st})G_1 & \phi_1(\gamma_{st}) \end{pmatrix}\\ &= \phi^{(1)}(\alpha_{st}, \beta_{st}) + \phi^{(1)}(\gamma_{st}, \delta_{st}) = \phi^{(1)}(A_{st}) + \phi^{(1)}(B_{st})\end{aligned}$$

which is the $(s,t) - th$ 8×8 block matrix of $\psi^{(1)}(A) + \psi^{(1)}(B)$.

(3) Let $A = (i,0)I_n$ and $B = (j,0)I_n$. Then, $AB = (k,0)I_n$ and

$$\begin{aligned}\psi^{(1)}(A) &= \psi^{(1)}((i,0)I_n) = \psi^{(1)}(diag((i,0),(i,0),\cdots,(i,0)))\\ &= diag\left(\begin{pmatrix} \phi_1(i) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(i) \end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(i) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(i), \end{pmatrix}\right)\\ &= diag\left(\begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i) \end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i), \end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}\psi^{(1)}(B) &= \psi^{(1)}((j,0)I_n) = \psi^{(1)}(diag((j,0),(j,0),\cdots,(j,0))) \\ &= diag\left(\begin{pmatrix} \phi_1(j) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(j) \end{pmatrix},\cdots,\begin{pmatrix} \phi_1(j) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(j), \end{pmatrix}\right) \\ &= diag\left(\begin{pmatrix} \phi_1(j) & O \\ O & \phi_1(j) \end{pmatrix},\cdots,\begin{pmatrix} \phi_1(j) & O \\ O & \phi_1(j), \end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}\psi^{(1)}(AB) &= \psi^{(1)}((k,0)I_n) = \psi^{(1)}(diag((k,0),(k,0),\cdots,(k,0))) \\ &= diag\left(\begin{pmatrix} \phi_1(k) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(k) \end{pmatrix},\cdots,\begin{pmatrix} \phi_1(k) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(k), \end{pmatrix}\right) \\ &= diag\left(\begin{pmatrix} \phi_1(k) & O \\ O & \phi_1(k) \end{pmatrix},\cdots,\begin{pmatrix} \phi_1(k) & O \\ O & \phi_1(k), \end{pmatrix}\right)\end{aligned}$$

Since $\phi_1(i)\phi_1(j) \neq \phi_1(k)$, we have $\psi^{(1)}(AB) \neq \psi^{(1)}(A)\psi^{(1)}(B)$.

(4) $\det(aA) = \det(\psi^{(1)}(aA)) = \det(a\psi^{(1)}(A)) = a^{8n}\det(\psi^{(1)}(A))$
$= a^{8n}\det(A)$.

(5) Let $A = (A_{st})_{n\times n}$, $A_{st} = (0,1)$, $C = (C_{st})_{8n\times 8n}$. Then,

$$C_{st} = \phi^{(1)}(A_{st}) = \phi^{(1)}(0,1) = \begin{pmatrix} \phi_1(0) & -\phi_1(1)G_1 \\ \phi_1(1)G_1 & \phi_1(0) \end{pmatrix} = \begin{pmatrix} O & -G_1 \\ G_1 & O \end{pmatrix}$$

and so $\det(A) = \det(\psi^{(1)}(A)) = \det(C) = 0$.
Also, $AA = nB$, where $B = (B_{st})_{n\times n}$, $B_{st} = (1,0)$. Hence

$$\det(AA) = \det(\psi^{(1)}(AA)) = \det(\psi^{(1)}(nB)) = n^{8n}\det(I_{8n}) = n^{8n}.$$

Thus, $\det(AA) \neq \det(A)\det(A)$. □

Note that every element $\alpha \in H^*$ can be uniquely expressed as $\alpha = \alpha_1 + \alpha_2 k$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$. Thus, every element $(\alpha,\beta) \in H_8^*$ can be uniquely expressed as $(\alpha,\beta) = (\alpha_1 + \alpha_2 k, \beta_1 + \beta_2 k)$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. Let

$$G = \{(\alpha_1 + \alpha_2 k, \beta_1 + \beta_2 k) \in H_8^* \mid \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}\}$$

and define the map $\theta : G \longrightarrow G$ by $\theta(\gamma,\eta) = (\alpha,\beta)(\gamma,\eta)$ for all $(\gamma,\eta) \in G$. Then,

$$\begin{array}{l}\theta(1,0) = \alpha_1(1,0) + \alpha_2(k,0)) + \beta_1(0,1) + \beta_2(0,k), \\ \theta(k,0) = (-\alpha_2)(1,0) + \alpha_1(k,0)) + (-\beta_2)(0,1) + \beta_1(0,k), \\ \theta(0,1) = (-\beta_1)(1,0) + (-\beta_2)(k,0) + \alpha_1(0,1) + \alpha_2(0,k), \\ \theta(0,k) = \beta_2(1,0) + (-\beta_1)(k,0) + (-\alpha_2)(0,1) + \alpha_1(0,k)\end{array}.$$

Thus, we establish the function $\phi^{(1)}_{\mathbb{C}} : G \longrightarrow M_{4\times 4}(\mathbb{C})$ defined by

$$\phi^{(1)}_{\mathbb{C}}((\alpha,\beta)) = \begin{pmatrix} \alpha_1 & -\alpha_2 & -\beta_1 & \beta_2 \\ \alpha_2 & \alpha_1 & -\beta_2 & -\beta_1 \\ \beta_1 & -\beta_2 & \alpha_1 & -\alpha_2 \\ \beta_2 & \beta_1 & \alpha_2 & \alpha_1 \end{pmatrix}.$$

THEOREM 3.4. *Let $(\alpha,\beta), (\gamma,\eta) \in G$. Then the followings are satisfied:*

(1) $\phi^{(1)}_{\mathbb{C}}(a(\alpha,\beta)) = a\phi^{(1)}_{\mathbb{C}}(\alpha,\beta)$ *for all* $a \in \mathbb{R}$.
(2) $\phi^{(1)}_{\mathbb{C}}((\alpha,\beta)+(\gamma,\eta)) = \phi^{(1)}_{\mathbb{C}}(\alpha,\beta)+\phi^{(1)}_{\mathbb{C}}(\gamma,\eta)$.
(3) $\phi^{(1)}_{\mathbb{C}}((\alpha,\beta)(\gamma,\eta)) \neq \phi^{(1)}_{\mathbb{C}}(\alpha,\beta)\phi^{(1)}_{\mathbb{C}}(\gamma,\eta)$ *in general.*

Proof. (1) and (2) are straightforward and we shall prove (3). Let $\alpha = \alpha_1 + \alpha_2 k = i$ and $\gamma = \gamma_1 + \gamma_2 k = j$ for some $\alpha_t, \gamma_t \in \mathbb{C},\ t = 1,2$. Then $\alpha_1 = i,\ \alpha_2 = 0,\ \gamma_1 = 0,\ \gamma_2 = i$. Thus,

$$\begin{aligned} \phi^{(1)}_{\mathbb{C}}((\alpha,0)(\gamma,0)) &= \phi^{(1)}_{\mathbb{C}}((k,0)) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &\neq \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ &= \phi^{(1)}_{\mathbb{C}}((\alpha,0))\phi^{(1)}_{\mathbb{C}}((\gamma,0)) \end{aligned}$$

and so $\phi^{(1)}_{\mathbb{C}}((\alpha,\beta)(\gamma,\eta)) \neq \phi^{(1)}_{\mathbb{C}}((\alpha,\beta))\phi^{(1)}_{\mathbb{C}}((\gamma,\eta))$ in general. □

Let $\mathbb{C}^2 = \{(\alpha,\beta) \mid \alpha \in \mathbb{C}\}$. Then, $A \in M_{n\times n}(H^*_8)$ is uniquely expressed by $A = A_1 + A_2 k$ for some $A_1, A_2 \in M_{n\times n}(\mathbb{C}^2)$.

DEFINITION 3.5. Let $A \in M_{n\times n}(H^*_8)$. Then, $\lambda \in H^*_8$ is a left eigenvalue of A if $AX = \lambda X$ for some $X \neq O \in M_{n\times 1}(H^*_8)$.

THEOREM 3.6. *Let $A = A_1 + A_2 k \in M_{n\times n}(H^*_8)$ for some $A_1, A_2 \in M_{n\times n}(\mathbb{C}^2)$ and $\lambda = \lambda_1 + \lambda_2 k$ for some $\lambda_1, \lambda_2 \in \mathbb{C}^2$. If λ is a left eigenvalue of A if and only if there exists a nonzero matrix $X = X_1 + X_2 k \in M_{n\times 1}(H^*_8)$ for some $X_1, X_2 \in M_{n\times 1}(\mathbb{C}^2)$ such that*

$$\begin{pmatrix} A_1 - \lambda_1 I_n & -A_2 + \lambda_2 I_n \\ A_2 - \lambda_2 I_n & A_1 - \lambda_1 I_n \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} O \\ O \end{pmatrix}.$$

Proof. Note that

$$(A_2k)(X_2k) = -A_2X_2,\ (\lambda_2k)(X_2k) = -\lambda_2X_2,$$
$$A_1(X_2k) + (A_2k)X_1 = (A_1X_2 + A_2X_1)k,$$
$$\lambda_1(X_2k) + (\lambda_2k)X_1 = (\lambda_1X_2 + \lambda_2X_1)k.$$

Thus, $AX = \lambda X$ is equivalent to

$$A_1X_1 - A_2X_2 = \lambda_1X_1 - \lambda_2X_2, \quad A_1X_1 - A_2X_2 = \lambda_1X_1 - \lambda_2X_2$$

which is equivalent to $\begin{pmatrix} A_1 - \lambda_1 I_n & -A_2 + \lambda_2 I_n \\ A_2 - \lambda_2 I_n & A_1 - \lambda_1 I_n \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} O \\ O \end{pmatrix}$.

□

References

[1] F. Catoni, R. Cannata, P. Zampetti, *An introduction to Commutative quaternions*, Adv. Appl., Clifford Algebras **16** (2005), 1-28.

[2] H. Kosal, M. Tosun, *Commutative quaternion matrices*, Adv. Appl., Clifford Algebras **24** (2014), 769-779.

[3] C. Segre, *The real representations of complex elements and extension to bicomplex systems*, Math. Ann., **40** (1892), 413-467.

[4] Y. Song, *Matrices of Nonassociative Commutative Quaternions*, FJMS **137**, (2022), 39-52.

[5] Y. Song, *Construction of Commutative Number Systems*, Linear and Multilinear Algebra **70**, (2020), 2382-2396.

[6] F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra. Appl., **251** (1997), 21-57.

Youngkwon Song
Department of Mathematics
Kwangwoon University, Seoul 139-701
Republic of Korea
E-mail: yksong@kw.ac.kr