JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 4, November 2023
http://dx.doi.org/10.14403/jcms.2023.36.4.245

SEMIALGEBRAIC ORBIT STRUCTURES

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Abstract. Let G be a semialgebraic group not necessarily compact. In this paper, we prove that the orbit space of every proper semialgebraic G-set has a semialgebraic structure.

1. Introduction

For semialgebraic transformation groups we consider semialgebraic groups G acting semialgebraically on semialgebraic sets M, i. e., the action map $\theta\colon G\times M\to M$ is semialgebraic. Note that a semialgebraic set is a subset of some $\mathbb{R}^n$ defined by finite number of polynomial equations and inequalities, and a semialgebraic map between semialgebraic sets is a map whose graph is a semialgebraic set. See Section 2 for some basic material for semialgebraic category.

Working in semialgebraic category requires a lot of nontrivial efforts to establish some of the properties which are easy or well-known in topological or smooth category. One of such properties is the existence of semialgebraic structure on the orbit space of a semialgebraic G-set. Namely, it is not quite obvious whether the orbit space M/G of a semialgebraic G-set M has a semialgebraic set structure such that the orbit map is semialgebraic. This is settled in the following theorem which is a restatement of Theorem 3.5.

Theorem. *Let G be a semialgebraic group and M a proper semialgebraic G-set. Then M/G has a unique semialgebraic structure such that the orbit map $M\to M/G$ is a semialgebraic quotient map.*

This theorem is proved by Brumfiel when G is compact in [2] and by Scheiderer when M is locally compact in [10].

Received November 01, 2023; Accepted November 21, 2023.
2020 Mathematics Subject Classification: Primary 57Sxx, 57S99, 57Q10; Secondary 14P10, 19L47.
Key words and phrases: noncompact transformation groups, semialgebraic, orbit space.

2. Semialgebraic actions

2.1. Semialgebraic maps

The class of *semialgebraic sets* in $\mathbb{R}^n$ is the smallest collection of subsets containing all subsets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for a real valued polynomial $p(x) = p(x_1, \ldots, x_n)$, which is stable under finite union, finite intersection and complement. A map $f\colon X \to Y$ between semialgebraic sets X ($\subset \mathbb{R}^m$) and Y ($\subset \mathbb{R}^n$) is called a *semialgebraic map* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. From now on we impose "Euclidian topology" on semialgebraic sets and mainly consider continuous semialgebraic maps. We summarize some results about semialgebraic sets and maps in the following proposition. For more details, see [1], [3], [7] and [8].

PROPOSITION 2.1. (1) *Every semialgebraic set has a finite number of connected components, which are also semialgebraic.*
(2) *Composition of two semialgebraic maps is semialgebraic.*
(3) *Let $f\colon X \to Y$ be a semialgebraic map. If $A \subset X$ is semialgebraic, then its image $f(A)$ is semialgebraic. If $B \subset Y$ is semialgebraic, then its inverse image $f^{-1}(B)$ is semialgebraic.*
(4) *If $A \subset X$ is semialgebraic, then the closure $\overline{A}$ and the interior $\mathring{A}$ in X are all semialgebraic.*
(5) *Let $f\colon X \to Q$ and $g\colon X \to Y$ be semialgebraic. Assume f is surjective. If $h\colon Q \to Y$ is a map such that $h \circ f = g$, then h is semialgebraic.*
(6) *If $A \subset \mathbb{R}^n$ is a nonempty semialgebraic set, then the map $x \mapsto \operatorname{dist}(x, A)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is semialgebraic.*

A continuous semialgebraic map $f\colon X \to Y$ is called *semialgebraically proper* if $f^{-1}(C)$ is compact for every compact semialgebraic subset C of Y. Since C should be semialgebraic in the definition, this notion is weaker than the condition that f is semialgebraic and proper. Similarly we say a semialgebraic map $f\colon X \to Y$ is *semialgebraically closed* if f maps every closed semialgebraic subset of X to a closed semialgebraic subset of Y.

As a sequence or net plays prominent roles in the study of topological spaces, we can handle the semialgebraic properness more efficiently by use of a curve germ which was contrived by Brumfiel, see [1] or [2]. A *curve germ* in a semialgebraic set X is represented by a continuous semialgebraic map $\alpha\colon (0, \epsilon] \to X$ for some $\epsilon > 0$. Two curve germs are considered same if they agree on a common subinterval $(0, \delta]$ for some

$\delta > 0$. If a curve germ α extends to a continuous map $\alpha\colon [0,\epsilon] \to X$, we say α is *completable*, and we write $\alpha \to x$ where $x = \alpha(0)$.

PROPOSITION 2.2 ([2, p.73]). *Let X, Y, X_1 and X_2 be semialgebraic sets.*

(1) *(Completeness) Every curve germ in a compact semialgebraic set has a completion.*
(2) *(Curve selection) If x belongs to the closure of a semialgebraic subset K of X, then there is a curve germ α in K with $\alpha \to x$.*
(3) *(Lifting) If $f\colon X \to Y$ is surjective, then every curve germ α in Y lifts to a curve germ $\tilde{\alpha}$ in X, that is $f\tilde{\alpha} = \alpha$.*
(4) *(Irreducibility) If $X = X_1 \cup X_2$, then any curve germ in X is a curve germ in either X_1 or X_2.*

PROPOSITION 2.3 ([2, p.73]). *Let $f\colon X \to Y$ be a semialgebraic map between semialgebraic sets.*

(1) *f is continuous if and only if for any completable curve germ α by x in X the curve germ $f \circ \alpha$ is also completable by $f(x)$ in Y.*
(2) *Suppose f is continuous. Then f is semialgebraically proper if and only if the following condition holds: given a curve germ $\tilde{\alpha}$ in X such that $f\tilde{\alpha} = \alpha$ is completable in Y, then $\tilde{\alpha}$ is completable in X.*
(3) *If f is semialgebraically proper, then it is semialgebraically closed.*

2.2. Semialgebraic actions

The definition of a semialgebraic group is given obviously, i.e., a semialgebraic set $G \subset \mathbb{R}^n$ is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are semialgebraic. A semialgebraic homomorphism between two semialgebraic groups is a semialgebraic map that is a group homomorphism. If H is a subgroup and semialgebraic subset, then H is called a *semialgebraic subgroup.*

By a *semialgebraic transformation group* we mean a triple (G, M, θ), where G is a semialgebraic group, M is a semialgebraic set, and $\theta\colon G \times M \to M$ is a continuous semialgebraic map such that

(1) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in M$
(2) $\theta(e, x) = x$ for all $x \in M$, where e is the identity of G.

In this case M is called a *semialgebraic G-set*, and θ is called the *action map.* As usual we shortly write gx for $\theta(g, x)$. A *semialgebraic G-subset* of a semialgebraic G-set M is a G-invariant semialgebraic subset of M. A continuous semialgebraic map $f\colon M \to N$ between semialgebraic G-sets is called a *semialgebraic G-map* if it is G-equivariant, i.e., $f(gx) = gf(x)$ for all $g \in G$ and $x \in M$.

Recall that a semialgebraic map $f\colon X \to Y$ is semialgebraically proper if the inverse image of every semialgebraic compact subset is compact. Now we define semialgebraically proper actions as follows. Let G be a semialgebraic group not necessarily compact. A semialgebraic action of G on M is *proper* if the augmented action map

$$\vartheta_*\colon G \times M \to M \times M, \quad (g, m) \mapsto (gm, m)$$

is semialgebraically proper. In this case M is called a proper semialgebraic G-set. In terms of semialgebraic curve germs, this can be rewritten as follows by Proposition 2.3.

Proposition 2.4. *Let G be a semialgebraic group and M a semialgebraic G-set. Then the semialgebraic action is proper if and only if for every curve germs γ in G and α in M if α and $\gamma\alpha$ is completable then γ is completable.*

For topological proper actions the following proposition appears in [4, p. 174] whose proofs are straightforward.

Proposition 2.5. *Let G be a semialgebraic group and M a semialgebraic G-set.*

(1) *If G is compact, the action is semialgebraically proper.*
(2) *If G is not compact but M is compact, the action is not semialgebraically proper.*
(3) *If the semialgebraic action is proper, its restriction to any semialgebraic subgroup $H \subset G$ is a proper semialgebraic H-action on M, and its restriction to any invariant semialgebraic subset U of M is a proper semialgebraic G-action on U.*
(4) *If the semialgebraic action is proper, the evaluation map*

$$\mathrm{ev}_m\colon G \to M, \quad g \mapsto gm,$$

is semialgebraically proper, for each $m \in M$.
(5) *A semialgebraic G-action on M is proper if and only if for every two compact semialgebraic subsets K_1 and K_2 of M, the subset*

$$\{g \in G \mid K_1 \cap gK_2 \neq \emptyset\}$$

of G is compact.

We want to compare with this notion of proper semialgebraic action with that of Palais. Let M be a semialgebraic G-set. For two subsets

$U, V \subset M$, define $((U,V))$ by

$$((U,V)) = \{g \in G \mid U \cap gV \neq \emptyset\}.$$

We say a point $x \in M$ is *proper* if x has a (semialgebraic) neighborhood U of x in M satisfying that for any $y \in M$ there exists a (semialgebraic) neighborhood V of y in M such that the closure of $((U,V))$ is compact.

Proposition 2.6. *A semialgebraic G-set M is proper if and only if every point of M is proper.*

Proof. Suppose M is a proper semialgebraic G-set. For any $x \in M$ it is clear from the definition of proper action that the isotropy subgroup G_x is compact. Since G is locally compact, by Proposition 2.5 (3), there exists an open (semialgebraic) neighborhood W of G_x in G whose closure is compact. By Proposition 2.3 (3) the augmented action map ϑ_* is a closed map. Hence the image $\vartheta_*((G-W)\times M)$ is closed in $M \times M$. Since $(x,x) \notin \vartheta_*((G-W)\times M)$, there exist semialgebraic open neighborhoods U and U_1 of x such that

$$U \times U_1 \cap \vartheta_*((G-W)\times M) = \emptyset.$$

It follows that $((U,U_1)) \subset W$. Indeed, if $g \in ((U,U_1))$, then there exists $y \in U_1$ such that $gy \in U$. Therefore $\vartheta_*(g,y) = (gy,y) \in U \times U_1$. But since $U \times U_1 \cap \vartheta_*((G-W)\times M) = \emptyset$, we see that $g \notin G - W$, hence $g \in W$. This shows that $((U,U_1)) \subset W$. By taking smaller U, assume $\overline{U} \subset U_1$.

We now claim that for each $y \in M$ there is a semialgebraic neighborhood V of y in M such that $((U,V))$ has compact closure. For $y \in M$ with $G(y) \cap \overline{U} = \emptyset$, take $V = M - G(\overline{U})$. Then $((U,V)) = \emptyset$. For $y \in M$ with $G(y) \cap \overline{U} \neq \emptyset$, then $gy \in \overline{U} \subset U_1$ for some $g \in G$. Take $V = g^{-1}U_1$. Then we have $((U,V)) = ((U,U_1))g \subset Wg$. Since W has the compact closure, so does $((U,V))$.

Conversely, suppose every point of M is proper. Let K_1 and K_2 be two compact semialgebraic subset of M. Since every point of M is proper we can find finite semialgebraic open covers $\{U_1,\ldots,U_n\}$ of K_1 and $\{V_1,\ldots,V_m\}$ of K_2 such that $((U_i,V_j))$ has the compact closure for every i and j. Since $((K_1,K_2)) \subset \bigcup_{i,j}((U_i,V_j))$ the set $((K_1,K_2))$ has the compact closure. Hence Proposition 2.5 (5) implies that M is a proper semialgebraic G-set. □

Note that in [6] Palais defined a proper G-space M to be a G-space that is proper at every point.

3. Semialgebraic orbit spaces

In this section we shall show that the orbit space of every proper semialgebraic G-set has a unique semialgebraic structure in the natural sense. In fact, we shall show that the semialgebraic structure of the orbit space of a proper semialgebraic G-set M can be defined by the quotient of some smaller subset Y of M.

Definition 3.1. Let M be a semialgebraic set and $E \subset M \times M$ an equivalence relation of M. Assume E is a closed and semialgebraic subset in $M \times M$. Let $p_1,\ p_2\colon E \subset M \times M \to M$ be the restrictions of the natural projections $M \times M \to M$ to each factors. We call a subset Y of M a *section-like* subset of M for a closed semialgebraic equivalence relation $E \subset M \times M$, if the map $p_1|_{p_2^{-1}(Y)}\colon p_2^{-1}(Y) \to M$ is semialgebraically proper and surjective.

Proposition 3.2 ([2, 10]). *Suppose $E \subset M \times M$ is a closed, semialgebraic equivalence relation. If M has a section-like subset Y, then there exists a unique semialgebraic structure on M/E such that the quotient map $M \to M/E$ is semialgebraic. Moreover the restriction of the quotient map $Y \to Y/E_Y$ is semialgebraically proper and Y/E_Y is semialgebraically homeomorphic to M/E where $E_Y = E \cap (Y \times Y)$.*

Proof. The results are basically from [2] and [10]. Brumfiel first showed the result when M itself is a section-like subset of M, i.e., when $p_1\colon E \to M$ is semialgebraically proper and surjective.

Scheiderer proved the general case. More precisely, he proved that if M is locally compact, then the existence of a section-like subset is a necessary and sufficient condition for M/E to have the desired semialgebraic structure in [10, Theorem 4.1]. He also pointed out that the sufficiency holds without the assumption of locally compactness, which is the claim of the proposition.

The uniqueness of the semialgebraic structure is obtained as follows: Any two semialgebraic structures on M/E have the same topology, and hence they are homeomorphic. Then Proposition 2.1 (5) implies that it is a semialgebraic homeomorphism. In [2] and [10] both authors constructed a semialgebraic structure on Y/E_Y as a model semialgebraic structure of M/E, hence Y/E_Y is semialgebraically homeomorphic to M/E. We now apply Theorem 1.4 of [2] to Y and E_Y to see that the map $Y \to Y/E_Y$ is semialgebraically proper. This completes the proof. □

From Proposition 3.2 we have the following corollary.

Corollary 3.3. *Let M and N be semialgebraic sets. And let A be a closed semialgebraic subset of M. Let $f\colon A \to N$ be a semialgebraically proper map. Then $M \cup_f N$ has a unique semialgebraic set structure.*

For a semialgebraic G-set M, let $E = E_G \subset M \times M$ be a semialgebraic equivalence relation corresponding to the action, that is, $E_G = \{(gx, x) \in M \times M \mid g \in G,\ x \in M\}$. Then E_G is the image of the augmented action map $\vartheta_*\colon G \times M \to M \times M$ which is semialgebraically proper, hence semialgebraically closed. Thus E_G is closed in $M \times M$. The following lemma gives a criterion for a semialgebraic subset Y of M to be section-like.

Lemma 3.4. *Let M be a semialgebraic G-set and Y a semialgebraic subset of M. Let $p_i\colon E_G \subset M \times M \to M$ $(i = 1, 2)$ be the projections to each factors. Let $f = p_1|_{p_2^{-1}(Y)}\colon p_2^{-1}(Y) \to M$.*

(1) *f is surjective if and only if any curve germ α in M can be written as $\alpha = \gamma\beta$ for some curve germs γ in G and β in Y.*
(2) *f is semialgebraically proper if and only if any curve germ β in Y such that $\gamma\beta$ is completable in M for some curve germ γ in G is completable in Y.*

Proof. The sufficiency of (1) is clear if we take α as constant curve germs. For necessity, apply Proposition 2.2 (3) to find β in Y such that $\alpha \times \beta$ is in E_G. By applying Proposition 2.2 (3) to $G \times M \to E_G$, $(g, m) \mapsto (gm, m)$, we can find γ in G such that $\alpha = \gamma\beta$.

(2) follows from Proposition 2.3 (2) since $f(\gamma\beta, \beta) = \gamma\beta$. □

Theorem 3.5. *Let G be a semialgebraic group and M a proper semialgebraic G-set. Then there exists a semialgebraic section-like subset Y for E_G, thus M/G has a unique semialgebraic structure such that $M \to M/G$ is a semialgebraic quotient map.*

Proof. By the definition of a semialgebraic set, M is a semialgebraic subset of $\mathbb{R}^n$ for some n. First we construct a new imbedding of M which sends some "bad" points to infinity. We slightly extend the notation $((U, V))$. For $U \subset \mathbb{R}^n$ and $V \subset M$ let us define the subset $((U, V))$ of G as

$$((U, V)) = \{g \in G \mid U \cap gV \neq \emptyset\},$$

thus $((U, V))$ means the set of those translators of V which meet with U. Let $\overline{M}$ denote the closure of M in $\mathbb{R}^n$. We say that $x \in \overline{M}$ has the "good" property if there is a neighborhood U of x in $\overline{M}$ such that for any $y \in M$ there exists a neighborhood V of y in M such that $((U, V)) \subset G$

has the compact closure. It is clear that the set of all points in $\overline{M}$ having the *good* property is open in $\overline{M}$ and contains M. Let A denote the complement of this set in $\overline{M}$, we call it the set of *bad* points.

Consider a map $M \to \mathbb{R}^{n+1}$ defined by $x \mapsto (x, 1/\operatorname{dist}(x, A))$. By this semialgebraic map we can assume M is in $\mathbb{R}^{n+1}$ and that every point in $\overline{M} \subset \mathbb{R}^{n+1}$ has the *good* property.

Let $m = n+1$. First we show that every orbit $G(x)$ of $x \in M$ is closed in $\mathbb{R}^m$. Let α be a completable curve germ such that $\alpha(0,1] \subset G(x)$ and $\alpha(0) \in \mathbb{R}^m$. By applying Proposition 2.2 (3) to the surjective map $G \times \{x\} \to G(x)$, we can set $\alpha(t) = g(t)x$ for $0 < t \le \epsilon$ for some $\epsilon > 0$, where $g(t)$ is a curve germ in G. Let U be a semialgebraic neighborhood of $\alpha(0)$ and let V be a semialgebraic neighborhood of x such that $((U, V))$ has a compact closure. Since $g(t)$ lies on $((U, V))$ by taking smaller ϵ, it is completable, say by g_0. Then $\alpha(0) = g_0 x \in G(x)$, which implies that $G(x)$ is closed.

From each orbit $G(x)$ in M choose a point $s(x)$ which is the closest to the origin of $\mathbb{R}^m$. Since $G(x)$ is closed we can see easily that such a point $s(x)$ exists. If there are more than one such points, choose a point whose coordinate is the smallest with respect to the lexicographical order of the coordinates of $\mathbb{R}^m$. Then the mapping $s\colon M \to M$, $x \mapsto s(x)$ may not be continuous but semialgebraic. Indeed, we shall see that the graph of s is defined using a first-order formula of the language of $\mathbb{R}$. First consider a semialgebraic set

$$X = \{(x, y) \in \mathbb{R}^{2m} \mid \exists g \in G\ y = gx \text{ and } \forall g' \in G\ d(0, y) \le d(0, g'x)\}.$$

Next, the graph of s is defined by

$$\{(x, y_1, \dots, y_m) \in X \mid \forall (x, z_1, \dots, z_m) \in X\ y_1 \le z_1, \dots, y_m \le z_m\}.$$

Hence the image $S = \{s(x) \mid x \in M\}$ is a semialgebraic subset of M.

Take $Y = \overline{S}$ the closure of S in M. We shall show that Y is section-like. The surjectivity of $p_1|_{p_2^{-1}(Y)}$ is clear. To apply Lemma 3.4 (2), take curve germs α in M, β in Y and γ in G such that $\alpha = \gamma\beta$ and α is completable. Since α is completable it is contained in a large ball, thus, so is β because $\operatorname{dist}(0, \beta(t)) \le \operatorname{dist}(0, \alpha(t))$. Since β is bounded in $\mathbb{R}^m$, it is completable in $\mathbb{R}^m$. We hope $\beta(0) \in Y$. Take U and V neighborhoods of x and y respectively such that $((U, V))$ (in the extended sense) has a compact closure. Since γ lies in $((U, V))$, it follows γ is completable. Since $\alpha(0) = \gamma(0)\beta(0)$ or $\beta(0) = (\gamma(0))^{-1}\alpha(0)$ we have that $\beta(0) \in M$. Since Y is closed in M, so $\beta(0) \in Y$. □

Remark 3.6. If M is locally compact, the above result can be deduced from [10, 6.2], since the action map is always open which is needed in the condition of [10, 6.2]. We have seen in Theorem 3.5 the condition of local compactness could be dropped, but if the equivalence relation does not occur from an action, there is an example such that even if $E \subset M \times M$ is a closed semialgebraic equivalence relation which is open over M but the quotient M/E could not be semialgebraic (see the example below [10, 6.2]).

References

[1] J. Bochnak, M. Coste, and M.-F. Roy, *Real Agebraic Geometry*, **36** of Erg. der Math. und ihrer Grenzg, Springer-Verlag, Berlin Heidelberg, 1998.

[2] G. W. Brumfiel, *Quotient space for semialgebraic equivalence relation*, Math. Z., **195** (1987), 69-78.

[3] H. Delfs and M. Knebusch, *Locally Semialgebraic Spaces*, volume 1173 of Lecture Notes in Math., Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[4] V. Guillemin, V. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and hamiltonian group actions*, volume 98 of Mathematical surveys and monographs., Amer. Math. Soc., 1998.

[5] D. Montgomery and L. Zippin, *A theorem on Lie groups*, Bull. Amer. Math. Soc., **8** (1942), 448-452.

[6] R. S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math., **73** (1961), no. 2, 295-323.

[7] D. H. Park, *A note on semialgebraically proper maps*, Ukrainian Math. J., **66** (2015), no. 9, 1414-1422.

[8] D. H. Park, *Equivariant semialgebraic embeddings*, J. Chungcheong Math. Soc., **35** (2022), no. 1, 13-23.

[9] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Applied Algebra **53** (1988), 239-255.

[10] C. Scheiderer, *Quotients of semi-algebraic spaces*, Math. Z., **201** (1989), 249-271.

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