JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 4, November 2023
http://dx.doi.org/10.14403/jcms.2023.36.4.223

A RATIONAL HOMOLOGY DISK SMOOTHING OF $W_{p,q,r}$

HEESANG PARK AND DONGSOO SHIN

ABSTRACT. We prove that a rational homology disk smoothing of the singularity $W_{p,q,r}$ is $\mathbb{Q}$-Gorenstein by using its sandwiched structure.

1. Introduction

The concept of $\mathbb{Q}$-Gorensteiness plays a crucial role in the study of the Kollár–Shepherd-Barron–Alexeev compactifications of the moduli spaces of complex surfaces of general type. We say that a smoothing $\mathcal{Z} \to \Delta$ of a normal surface singularity Z is $\mathbb{Q}$-*Gorenstein* if some multiple of the canonical class of $\mathcal{Z}$ is Cartier. Originally, Wahl [6] and Kollár–Shepherd-Barron [3] classified quotient surface singularities that allow $\mathbb{Q}$-Gorenstein smoothings. These are cyclic quotient surface singularities of the form $\frac{1}{dn^2}(1, dna-1)$, where integers $d \geq 1$, $n > a \geq 1$, and $(n, a) = 1$. Such singularities are referred to as *singularities of class* T. More recently, Bhupal–Stipsicz [1] classified the resolution graphs of weighted homogeneous surface singularities that admit smoothings with the rational homology of the 4-disk.

One of the earliest examples that can be smoothed with rational homology disks is the singularity $W_{p,q,r}$, initially discovered by Wahl [6]. Here, $W_{p,q,r}$ is a singularity classified by Bhupal–Stipsicz [1], and its dual graph is illustrated in Figure 1.

In this paper, we prove that a rational homology disk smoothing of $W_{p,q,r}$ is indeed $\mathbb{Q}$-Gorenstein. It's worth noting that Wahl [7] had already established the $\mathbb{Q}$-Gorenstein property of rational homology disk smoothings of all the singularities given in Bhupal–Stipsicz [1]. However,

Received September 12, 2023; Accepted October 27, 2023.
2020 Mathematics Subject Classification: 14B07.
Key words and phrases: deformation, $\mathbb{Q}$-Gorenstein, rational homology disk smoothing.

in this paper, we present a different, simpler method to prove its $\mathbb{Q}$-Gorensteiness.

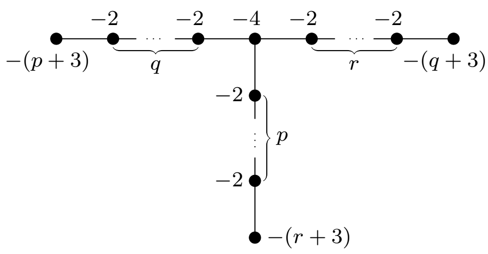

FIGURE 1. $W(p, q, r)$, where $p, q, r \geq 0$

THEOREM 1.1. *The smoothing of $W_{p,q,r}$ whose Milnor fiber is a rational homology disk is $\mathbb{Q}$-Gorenstein.*

We briefly sketch the idea of the proof. We first show that $W_{p,q,r}$ is a sandwiched surface singularity; See Section 2 for details. Then one can apply the $\mathbb{Q}$-Gorensteinness criterion given by de Jong–van Straten [2]. In details, for each smoothing of a sandwiched surface singularity, there is a certain matrix, called an *incidence matrix*, that corresponds to the given smoothing. de Jong–van Straten [2] showed that a smoothing of a sandwiched surface singularity is $\mathbb{Q}$-Gorenstein if and only if its incidence matrix satisfies a certain simple condition; Proposition 3.1. So we first find the incidence matrix corresponding to the rational homology disk smoothing of $W_{p,q,r}$ and we then show that the incidence matrix satisfies the criterion.

Acknowledgements

The authors would like to thank the referee for careful reading and valuable comments. Heesang Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education: NRF-2021R1F1A1063959.

2. Sandwiched surface singularities

A *sandwiched surface singularity* $(X, 0)$ is a normal surface singularity admitting a birational morphism to $\rho\colon (X, 0) \to (\mathbb{C}^2, 0)$. Sandwiched surface singularities are rational singularities characterized by their dual resolution graphs, so-called *sandwiched graphs*:

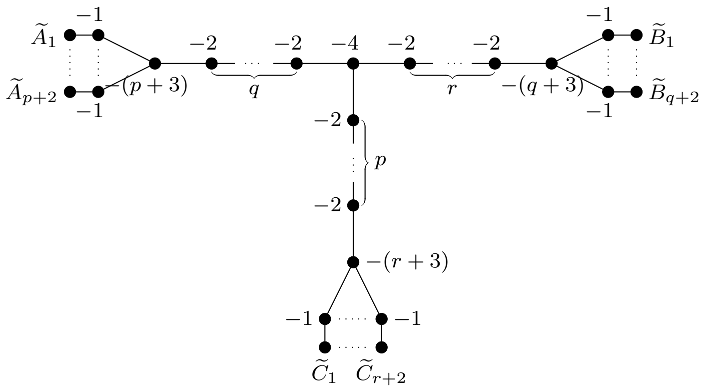

FIGURE 2. A sandwiched structure for $W_{p,q,r}$

DEFINITION 2.1 (Spivakovsky [5, Definition 1.9]). A graph Γ is called a *sandwiched graph* if it is the dual resolution graph of a rational surface singularity that can be blown down to a smooth point by adding new vertices with weights (-1) on the proper places.

PROPOSITION 2.2 (Spivakovsky [5, Proposition 1.11]). *A normal surface singularity is sandwiched if and only if its dual resolution graph is sandwiched.*

LEMMA 2.3. *The singularity $W_{p,q,r}$ is sandwiched.*

Proof. It is already proved in Park–Shin [4]. Its dual graph is sandwiched as given in Figure 2. □

2.1. Decorated curves

In their work on sandwiched surface singularities (X, p), de Jong and van Straten [2] introduced a pair (C, l). This pair consists of a plane curve singularity $C = \cup_{i=1}^{s} C_i$ and an assignment function $l\colon \{C_i \mid i = 1, \dots, e\} \to \mathbb{N}$ that characterizes the singularity X in terms of (C, l). Let's briefly revisit how one can derive (C, l) from (X, p). For a more in-depth explanation, please refer to de Jong-van Straten [2].

DEFINITION 2.4 (de Jong-van Straten [2, Definition 1.4]). A *decorated germ* is a pair (C, l), comprising a plane curve singularity $C = \cup_{i=1}^{s} C_i \subset \mathbb{C}^2$ passing through the origin and an assignment function $l\colon \{C_i \mid i = 1, \dots, s\} \to \mathbb{N}$. This assignment function must satisfy the condition $l(C_i) \geq m(C_i)$, where $m(C_i)$ denotes the sum of multiplicities of branch C_i in the multiplicity sequence of the minimal resolution of C.

Using a sandwiched graph structure, it is possible to construct a decorated curve (C, l) for (X, p). The dual graph of the minimal resolution (V, E) of (X, p) is, in fact, sandwiched and can be transformed into a smooth point by introducing some (-1)-vertices. Simultaneously, one can embed (V, E) into a blow-up space $(\widetilde{\mathbb{C}}^2, F)$ of $\mathbb{C}^2$ centered at the origin (including its infinitely near point), where F represents the set of exceptional divisors. For each (-1)-curve $F_i \in F$, one selects a *curvetta* $\widetilde{C}_i$ (essentially, a small segment of a curve) that intersects F_i transversely. The union of these $\widetilde{C}_i$ segments, denoted as $\widetilde{C}$, is then mapped to $C = \rho(\widetilde{C}) = \bigcup_{F_i \leq F} C_i$, where each C_i is the image of $\widetilde{C}_i$ under the map ρ. At this point, C can be regarded as a germ of plane curves passing through the origin 0. To decorate C_i, assign the number l_i, which represents the sum of multiplicities of the blowing-up points located on the strict transform of C_i.

A decorated curve for $W_{p,q,r}$ is given in Figure 2.

Lemma 2.5. *For* $W_{p,q,r}$, $\widetilde{C} = (\cup \widetilde{A}_i) \bigcup (\cup \widetilde{B}_j) \bigcup (\cup \widetilde{C}_k)$ *and* $l(A_i) = q + 3$, $l(B_j) = r + 3$, $\quad l(C_k) = p + 3$.

For details, refer Park–Shin [4, §15.2].

2.2. Picture deformations

A one-parameter deformation of a sandwiched surface singularity (X, p) can be traced back to a one-parameter deformation of its decorated curve (C, l). The detailed deformation theory is outlined in de Jong-van Straten [2].

The decoration l associated with a decorated curve (C, l) can be thought of as the combination of unique subschemes, each having length l_i, supported on the preimage of the origin 0 on the normalization of C_i.

Definition 2.6 (de Jong-van Straten [2, Definition 4.2]). Suppose (C, l) is a decorated curve associated with a sandwiched surface singularity (X, p). Then, a *picture deformation* $(\mathcal{C}, \mathcal{L})$ of (C, l) over a small disk Δ centered at the origin 0 consists of the following:

(1) A δ-constant deformation $\mathcal{C} \to \Delta$ of C, meaning that $\delta(C_t)$ is constant for all $t \in \Delta$, where C_t is a fiber over t.
(2) A flat deformation $\mathcal{L} \subset \mathcal{C}$ over Δ of the scheme l.
(3) $\mathcal{M} \subset \mathcal{L}$, where the relative total multiplicity scheme $\mathcal{M}$ of $\mathcal{C} \to \Delta$ is defined as the closure $\bigcup_{t \in \Delta \setminus 0} m(C_t)$.
(4) For generic $t \in T \setminus 0$ the divisor l_t on $\widetilde{C}_t$ is reduced.

PROPOSITION 2.7 (de Jong-van Straten [2, Theorem 4.4]). *For any one-parameter smoothing of X, there corresponds to a picture deformation of its decorated curve (C, l), and vice versa.*

2.3. Incidence matrices

The combinatorial aspects of picture deformations for (C, l) can be captured using specific matrices. Let's consider a picture deformation $(\mathcal{C}, \mathcal{L})$ of (C, l). Assuming that C is composed of components $C = \cup_{i=1}^{s} C_i$ and that $C_t = \cup_{i=1}^{s} C_{i,t}$ for the deformed curve C_t, we denote by $P_1, \ldots, P_n$ the images in C_t of the points in the support of l_t.

DEFINITION 2.8 (de Jong-van Straten [2, p. 483]). The *incidence matrix* of a picture deformation $(\mathcal{C}, \mathcal{L})$ is represented as $I(\mathcal{C}, \mathcal{L}) \in M_{s,n}(\mathbb{Z})$, where $M_{s,n}(\mathbb{Z})$ is the set of $s \times n$ matrices whose entries are integers. In this matrix, the entry at position (i, j) is equal to the multiplicity of P_j as a point on $C_{i,t}$ for $t \neq 0$.

LEMMA 2.9. *The incidence matrices that corresponds to the rational homology disk smoothing of $W_{p,q,r}$ is given as follows:*

$$M = \begin{array}{c} \\ A(I) \\ \\ \\ B(I) \\ \\ \\ C(I) \\ \\ \end{array} \left[\begin{array}{ccc|ccc|ccc} \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{q+2}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{p+2}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{r+2}} \\ 1 & \cdots & 1 & 1 & & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & & 1 & 0 & \cdots & 0 \\ \hline 1 & & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & & 1 \end{array}\right]$$

In addition, if $p = q = r$, then we have one more incidence matrix:

$$M' = \begin{array}{c} \\ A(I) \\ \\ \\ B(I) \\ \\ \\ C(I) \\ \\ \end{array} \left[\begin{array}{ccc|ccc|ccc} 1 & \cdots & 1 & 1 & & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & & 1 \\ \hline 1 & & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array}\right]$$

Proof. See Park–Shin [4, §16.3]. □

3. $\mathbb{Q}$-Gorensteinness

de Jong and van Straten [2] provided a straightforward criterion for determining whether a one-parameter smoothing of a sandwiched surface singularity is $\mathbb{Q}$-Gorenstein.

Proposition 3.1 (de Jong and van Straten [2, Corollary 5.12]). *A one-parameter smoothing $\mathcal{X} \to \Delta$ is $\mathbb{Q}$-Gorenstein if and only if $(1, 1, \ldots, 1)$ is a rational linear combination of the rows of the corresponding incidence matrix.*

Proof of Theorem 1.1. We can verify that the matrix M has full rank, meaning that there is always a solution to the matrix equation $M^T Y = (1, \ldots, 1)^T$. It's important to note that the determinants of M^T and its submatrices are always rational numbers, implying the existence of a rational solution Y. Consequently, we can express $(1, 1, \ldots, 1)$ as a rational linear combination of the rows of M. Similarly, we can prove the criterion to the matrix M'. □

References

[1] M. Bhupal, A. Stipsicz, *Weighted homogeneous singularities and rational homology disk smoothings*, Amer. J. Math., **133** (2011), no. 5, 1259–1297.

[2] T. de Jong, D. van Straten, *Deformation theory of sandwiched surface singularities*, Duke Math. J., **95** (1998), no. 3, 451–522.

[3] J. Kollár, N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math., **91** (1988), no. 2, 299–338.

[4] H. Park, D. Shin, *Deformations of sandwiched surface singularities and the minimal model program*, arXiv:2205.11165.

[5] M. Spivakovsky, *Sandwiched surface singularities and desingularization of surfaces by normalized Nash transformations*, Ann. Math., **131** (1990), 411–491.

[6] J. Wahl, *Smoothings of normal surface singularities*, Topology **20** (1981), no. 3, 219–246.

[7] J. Wahl, *Log-terminal Smoothings of Graded Normal Surface Singularities*, Michigan Math. J., **62** (2013), 475–489.

Heesang Park
Department of Mathematics
Konkuk University, Seoul 05029, Republic of Korea
E-mail: HeesangPark@konkuk.ac.kr

Dongsoo Shin
Department of Mathematics
Chungnam National University, Daejeon 34134, Republic of Korea
E-mail: dsshin@cnu.ac.kr