J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.4.407 ISSN(Online) 2287-6081
Volume 30, Number 4 (November 2023), Pages 407–416

C^*-ALGEBRA-VALUED EXTENDED QUASI b-METRIC SPACES AND FIXED POINT THEOREMS WITH AN APPLICATION

Qusuay H. Alqifiary [a] and Jung Rye Lee [b,*]

Abstract. In this paper, we introduce the concept of C^*-algebra-valued quasi b-metric space and prove some existence and uniqueness theorems. Furthermore, we prove the Hyers-Ulam stability results for fixed point problems via C^*-algebra-valued extended quasi b-metric space.

1. Introduction and Preliminaries

In 2014, Ma *et al.* [8] introduced the concept of C^*-algebra-valued metric spaces by replacing the range of real numbers with an unital C^*-algebra-valued metric space. In 2015, Ma *et al.* [7] introduced a generalized C^*-algebra-valued metric space. For more information about C^*-algebra, see [4]. Many researchers have obtained fixed point theorems in C^*-algebra-valued metric spaces (see [2, 6, 10, 14]). Samet *et al.* [13] introduced α-Ψ-contractive mappings in metric spaces and then developed in b-metric spaces [12]. Many authors have introduced several results related to α-admissible and α-Ψ-contractive mappings [3, 5, 9].

In this paper, we present the notion of an extended quasi-b-metric space in C^*-algebra and obtain some new results associated with the fixed point theorem and application to the Hyers-Ulam stability.

Suppose that Ω is a unital C^*-algebra with a unit 1_Ω, a partial ordering $\preceq$ on Ω as $a \preceq b$ if $b - a \succeq 0_\Omega$, where 0_Ω means the zero element in Ω and $\Omega^+ = \{x \in \Omega : x \succeq 0_\Omega\}$.

Definition 1.1 ([8]). Let X be a nonempty set. A mapping $q : X \times X \to \Omega$ is called a *C^*-algebra-valued metric* on X if it satisfies the following: For all $a, b, c \in X$,

Received by the editors May 14, 2023. Revised September 12, 2023. Accepted Sept. 14, 2023.
2020 *Mathematics Subject Classification.* 47H10, 93D15, 39B82.
Key words and phrases. C^*-algebra-valued metric space, extended quasi b-metric space, fixed point, functional equation, Cauchy sequence, Hyers-Ulam stability.
*Corresponding author.

(1) $q(a,b) \succeq 0_\Omega$ and $q(a,b) = 0_\Omega$ if and only if $a = b$;
(2) $q(a,b) = q(b,a)$;
(3) $q(a,b) \preceq [q(a,c) + q(c,b)]$.

Then the triplet (X, Ω, q) is said to be a *C^*-algebra-valued metric space.*

In 2015, Ma *et al.* [11] introduced the notion of C^*-algebra-valued b-metric space.

Definition 1.2 ([11]). Let X be a nonempty set. A mapping $q : X \times X \to \Omega$ is called a *C^*-algebra-valued b-metric* on X if it satisfies the following: For all $a, b, c \in X$,

(1) $q(a,b) \succeq 0_\Omega$ and $q(a,b) = 0_\Omega$ if and only if $a = b$;
(2) $q(a,b) = q(b,a)$;
(3) $q(a,b) \preceq s[q(a,c) + q(c,b)]$.

Then the triplet (X, Ω, q) is said to be a *C^*-algebra-valued b-metric space.*

In 2020, Asim and Imdad [1] introduced the following definition of C^*-algebra-valued extended b-metric space.

Definition 1.3. Let X be a nonempty set and $\mu : X \times X \to \Omega^+$ be a mapping. A mapping $L_\mu : X \times X \to \Omega$ is called a *C^*-algebra-valued extended b-metric* on X if it satisfies the following: For all $a, b, c \in X$,

(1) $L_\mu(a,b) \succeq 0_\Omega$ and $L_\mu(a,b) = 0_\Omega$ if and only if $a = b$;
(2) $L_\mu(a,b) = L_\mu(b,a)$;
(3) $L_\mu(a,b) \preceq \mu(a,b)[L_\mu(a,c) + L_\mu(c,b)]$.

Then the triplet (X, Ω, L_μ) is said to be a *C^*-algebra-valued extended b-metric space.*

In this paper, we introduce another type of generalized C^*-algebra-valued metric space, which is called a C^*-algebra-valued extended quasi b-metric space (in short, C^*-avEqbms) as follows:

Definition 1.4. Let X be a nonempty set and $\mu : X \times X \to \Omega^+$ be a mapping. A mapping $L_\mu : X \times X \to \Omega$ is a C^* -avEqbm on X if it satisfies the following: For all $a, b, c \in X$,

(1) $L_\mu(a,b) \succeq 0_\Omega$ and $L_\mu(a,b) = 0_\Omega$ if and only if $a = b$;
(2) $L_\mu(a,b) \preceq \mu(a,b)[L_\mu(a,c) + L_\mu(c,b)]$.

Then the triplet (X, Ω, L_μ) is said to be a *C^*-algebra-valued extended quasi b-metric space.*

Remark 1.5. Observe that if $\mu(a,b) = s \succeq 1_\Omega$, then (X, Ω, L_μ) is a C^*-algebra-valued extended quasi b-metric space.

Definition 1.6. Let (X, Ω, L_μ) be a C^*-avEqbms. A sequence $\{y_i\}$ in X is said to be

(*i*) *convergent* if for all $c \in \Omega$ with $C \succeq 0_\Omega$, there exists a natural number $N = N(c)$ such that $L_\mu(y, y_n) \preceq c$ and $L_\mu(y_n, y) \preceq c$ for all $n > N$;

(*ii*) a *left Cauchy sequence* if for all $c \in \Omega$ with $C \succeq 0_\Omega$, there exists a natural number $N = N(c)$ such that $L_\mu(y_n, y_m) \preceq c$ for all $n \geq m \geq N$;

(*iii*) a *right Cauchy sequence* if for all $c \in \Omega$ with $C \succeq 0_\Omega$, there exists a natural number $N = N(c)$ such that $L_\mu(y_m, y_n) \preceq c$ for all $n \geq m \geq N$;

(*iv*) a *Cauchy sequence* if for all $c \in \Omega$ with $C \succeq 0_\Omega$, there exists a natural number $N = N(c)$ such that $L_\mu(y_m, y_n) \preceq c$ for all $n, m \geq N$.

Remark 1.7. We say that a C^*-avEqbms (X, Ω, L_μ) is a complete C^*-algebra-valued extended quasi b-metric space if every Cauchy sequence is convergent with respect to Ω.

Definition 1.8. Let $X \neq \emptyset$ and $\alpha_\Omega : X \times X \longrightarrow (\Omega')^+$ be a mapping. A self mapping $T : X \to X$ is called α_Ω*-admissible* if for every $(a, b) \in X \times X$ such that $\alpha_\Omega(a, b) \succeq 1_\Omega$, $\alpha_\Omega(Ta, Tb) \succeq 1_\Omega$.

Definition 1.9. Let (X, Ω, L_μ) be a complete C^*-avEqbms. A mapping $T : X \to X$ is said to have a *generalized Lipschitz condition* if there exists $b \in \Omega$ such that $\|b\| < 1$ and $L_\Omega(Tx, Ty) \preceq b^* L_\Omega(x, y) b$ for all $x, y \in X$.

Lemma 1.10. *Let (X, Ω, L_μ) be a C^*- avEqbms. If L_μ is continuous, then every convergent sequence has a unique limit.*

2. Main Results

Theorem 2.1. *Let (X, Ω, L_μ) be a complete C^*-avEqbms with $\mu : X \times X \to \Omega^+$. Suppose that a mapping $T : X \longrightarrow X$ satisfies the following:*

(*i*) *T is a generalized Lipschitz contraction;*

(*ii*) *T is continuous.*

Then T has a unique fixed point.

Proof. Let $y_0 \in X$ and define a sequence $\{y_n\}$ in X such that $y_n = T(y_{n-1})$ for all n. If $y_n = y_{n+1}$ for some n, then y_n is a fixed point for T. Assume that $y_n \neq y_{n+1}$ for all n. Since T is a generalized Lipschitz contraction, we have

$$\begin{aligned}
Ł_\mu(y_n, y_{n+1}) = Ł_\mu(Ty_{n-1}, Ty_n) &\preceq b^* Ł_\mu(y_{n-1}, y_n) b \\
&\preceq (b^*)^2 Ł_\mu(y_{n-2}, y_{n-1}) b^2 \\
&\vdots \\
&\preceq (b^*)^n Ł_\mu(y_0, y_1) b^n \\
&\preceq (b^*)^n \Upsilon b^n,
\end{aligned}$$

where $\Upsilon = Ł_\mu(y_0, y_1)$.

Now, we will show that $\{y_n\}$ is a Cauchy sequence. For $n, m \in \mathbb{N}$ such that $n < m$, we have

$$\begin{aligned}
&Ł_\mu(y_n, y_m) \preceq \mu(y_n, y_m)[Ł_\mu(y_n, y_{n+1}) + Ł_\mu(y_{n+1}, y_m)] \\
&\preceq \mu(y_n, y_m) Ł_\mu(y_n, y_{n+1}) + \mu(y_n, y_m)\mu(y_{n+1}, y_m) Ł_\mu(y_{n+1}, y_{n+2}) \\
&\quad + \cdots + \mu(y_n, y_m)\mu(y_{n+1}, y_m) \cdots \mu(y_{m-2}, y_m)\mu(y_{m-1}, y_m) Ł_\mu(y_{m-1}, y_m) \\
&\preceq \mu(y_n, y_m)(b^*)^n \Upsilon (b)^n + \mu(y_n, y_m)\mu(y_{n+1}, y_m)(b^*)^{n+1} \Upsilon (b)^{n+1} \\
&\quad + \cdots + \mu(y_n, y_m)\mu(y_{n+1}, y_m) \cdots \mu(y_{m-2}, y_m)\mu(y_{m-1}, y_m)(b^*)^{m-1} \Upsilon (b)^{m-1} \\
&= \mu(y_n, y_m)(b^*)^n \Upsilon^{\frac{1}{2}} \Upsilon^{\frac{1}{2}} (b)^n + \mu(y_n, y_m)\mu(y_{n+1}, y_m)(b^*)^{n+1} \Upsilon^{\frac{1}{2}} \Upsilon^{\frac{1}{2}} (b)^{n+1} \\
&\quad + \cdots + \mu(y_n, y_m)\mu(y_{n+1}, y_m) \cdots \mu(y_{m-2}, y_m)\mu(y_{m-1}, y_m)(b^*)^{m-1} \Upsilon^{\frac{1}{2}} \Upsilon^{\frac{1}{2}} (b)^{m-1} \\
&= \mu(y_n, y_m)(\Upsilon^{\frac{1}{2}} b^n)^* (\Upsilon^{\frac{1}{2}} (b)^n) + \mu(y_n, y_m)\mu(y_{n+1}, y_m)(\Upsilon^{\frac{1}{2}} b^{n+1})^* (\Upsilon^{\frac{1}{2}} (b)^{n+1}) \\
&\quad + \cdots + \mu(y_n, y_m)\mu(y_{n+1}, y_m) \cdots \mu(y_{m-2}, y_m)\mu(y_{m-1}, y_m)(\Upsilon^{\frac{1}{2}} b^{m-1})^* (\Upsilon^{\frac{1}{2}} (b)^{m-1}) \\
&= \mu(y_n, y_m)|\Upsilon^{\frac{1}{2}} b^n|^2 + \mu(y_n, y_m)\mu(y_{n+1}, y_m)|\Upsilon^{\frac{1}{2}} b^{n+1}|^2 \\
&\quad + \cdots + \mu(y_n, y_m)\mu(y_{n+1}, y_m) \cdots \mu(y_{m-2}, y_m)\mu(y_{m-1}, y_m)|\Upsilon^{\frac{1}{2}} b^{m-1}|^2 \\
&= \sum_{j=0}^{m-1} |\Upsilon^{\frac{1}{2}} b^{n+j}|^2 \prod_{k=0}^{j} \mu(y_{n+k}, y_m) \preceq \| \sum_{j=0}^{m-1} |\Upsilon^{\frac{1}{2}} b^{n+j}|^2 \| \prod_{k=0}^{j} \mu(y_{n+k}, y_m) \\
&\preceq \sum_{j=0}^{m-1} \| \Upsilon^{\frac{1}{2}} \| \| b^{n+j} \|^2 \prod_{k=0}^{j} \mu(y_{n+k}, y_m) \preceq \| \Upsilon^{\frac{1}{2}} \| \sum_{j=0}^{m-1} \| b^{n+j} \|^2 \prod_{k=0}^{j} \mu(y_{n+k}, y_m) \\
&\preceq \| \Upsilon^{\frac{1}{2}} \| \sum_{j=0}^{m-1} \| b^j \|^2 \prod_{k=0}^{j} \mu(y_k, y_m).
\end{aligned}$$

By the ratio test, we have

$$\lim_{j\to\infty} \frac{\| b^{j+1} \|^2 \prod_{k=0}^{j+1} \mu(y_k, y_m)}{\| b^j \|^2 \prod_{k=0}^{j} \mu(y_k, y_m)} \preceq \lim_{j\to\infty} \mu(y_k, y_m) \| b \|^2 \prec 1_\Omega.$$

Now, for all $m \geq 1$, we have

$$S_n = \sum_{j=0}^{n} \| b^j \|^2 \prod_{k=0}^{j} \mu(y_k, y_m)$$

and

$$S = \sum_{j=0}^{\infty} \| b^j \|^2 \prod_{k=0}^{j} \mu(y_k, y_m).$$

Thus we obtain

$$Ł_\mu(y_n, y_m) \preceq \| \Upsilon \| \| b^{2n} \| [S_{n-1} - S_n].$$

So the sequence $\{y_n\}$ is a left Cauchy sequence in Ω. Similarly, we prove that the sequence $\{y_n\}$ is a right Cauchy sequence in Ω. Hence it is a Cauchy sequence. Since Ω is complete, there exists $y \in \Omega$ such that

$$\lim_{n\to\infty} Ł_\mu(y_n, y) = \lim_{n\to\infty} Ł_\mu(y, y_n) = 0_\Omega.$$

Now, we will show that y is a fixed point of T. For every $n \in \mathbb{N}$, we have

$$\begin{aligned} Ł_\mu(Ty, y) &\preceq \mu(Ty, y)[Ł_\mu(Ty, y_{n+1}) + Ł_\mu(y_{n+1}, y)] \\ &= \mu(Ty, y)[Ł_\mu(Ty, Ty_n) + Ł_\mu(y_{n+1}, y)] \\ &\preceq \mu(Ty, y)[b^* Ł_\mu(y, y_n) b + Ł_\mu(y_{n+1}, y)] \\ &\to 0_\Omega \end{aligned}$$

as $n \to \infty$.

Therefore, y is a fixed point of T.

For the uniqueness, suppose that $a, b \in X$ such that $Ta = a$ and $Tb = b$. Then we have

$$Ł_\mu(a, b) = Ł_\mu(Ta, Tb) \preceq b^* Ł_\mu(a, b) b$$

and so

$$\begin{aligned} \| Ł_\mu(a, b) \| &\preceq \| b^* \| \| Ł_\mu(a, b) \| \| b \| \\ &= \| b \|^2 \| Ł_\mu(a, b) \| \\ &< \| Ł_\mu(a, b) \| . \end{aligned}$$

This is a contradiction. Hence $a = b$. That is. T has a unique fixed point . □

Definition 2.2. Suppose that A and B are unital C^*-algebras with units 1_A and 1_B, respectively. We say that a mapping $\Gamma : A \to B$ is *C^*-algebra homomorphism* if for all $a_1, a_2 \in \mathbb{C}$ and $\zeta, \xi \in A$,

(i) $\Gamma(a_1\zeta + a_2\xi) = a_1\Gamma(\zeta) + a_2\Gamma(\xi)$;
(ii) $\Gamma(\zeta\xi) = \Gamma(\zeta)\Gamma(\xi)$;
(iii) $\Gamma(\zeta^*) = \Gamma(\zeta)^*$;
(iv) $\Gamma(1_A) = 1_B$.

Definition 2.3. Let Ψ_Ω be the set of positive functions $\Gamma_\Omega : \Omega^+ \to \Omega^+$, which satisfy the following:

(i) Γ_Ω is continuous and nondecreasing;
(ii) $\Gamma_\Omega(a) = 0_\Omega$ if and only if $a = 0_\Omega$;
(iii) $\sum_{n=1}^{\infty} \Gamma_\Omega^n(a) < \infty$, $\lim\limits_{n\to\infty} \Gamma_\Omega^n(a)$ for each $a \succeq 0_A$;
(iv) the series $\sum_{k=0}^{\infty} b^k \Gamma_\Omega^k(a) < \infty$ for $a \succeq 0_A$.

Remark 2.4. We can conclude the following:
(1) Every C^*-algebra homomorphism is contractive and hence bounded.
(2) Every C^*-algebra homomorphism is positive.

Definition 2.5. Let $(X, \Omega, Ł_\mu)$ be a C^*-algebra-valued Eqbms and $T : X \to X$ be a mapping. Then we say that T is a *α-Γ_Ω-contractive mapping* if there exist two functions $\alpha : X \times X \to \Omega^+$ and $\Gamma_\Omega \in \Psi_\Omega$ such that

$$\alpha(a,b)Ł_\mu(Ta,Tb) \preceq \Gamma_\Omega(Ł_\mu(a,b)), \forall a,b \in X.$$

Theorem 2.6. *Let $(X, \Omega, Ł_\mu)$ be a complete C^*-algebra-valued Eqbms and $T : X \to X$ be an α-Γ_Ω-contractive mapping satisfying the following conditions:*

(a) *T is α-admissible;*
(b) *there exists $y_0 \in X$ such that $\alpha(y_0, Ty_0) \succeq 1_\Omega$ and $\alpha(Ty_0, y_0) \succeq 1_\Omega$;*
(c) *T is continuous.*

Then T has a fixed point in X.

Proof. Let $y_0 \in X$ such that $\alpha(y_0, Ty_0) \succeq 1_\omega$ and define a sequence $\{y_n\}$ such that $y_{n+1} = Ty_n$, $\forall n \in \mathbb{N}$.
If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then y_n is a fixed point for T.

Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we get

$$\alpha(y_0, y_1) = \alpha(y_0, Ty_0) \succeq 1_\Omega \Rightarrow \alpha(Ty_0, Ty_1) = \alpha(y_1, y_2) \succeq 1_\Omega. \tag{2.1}$$

By induction, we have

$$\alpha(y_n, y_{n+1}) \succeq 1_\Omega, \forall n \in \mathbb{N}. \tag{2.2}$$

By (2.1) and (2.2), we get

$$Ł_\mu(y_n, y_{n+1}) = Ł_\mu(Ty_{n-1}, Ty_n) \preceq \alpha(y_{n-1}, y_n)Ł_\mu(Ty_{n-1}, Ty_n) \preceq \Gamma_\Omega(Ł_\mu(y_{n-1}, y_n).$$

By induction, we obtain

$$Ł_\mu(y_n, y_{n+1}) \preceq \Gamma_\Omega^n(Ł_\mu(y_0, y_1) \tag{2.3}$$

for all $n \in \mathbb{N}$.
For $n < m$, we have

$$\begin{aligned}
&Ł_\mu(y_n, y_m) \preceq \mu(y_n, y_m)[Ł_\mu(y_n, y_{n+1}) + Ł_\mu(y_{n+1}, y_m)]\\
&\preceq \mu(y_n, y_m)\Gamma_\Omega^n(Ł_\mu(y_0, y_1) + \mu(y_n, y_m)\mu(y_{n+1}, y_m)[Ł_\mu(y_{n+1}, y_{n+2}) + Ł_\mu(y_{n+2}, y_m)]\\
&\preceq \mu(y_n, y_m)\Gamma_\Omega^n(Ł_\mu(y_0, y_1) + \mu(y_n, y_m)\mu(y_{n+1}, y_m)\Gamma_\Omega^{n+1}(Ł_\mu(y_0, y_1) + \cdots\\
&\preceq \Gamma_\Omega^n(Ł_\mu(y_0, y_1)[\mu(y_1, y_m)\mu(y_2, y_m) \cdots \mu(y_{n-1}, y_m)\mu(y_n, y_m)\\
&\quad + \mu(y_1, y_m)\mu(y_2, y_m) \cdots \mu(y_n, y_m)\mu(y_{n+1}, y_m)\Gamma_\Omega(Ł_\mu(y_0, y_1)) + \cdots\\
&\quad + \mu(y_1, y_m)\mu(y_2, y_m)\cdots\mu(y_n, y_m)\mu(y_{n+1}, y_m) \cdots \mu(y_{m-1}, y_m)\Gamma_\Omega^{m-n-1}(Ł_\mu(y_0, y_1))]\\
&= \Gamma_\Omega^n(Ł_\mu(y_0, y_1) \sum_{j=n}^{m-1} \Gamma^{j-n}(Ł_\mu(y_0, y_1)) \prod_{i=1}^{j} \mu(y_i, y_m).
\end{aligned}$$

Since $\lim_{n\to\infty} \Gamma^n(Ł_\mu(y_0, y_1)) = 0$, using Definition 2.3, we obtain $\lim_{n\to\infty} Ł_\mu(y_n, y_m) = 0$. Thus $\{y_n\}$ is a left Cauchy sequence in X. Similarly, by taking $\alpha(Ty_0, y_0) \succeq 1_\Omega$ for $M > n$, we can prove that $\{y_n\}$ is a right Cauchy sequence in X. Hence $\{y_n\}$ is a Cauchy sequence in X. Since $(X, \Omega, Ł_\mu)$ is complete, there exists $y \in X$ such that $y_n \to y$ as $n \to \infty$. From continuity of T, it follows that $y_{n+1} = Ty_n \to Ty$ as $n \to \infty$. By uniqueness of the limit, we get $Ty = y$, which is a fixed point of T. □

To prove the uniqueness of the fixed point, we will consider the condition:

H: For all $a, b \in X$ there exists $c \in X$ such that $\alpha(a, c) \succeq 1_\Omega$, $\alpha(b, c) \succeq 1_\Omega$ or $\alpha(c, a) \succeq 1_\Omega$, $\alpha(c, b) \succeq 1_\Omega$.

Theorem 2.7. *Suppose that (H) and all the assumptions of Theorem 2.6 hold. Then we obtain the uniqueness of fixed points of T.*

Proof. Suppose that a and b are two fixed points of T. From (H), there exists $c \in X$ such that $\alpha(a, c) \succeq 1_\Omega$ and $\alpha(b, c) \succeq 1_\Omega$. Since T is α-admissible, we get

$\alpha(a, T^n c) \succeq 1_\Omega$,$\alpha(b, T^n c) \succeq 1_\Omega$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}Ł_\mu(a, T^n c) = Ł_\mu(Ta, T(T^n c)) &\preceq \alpha(a, T^n c) Ł_\mu(Ta, T(T^n c))\\ &\preceq \Gamma_\Omega^n(Ł_\mu(a, c))\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\Gamma_\Omega^n(Ł_\mu(a, c)) \to 0_\Omega$ as $n \to \infty$, $T^n c = a$. Similarly, $T^n c = b$ as $n \to \infty$. The uniqueness of the limit gives $a = b$. □

3. Application of Fixed Point Theorems

Let $(X, \Omega, Ł_\mu)$ be a C^*-algebra-valued Eqbms. and $T : X \to X$ be a mapping. Let us consider the fixed point equation

$$y = Ty. \tag{3.1}$$

We say that the fixed point problem (3.1) is Hyers-Ulam stable via C^*-algebra-valued extended quasi b-metric space if for each $\epsilon \succ 0_\Omega$ and $v \in X$ satisfying

$$Ł_\mu(Tv, v) \preceq \epsilon, \tag{3.2}$$

there exists $K \succ 0_\Omega$ and $u \in X$ satisfying the fixed point equation (3.1) such that

$$Ł_\mu(u, v) \preceq K\epsilon. \tag{3.3}$$

Theorem 3.1. *Let $(X, \Omega, Ł_\mu)$ be a complete C^*-algebra-valued Eqbms. Suppose that all the assumptions of Theorem 2.1 (resp., Theorem 2.7) hold. If $\mu(u, v)\alpha(u, v) \succeq 1_\Omega$ for all ϵ-solutions u, v, then the fixed point equation (3.1) is Hyers-Ulam stable.*

Proof. By Theorem 2.1, we have a unique $u^* \in X$ satisfying (3.1). Let $\epsilon \succ 0_\omega$ and $v^* \in X$ be a solution of (3.2), that is, $u^* = Tu^*$ and $Ł_\mu(Tv^*, v^*) \preceq \epsilon$. Since $Ł_\mu(Tu^*, u^*) = Ł_\mu(u^*, u^*) = o_\Omega \preceq \epsilon$, by hypothesis, we have $\alpha(u^*, v^*) \succeq 1_\Omega$. Thus

$$\begin{aligned}Ł_\mu(u^*, v^*) &= Ł_\mu(Tu^*, v^*)\\ &\preceq \mu(Tu^*, v^*)[Ł_\mu(Tu^*, Tv^*) + Ł_\mu(Tv^*, v^*)]\\ &\preceq \mu(u^*, v^*)\alpha(u^*, v^*)Ł_\mu(Tu^*, Tv^*) + \mu(u^*, v^*)\epsilon\\ &\preceq \mu(u^*, v^*)\alpha(u^*, v^*)Ł_\mu(u^*, v^*) + \mu(u^*, v^*)\epsilon.\end{aligned}$$

So $1 - \mu(u^*, v^*)\alpha(u^*, v^*)Ł_\mu(u^*, v^*) \preceq \mu(u^*, v^*)\epsilon$. Thus we deduce

$$Ł_\mu(u^*, v^*) \preceq \frac{\mu(u^*, v^*)}{(1 - \mu(u^*, v^*)\alpha(u^*, v^*)}\epsilon = K\epsilon,$$

where $K = \frac{\mu(u^*, v^*)}{1 - \mu(u^*, v^*)\alpha(u^*, v^*)} \succ 0_\Omega$. □

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

References

1. M. Asim & M. Imdad: C^*-Algebra valued extended b-metric spaces and fixed point results with an application. U.P.B. Sci. Bull. Ser. A. **82** (2020), no.1, 207-217.
2. R. Chaharpashlou, D. O'Regan, C. Park & R. Saadati: C^*-Algebra valued fuzzy normed spaces with application of Hyers-Ulam stability of a random integral equation. Adv. Difference Equ. **2020** (2020), Paper No. 326. `https://doi.org/10.1186/s13662-020-02780-0`
3. I. Ćirić, V. Parvaneh & N. Hussain: Fixed point results for weakly α-admissible pairs. Filomat **30** (2016), no. 14, 3697-3713. `https://doi.org/10.2298/FIL1614697C`
4. K.R. Davidson: C^*-Algebras by Example. Fields Institute Monographs **6**, Am. Math. Soc., Providence, 1996.
5. Z. Islam, M. Sarwar & C. Tunç: Coincidence point results for (Φ, Ψ)-weak contractive mappings in cone 2-metric spaces. Honam Math. J. **43** (2021), no. 2, 305-323. `https://doi.org/10.5831/hmj.2021.43.2.305`
6. C. Kongban & P. Kumam: Quadruple random common fixed point results of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras. J. Nonlinear Sci. Appl. **11** (2018), 131-149. `https://doi.org/10.22436/jnsa.011.01.10`
7. Z. Ma & L. Jiang: C^*-Algebra-valued b-metric space and related fixed point theorems. Fixed Point Theory Appl. **2015** (2015), Paper No. 222. `https://doi.org/10.1155/2018/3257189`
8. Z. Ma, L. Jiang & H. Sun: C^*-Algebra-valued metric space and related fixed point theorems. Fixed Point Theory Appl. **2014** (2014), Paper No. 206. `https://doi.org/10.1186/1687-1812-2014-206`
9. S.K. Malhotra, J.B. Sharma & S. Shukla: Fixed point of α-admissible mapping in cone metric spaces with Banach algebra. Int. J. Anal. Appl. **9** (2015), no. 1, 9-18.

10. B. Moeini, A.H. Ansari & C. Park: $\mathcal{JHR}$-operator pairs in C^*-algebra-valued modular metric spaces and related fixed point results via C_*-class functions. J. Fixed Point Theory Appl. **20** (2018), no. 1, Paper No. 17.
11. R. Mustafa, S. Omran & Q.N. Nguyen: Fixed point theory using Ψ contractive mapping in algebra valued b-metric space. Math. **9** (2021), Paper No. 92. https://doi.org/10.3390/math9010092
12. B. Samet: The class of (α, Ψ)-type contractions in b-metric space and fixed point theorems. Fixed Point Theory Appl. **2015** (2015), Paper No. 92. https://doi.org/10.1186/s13663-015-0344-z
13. B. Samet, C. Vetro & P. Vetro: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal. **75** (2012), no. 4, 2154-2165. https://doi.org/10.1016/j.na.2011.10.014
14. Q. Xin, L. Jiang & Z. Ma: Common fixed point theorems in C^*-algebra-valued metric spaces. J. Nonlinear Sci. Appl. **9** (2016), 4617-4627. https://doi.org/10.48550/arXiv.1506.05545

[a]PROFESSOR: DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF AL-QADISIYAH, AL-DIWANIYA, IRAQ
Email address: qusuay.alqifiary@qu.edu.iq

[b]PROFESSOR: DEPARTMENT OF DATA SCIENCE, DAEJIN UNIVERSITY, KYUNGGI 11159, KOREA
Email address: jrlee@daejin.ac.kr