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ON ISOMORPHISM THEOREMS AND CHINESE REMAINDER THEOREM IN HYPERNEAR RINGS

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Abstract. The purpose of this paper is to consider the abstract theory of hypernear rings. In this regard, we derive the isomorphism theorems for hypernear rings as well as Chinese Remainder theorem. Our results can be considered as a generalization for the cases of Krasner hyperrings, near rings and rings.

1. Introduction

Recent developments in various algebraic structures and the applications of those in different areas play an important role in Science and Technology. One of the best tools to study the non-linear algebraic systems is the theory of *near rings*. The interest in near rings and near-fields started at the beginning of the 20th century, when L. Dickson wanted to know whether the list of axioms for skew fields is redundant or not. He found in [6] that there do exist "near fields" which fulfill all axioms for skew fields except one distributive law. Since 1950, the theory of near rings had applications to several domains, for instance in the area of dynamical systems, graphs, homological algebra, universal algebra, category theory, geometry, and so on. A *near ring* is an algebraic structure similar to a ring but satisfying fewer axioms. Although near rings arise naturally in various ways, most near rings studied today arise as the endomorphisms of a group or cogroup object of a category. For details about near ring theory and applications, we refer to [1, 7].

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Marty [10] in 1934 at the eighth Congress of Scandinavian Mathematicians, where he generalized the notion of a group to that of a hypergroup. In a group, the composition of two elements is an element whereas

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in a hypergroup, the composition of two elements is a non-empty set. Since then, researchers were busy in defining and working on other types of hyperstructures such as hyperring, hypermodule, and many others. For example, Krasner [9] introduced a special type of hyperrings known as *Krasner hyperring* and he with other researchers worked on it and its properties. Since the hyperring is a very general hyperstructure, several researchers endowed it with more stronger or less strong axiom. As a result we are really dealing now with a big number of hyperrings. In recent years, an interest in hyperalgebraic system with binary hyperoperation of addition and operation of multiplication satisfying all hyperring axioms except possibly one of the distributive law and the commutativity of addition has arisen. Such systems are generally called *hypernear rings* and they were introduced by Dasic [5] during the Fourth international congress on algebraic hyperstructures and applications in 1990.

In our paper, we consider hypernear rings from theoretical point of view and it is organized as follows. After an Introduction, in Section 2, we present some preliminaries related to hypernear rings and we study some of their properties. In Section 3, we define factor hypernear rings and derive the three isomorphism theorems in hypernear rings. Finally, in Section 4, we derive Chinese Remainder Theorem in hypernear rings and deduce it for near rings.

2. Properties of Hypernear Rings

In this section, we present some definitions related to hypernear rings (see [2, 3, 5, 8]) that are used throughout the paper. Moreover, we present some examples and study some properties of hypernear rings that are used in Sections 3 and 4.

Definition 2.1 ([1])**.** Let R be a non-empty set. Then $(R, +, \cdot)$ is called a *near ring* if:

(1) $(R, +)$ is a group (not necessary abelian);
(2) $(R, \cdot)$ is a semigroup;
(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$ (Left distributive law).

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*.

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.2 ([2, 3])**.** A *quasi-canonical hypergroup* is a hypergroupoid $(R, +)$, where $0 \in R$ is an identity element, "+" maps $R \times R$ into the non-empty subsets of R, and the following axioms hold for all $x, y, z \in R$:

(1) $(x + y) + z = x + (y + z)$,
(2) $0 + x = x + 0 = x$,
(3) $-x \in R$ with $0 \in (-x + x) \cap (x + (-x))$,
(4) $x \in y + z$ implies $y \in x - z$ and $z \in -y + x$.

The following elementary facts about quasi-canonical hypergroup follow easily from the axioms: $-(-x) = x$, $-(x + y) = -y - x$.

Definition 2.3 ([9])**.** Let R be a non-empty set. Then $(R, +, \cdot)$ is called a *Krasner hyperring* if:

(1) $(R, +)$ is a commutative quasi-canonical hypergroup with identity 0;
(2) $(R, \cdot)$ is a semigroup with $x \cdot 0 = 0$ and $0 \cdot x = 0$ for all $x \in R$;
(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$ (Left distributive law);
(4) $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$ (Right distributive law).

Definition 2.4 ([5])**.** Let R be a non-empty set. Then $(R, +, \cdot)$ is called a (left) *hypernear ring* if:

(1) $(R, +)$ is a quasi-canonical hypergroup with identity 0;
(2) $(R, \cdot)$ is a semigroup and $x \cdot 0 = 0$ for all $x \in R$;
(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$ (Left distributive law).

Remark 1. In view of axiom 3 in Definition 2.4, one speaks more precisely about "left hypernear rings". Postulating $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$ instead of 3, one gets "right hypernear rings". The theory runs completely parallel in both cases, of course one can decide to use just one version. Moreover, one can easily construct a right/left hypernear ring $(R, +, \star)$ from a left/right hypernear ring $(R, +, \cdot)$ by using the operation $x \star y = y \cdot x$ for all $x, y \in R$.

Throughout our paper, we deal with left hypernear rings.

Remark 2. Every near ring is a hypernear ring.

Remark 3. Every Krasner hyperring is a hypernear ring.

Remark 4. Let $(R,+)$ be a quasi-canonical hypergroup and define "$\cdot$" as:

(1) $x \cdot y = 0$ for all $x, y \in R$. Then $(R,+,\cdot)$ is a hypernear ring called the *trivial hypernear ring*.

(2) $x \cdot y = y$ for all $x, y \in R$. Then $(R,+,\cdot)$ is a hypernear ring called the *trivial constant hypernear ring*.

(3) $x \cdot y = \begin{cases} 0 & \text{if } x = 0, \\ y & \text{otherwise.} \end{cases}$ for all $x, y \in R$. Then $(R,+,\cdot)$ is a hypernear ring called the *trivial zero-symmetric hypernear ring*.

Definition 2.5. Let $(R,+,\cdot)$ be a hypernear ring. If $0 \cdot x = 0$ for all $x \in R$ then R is called *zero symmetric hypernear ring*.

Definition 2.6. Let $(R,+,\cdot)$ be a hypernear ring. If $1 \in R$ and $1 \cdot x = x \cdot 1 = x$ for all $x \in R$ then R is called *unitary hypernear ring*.

Definition 2.7. Let $(R,+,\cdot)$ be a unitary hypernear ring. If $(R - \{0\}, \cdot)$ is a group then R is called *hypernear field*.

Definition 2.8 ([5])**.** Let $(R,+,\cdot)$ be a hypernear ring. A non-empty subset S of R is called a *subhypernear ring* of R if $(S,+,\cdot)$ is a hypernear ring.

Remark 5. To prove that $(S,+,\cdot)$ is a subhypernear ring of $(R,+,\cdot)$, it suffices to show that $x - y \subseteq S$ and $x \cdot y \in S$ for all $x, y \in S$.

Definition 2.9 ([5])**.** Let $(R,+,\cdot)$, $(S,+_1,\cdot_1)$ be hypernear rings and $f : R \to S$ be a function. Then f is called a *hypernear ring homomorphism* if for all $x, y \in R$

(1) $f(x + y) = f(x) +_1 f(y)$;

(2) $f(x \cdot y) = f(x) \cdot_1 f(y)$; and

(3) $f(0) = 0$.

If f is also bijective then it is called *hypernear ring isomomorphism* and we say that R and S are isomorphic hypernear rings ($R \cong S$).

Definition 2.10 ([5])**.** Let $(R,+,\cdot)$, $(S,+_1,\cdot_1)$ be hypernear rings and $f : R \to S$ be a hypernear ring homomorphism. Then Kernel of f is defined as

$$Ker(f) = \{x \in R : f(x) = 0\}.$$

Proposition 2.11. *Let* $(R, +, \cdot)$, $(S, +_1, \cdot_1)$ *be hypernear rings and* $f : R \to S$ *be a hypernear ring homomorphism. Then for all* $x, y \in R$

(1) $f(-y) = -f(y)$;
(2) $f(x - y) = f(x) - f(y)$.

Proof. The proof is straightforward. □

Lemma 2.12. *Let* $(H, +)$ *be a quasi-canonical hypergroup and let* $M(H)$ *be the set of all mappings* $f : H \to H$*. For all* $f, g \in M(H)$ *we define the hyperoperation* $f \oplus g$ *of mappings as follows:*

$$(f \oplus g)(x) = \{h \in M(H) : h(x) \in f(x) + g(x)\},$$

and the operation "·" as $(f \cdot g)(x) = g(f(x))$*. Then* $(M(H), \oplus, \cdot)$ *is a hypernear ring.*

Proof. The proof is straightforward. □

Lemma 2.13 ([5]). *Let* $(H, +)$ *be a quasi-canonical hypergroup and let* $M_0(H)$ *be the set of all mappings* $f : H \to H$ *such that* $f(0) = 0$*. For all* $f, g \in M_0(H)$ *we define the hyperoperation* $f \oplus g$ *of mappings as follows:*

$$(f \oplus g)(x) = \{h \in M_0(H) : h(x) \in f(x) + g(x)\},$$

and the operation "·" as $(f \cdot g)(x) = g(f(x))$*. Then* $(M_0(H), \oplus, \cdot)$ *is a hypernear ring.*

Remark 6. The hypernear ring $(M_0(H), \oplus, \cdot)$ of Lemma 2.13 is a zero-symmetric and unitary hypernear ring (the unity here is the identity map). Moreover, it is a subhypernear ring of the hypernear ring $(M(H), \oplus, \cdot)$ of Lemma 2.12.

Example 1. Let $(H, +)$ be the quasi-canonical hypergroup defined by the following table.

$+$	0	1
0	0	1
1	1	$\{0, 1\}$

Let f be the zero map, g be the identity map, h be the map defined as $h(0) = 1, h(1) = 0$, and i be the map defined as $i(0) = 1, i(1) = 1$. Using Lemmas 2.12

and 2.13, we get that $(M(H), \oplus, \cdot)$ is a hypernear ring and $(M_0(H), \oplus, \cdot)$ is a zero-symmetric and unitary hypernear ring and they are presented by the following tables.

$\oplus$	f	g	h	i
f	f	g	h	i
g	g	$\{f,g\}$	i	$\{h,i\}$
h	h	i	$\{f,h\}$	$\{g,i\}$
i	i	$\{h,i\}$	$\{g,i\}$	$\{f,g,h,i\}$

and

$\cdot$	f	g	h	i
f	f	f	i	i
g	f	g	h	i
h	f	h	g	i
i	f	i	f	i

$\oplus$	f	g
f	f	g
g	g	$\{f,g\}$

and

$\cdot$	f	g
f	f	f
g	f	g

Example 2. Let $(R, +, \cdot)$ be defined by the following tables. Then Remark 4 asserts that $(R, +, \cdot)$ is a constant hypernear ring. Moreover, it is not isomorphic to the hypernear ring $(M_0(H), \oplus, \cdot)$ of Example 1.

$+$	0	1
0	0	1
1	1	$\{0,1\}$

and

$\cdot$	0	1
0	0	1
1	0	1

Example 3. Let $(R, +, \cdot)$ be defined by the following tables. Then Remark 4 asserts that $(R, +, \cdot)$ is a constant hypernear ring.

$+$	0	1	2
0	0	1	2
1	1	$\{0,2\}$	$\{1,2\}$
2	2	$\{1,2\}$	$\{0,1\}$

and

$\cdot$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

Lemma 2.14. *Let* $(R,+)$ *be a quasi-canonical hypergroup and* $R[x]$ *be the set of all polynomials with coefficients from* R*. Then* $(R[x],+,\cdot)$ *is a hypernear ring, where for all* $p(x)=a_0+a_1x+\ldots+a_nx^n, q(x)=b_0+b_1x+\ldots+b_nx^n \in R[x]$*,* $p(x)+q(x)=\{h(x)=c_0+\ldots+c_nx^n : c_i \in a_i+b_i$ *for* $i=1,\ldots,n\}$ *and* $p(x)\cdot q(x)=q(p(x))$*.*

Proof. The proof is straightforward. □

Example 4. Let n be a natural number, $(R,+)$ be a quasi-canonical hypergroup and $P_n(R)$ be the set of all polynomials with degree less than or equal to n and with coefficients from R. Then $(P_n(R),+,\cdot)$ is a constant hypernear ring, where for all $p(x)=a_0+a_1x+\ldots+a_nx^n, q(x)=b_0+b_1x+\ldots+b_nx^n \in P_n(R)$, $p(x)+q(x)=\{h(x)=c_0+\ldots+c_nx^n : c_i \in a_i+b_i$ for $i=1,\ldots,n\}$ and $p(x)\cdot q(x)=q(x)$.

Example 5. Let n be a natural number, $(R,+)$ be a quasi-canonical hypergroup and $M_n(R)$ be the set of all $n\times n$ matrices with entries from R. Then $(M_n(R),+,\cdot)$ is a constant hypernear ring, where for all $A=(a_{ij}), B=(b_{ij})\in M_n(R)$, $A+B=\{C=(c_{ij}) : c_{ij}\in a_{ij}+b_{ij}\}$ and $A\cdot B=B$.

Example 6. Let $(R,+,\cdot)$ be defined by the following tables. Then Remark 4 asserts that $(R,+,\cdot)$ is a zero-symmetric hypernear ring.

$+$	0	1	2
0	0	1	2
1	1	$\{0,2\}$	$\{1,2\}$
2	2	$\{1,2\}$	$\{0,1\}$

and

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Example 7. Let $(R,+,\cdot)$ be defined by the following tables. Then Remark 4 asserts that $(R,+,\cdot)$ is a zero-symmetric hypernear ring.

$+$	0	1	2	3
0	0	1	2	3
1	1	$\{0,1\}$	2	3
2	2	2	$\{0,1,3\}$	$\{2,3\}$
3	3	3	$\{2,3\}$	$\{0,1,2\}$

and

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

Proposition 2.15. *Let $(R,+,\cdot)$ be a hypernear ring and $x,y\in R$. Then $x\cdot(-y)=-x\cdot y$.*

Definition 2.16. [5] Let $(R,+,\cdot)$ be a hypernear ring. A subhypernear ring I of R is called a *hyperideal* of R if:

(1) $x+y-x\subseteq I$ for all $x\in R$ and $y\in I$;
(2) $x\cdot y\in I$ for all $x\in R$ and $y\in I$ i.e., $RI\subseteq I$;
(3) $(x+a)\cdot y-x\cdot y\subseteq I$ for all $x,y\in R$ and $a\in I$.

Example 8. Let $(R,+,\cdot)$ be the hypernear ring in Example 7. Then $(\{0,1\},+,\cdot)$ is a subhypernear ring of R that is not a hyperideal (This is clear as $-3+0+3=3+3=\{0,1,2\}\nsubseteq\{0,1\}$).

Remark 7. Every hypernear ring $R\neq\{0\}$ has at least two hyperideals: R and the trivial hyperideal $\{0\}$.

Remark 8. If I is a hyperideal of a zero-symmetric hypernear ring R then $RI\cup IR\subseteq I$.

Proposition 2.17. *Let $(R,+,\cdot)$ be a hypernear ring and I a hyperideal of R. If*

(1) *R is unitary and $1\in I$ then $I=R$.*

(2) *R is a hypernear field then $I = \{0\}$ or $I = R$.*

Proof. The proof is straightforward. □

Proposition 2.18. *Let $(R, +, \cdot)$ be a hypernear ring and A, B be hyperideals of R. Then $A \cap B$ is a hyperideal of R.*

Proof. The proof is straightforward. □

Corollary 2.19. *Let $(R, +, \cdot)$ be a hypernear ring and A_i be a hyperideal of R for $i = 1, \ldots, n$. Then $\bigcap_{i=1}^{n} A_i$ is a hyperideal of R.*

Proof. The proof follows by using Proposition 2.18 and induction. □

Proposition 2.20. *Let $(R, +, \cdot)$ be a hypernear ring and A, B be hyperideals of R. Then $A + B = B + A$.*

Proof. Let $x \in A + B$. Then there exist $a \in A, b \in B$ such that $x \in a + b$. We get now that $x \in 0 + a + b \subseteq (b - b) + a + b = b + (-b + a + b) \subseteq B + A$. Thus, $A + B \subseteq B + A$. Similarly, we can prove $B + A \subseteq A + B$. □

Lemma 2.21. *Let $(R, +, \cdot)$ be a hypernear ring and A, B be hyperideals of R. Then $A + B$ is a hyperideal of R.*

Proof. Let $a + b, a_1 + b_1 \in A + B$ and $r, s \in R$. (1) We have $a + b - (a_1 + b_1) = a+b-b_1-a_1 \subseteq a+b-b_1-a+a-a_1 = (a+b-b_1-a)+a-a_1 \subseteq B+A = A+B$. (2) $r\cdot(a+b) = r\cdot a+r\cdot b \subseteq A+B$. (3) $-r+a+b+r \subseteq (-r+a+r)+(-r+b+r) \subseteq A+B$. (4) $(r + (a + b)) \cdot s - r \cdot s = ((r + a) + b) \cdot s - r \cdot s = (\bigcup_{t\in r+a} t + b) \cdot s - r \cdot s \subseteq \bigcup_{t\in r+a}[(t + b) \cdot s - t \cdot s + t \cdot s - r \cdot s] \subseteq \bigcup_{t\in r+a}[(t + b) \cdot s - t \cdot s] + \bigcup_{t\in r+a} t \cdot s - r \cdot s$. But $\bigcup_{t\in r+a} t - r \cdot s = (r + a) \cdot s - r \cdot s \subseteq A$ and $\bigcup_{t\in r+a}[(t + b) \cdot s - t \cdot s] \subseteq B$. Thus, $(r + (a + b)) \cdot s - r \cdot s = ((r + a) + b) \cdot s - r \cdot s \subseteq B + A = A + B$. □

Corollary 2.22. *Let $(R, +, \cdot)$ be a hypernear ring and A_i be a hyperideal of R. Then $\sum_{i=1}^{n} A_i$ is a hyperideal of R.*

Proof. The proof follows from Lemma 2.21 and using induction. □

3. Isomorphism Theorems in Hypernear Rings

In this section, we study some properties of hypernear ring homomorphism, define the factor hypernear ring, and derive the three isomorphism theorems in hypernear rings. The results in this section can be considered as a generalization for near rings [1] and for Krasner hyperrings as well [4].

Lemma 3.1. *Let $(R,+,\cdot)$ be a hypernear ring, $r \in R$, and I a hyperideal of R. Then*

(1) $I + I = I$;
(2) $r + I = I$ *if and only if* $r \in I$.

Proof. The proof of 1 is trivial, we prove 2.
Let $r \in I$. Having $0 \in I$ implies that $r = r + 0 \in r + I$. Thus, $I \subseteq r + I$. The latter and having $r + I \subseteq I$ implies that $r + I = I$. Conversely, let $r + I = I$. Then $r + 0 \in r + I = I$. □

Lemma 3.2. *Let $(R,+,\cdot)$ be a hypernear ring, $r, s \in R$, and I a hyperideal of R. Then*

(1) $(r + I) + (s + I) = (r + s) + I$;
(2) $(r + I) \cdot (s + I) + I = (r \cdot s) + I$.

Proof. Let $r, s \in R$.
(1): Having

$$(r + I) + (s + I) = \bigcup_{a,b \in I} (r + a) + (s + b) \subseteq \bigcup_{a,b \in I} (r + s - s + a + s + b)$$

and $-s + a + s \subseteq I$ implies that $(r + I) + (s + I) \subseteq r + s + I$. On the other hand, we have $(r+s)+I = (r+0)+(s+I) \subseteq (r+I)+(s+I)$. Thus, $(r+I)+(s+I) = (r+s)+I$.

(2): Having

$$(r + I) \cdot (s + I) + I = \bigcup_{a,b \in I} (r + a) \cdot (s + b) + I = \bigcup_{a,b \in I} (r + a) \cdot s + (r + a) \cdot b + I$$

and that $b \in I$ implies that $(r+I) \cdot (s+I)+I \subseteq \bigcup_{a \in I}(r \cdot s - r \cdot s + (r+a) \cdot s) + I$. The latter and having $-r \cdot s + (r + a) \cdot s \subseteq I$ implies that $(r + I) \cdot (s + I) + I \subseteq (r \cdot s) + I$. On the other hand, we have $(r \cdot s) + I = (r + 0) \cdot (s + 0) + I \subseteq (r + I) \cdot (s + I) + I$ □

Lemma 3.3. *Let $(R,+,\cdot)$ be a hypernear ring, I a hyperideal of R and $r, s \in R$. Then $r + I = s + I$ if and only if $(-s + r) \cap I \neq \emptyset$.*

Proof. Let $r + I = s + I$. Then there exist $i \in I$ such that $r \in s + i$. The latter implies that $i \in -s + r$. Thus, $-s + r \cap I \neq \emptyset$.

Let $-s + r \cap I \neq \emptyset$. Then there exists $i \in I$ such that $i \in -s + r$. The latter implies that $-s \in i - r$ and $r \in i + s$. Hence, $s \in r - i \subseteq r + I$ and $r \in I + s = s + I$. The latter implies that $r + I = s + I$. □

Let $(R, +, \cdot)$ be a hypernear ring and I a hyperideal of R. We define $(R/I, \oplus, \odot)$ as follows: For all $r, s \in R$,

$$(r + I) \oplus (s + I) = (r + s) + I \text{ and } (r + I) \odot (s + I) = (r \cdot s) + I.$$

Lemma 3.4. *Let $(R, +, \cdot)$ be a hypernear ring and I a hyperideal of R. Then the hyperoperation "$\oplus$" and the operation "$\odot$" are well defined.*

Proof. Lemma 3.2 asserts that the hyperoperation "$\oplus$" and the operation "$\odot$" are well defined. □

Theorem 3.5. *Let $(R, +, \cdot)$ be a hypernear ring and I a hyperideal of R. Then $(R/I, \oplus, \odot)$ is a hypernear ring and it is called the factor hypernear ring.*

Proof. Using Lemma 3.1 and Lemma 3.2, we can easily see that $(R/I, \oplus, \odot)$ is a hypernear ring with I as a zero and $-(r + I) = -r + I$ for all $r \in R$. □

Proposition 3.6. *Let $(R, +, \cdot)$, $(S, +_1, \cdot_1)$ be hypernear rings and $f : R \to S$ be a hypernear ring homomorphism. Then f is one-to-one if and only if $Ker(f) = \{0\}$.*

Proof. Let f be a one-to-one hypernear ring homomorphism and $x \in Ker(f)$. Then $f(x) = 0 = f(0)$. The latter implies that $x = 0$ and hence, $Ker(f) = \{0\}$.

Let $Ker(f) = \{0\}$ and $x, y \in R$ such that $f(x) = f(y)$. Then by using Proposition 2.11 we get that $0 \in -f(x) + f(x) = -f(x) + f(y) = f(-x + y)$. The latter implies that there exists $t \in -x + y$ such that $f(t) = 0$. Having $Ker(f) = \{0\}$ implies that $t = 0$. We get now that $0 \in -x + y$. Thus, $y = -(-x) = x$. □

Proposition 3.7. *Let $(R, +, \cdot)$, $(S, +_1, \cdot_1)$ be hypernear rings and $f : R \to S$ be a hypernear ring homomorphism. Then $Ker(f)$ is a subhypernear ring of R.*

Proof. The proof is straightforward. □

Proposition 3.8. *Let $(R, +, \cdot)$, $(S, +_1, \cdot_1)$ be hypernear rings and $f : R \to S$ be a hypernear ring homomorphism. Then $f(R)$ is a subhypernear ring of S.*

Proof. The proof is straightforward. □

Proposition 3.9. *Let $(R, +, \cdot)$, $(S, +_1, \cdot_1)$ be hypernear rings and $f : R \to S$ be a surjective hypernear ring homomorphism.*

(1) *If R is zero-symmetric then S is zero-symmetric.*
(2) *If R is unitary then S is unitary.*

(3) *If R is a hypernear field then S is a hypernear field.*

Proof. The proof is straightforward. □

Lemma 3.10. *Let $(R, +, \cdot)$ be a hypernear ring. Then every hyperideal of R is a kernel of a hypernear ring homomorphism.*

Proof. Let I be a hyperideal of R and $f : R \to R/I$ be defined by $f(r) = r + I$ for all $r \in R$. We prove that f is a hypernear ring homomorphism. (1) f is well defined. Let $r = s$. Having $0 \in (-r + s) \cap I$ implies that $r + I = s + I$ (by Lemma 3.3). Thus $f(r) = f(s)$. (2) $f(r + s) = (r + s) + I = (r + I) \oplus (s + I) = f(r) \oplus f(s)$. (3) $f(r \cdot s) = (r \cdot s) + I = (r + I) \odot (s + I) = f(r) \odot f(s)$. Lemma 3.1 asserts that $Ker(f) = \{r \in R : r + I = I\} = I$. This completes the proof. □

Theorem 3.11. (First Isomorphism Theorem) *Let $(R, +, \cdot)$, $(S, +_1, \cdot_1)$ be hypernear rings and $f : R \to S$ be a surjective hypernear ring homomorphism. If $Ker(f)$ is a hyperideal of R then $R/Ker(f) \cong S$.*

Proof. Let $\phi : R/Ker(f) \to S$ be defined as $\phi(r + Ker(f)) = f(r)$. We prove that ϕ is a hypernear ring isomorphism. (1) ϕ is well defined: Let $r+Ker(f) = s+Ker(f)$. Lemma 3.3 asserts that $(-r+s)\cap Ker(f) \neq \emptyset$. The latter implies that there exists $x \in -r+s$ such that $f(x) = 0$. We get now that $0 = f(x) \in f(-r+s) = -f(r)+f(s)$ (By Proposition 2.11). The latter implies that $f(r) = f(s)$ and hence, $\phi(r + Ker(f)) = \phi(s+Ker(f))$. (2) $\phi((r+Ker(f))\oplus(s+Ker(f))) = \phi((r+s)+Ker(f)) = f(r+s) = f(r)+f(s) = \phi(r+Ker(f))+\phi(s+Ker(f))$. (3) $\phi((r+Ker(f))\odot(s+Ker(f))) = \phi((r \cdot s) + Ker(f)) = f(r \cdot s) = f(r) \cdot f(s) = \phi(r + Ker(f)) \cdot \phi(s + Ker(f))$. (4) It is clear that ϕ is surjective. (5) Let $\phi(r+Ker(f)) = \phi(s+Ker(f))$. Then $f(r) = f(s)$. The latter implies that $0 \in -f(s) + f(r) = f(-s + r)$. We get now that there exists $t \in -s + r$ such that $f(t) = 0$. The latter implies that $t \in (-s + r) \cap Ker(f)$. Thus, $r + Ker(f) = s + Ker(f)$. □

Theorem 3.12. (Second Isomorphism Theorem) *Let $(R, +, \cdot)$ be a hypernear ring and A, B be hyperideals of R. Then $(A + B)/B \cong A/(A \cap B)$.*

Proof. Let $f : A \to (A + B)/B$ be defined as $f(x) = x + B$ for all $x \in A$. We prove that f is a surjective hypernear ring homomorphism. (1) f is well defined. Let $x = y \in A$. Having $0 \in (x-y)\cap B$ implies that $x+B = y+B$. (2) $f(x+y) = x+y+B = x + B \oplus y + B = f(x) \oplus f(y)$. (3) $f(x \cdot y) = x \cdot y + B = x + B \odot y + B = f(x) \odot f(y)$. (4) It is clear that f is surjective. Also, $Ker(f) = \{a \in A : a + B = B\} = A \cap B$.

Since $Ker(f)$ is a hyperideal of A, it follows by using the first isomorphism theorem that $A/(A \cap B) \cong (A+B)/B$. □

Lemma 3.13. *Let $(R, +, \cdot)$ be a hypernear ring and I, J be hyperideals of R with $J \subseteq I$. Then I/J is a hyperideal of R/J.*

Proof. Let $i_1 + J, i_2 + J \in I/J$ and $r + J, s + J \in R/J$. (1) Having I a hyperideal of R and $-i_1 + J \oplus i_2 + J = (-i_1 + i_2) + J$ implies that $-i_1 + J \oplus i_2 + J \subseteq I/J$. (2) $(r+J)\odot(i_1+J) = r \cdot i_1 + J \in I/J$. (3) $-r+J\oplus i_1+J\oplus r+J = (-r+i_1+r)+J \subseteq I/J$. $-(r+J)(s+J) \oplus (r+J \oplus i_1+J)(s+J) = -r \cdot s + (r+i_1)\cdot s + J \subseteq I/J$. Therefore, I/J is a hyperideal of R/J. □

Theorem 3.14. (Third Isomorphism Theorem) *Let $(R, +, \cdot)$ be hypernear ring and I, J be hyperideals of R with $J \subseteq I$. Then $(R/J)/(I/J) \cong R/I$.*

Proof. Let $f : R/J \to R/I$ be defined as $f(r+J) = r+I$ for all $r \in R$. We prove that f is surjective hypernear ring homomorphism. (1) f is well defined. Let $r+J = s+J$. Then $r-s \cap J \neq \emptyset$. Having $J \subseteq I$ implies that $\emptyset \neq (r-s)\cap J \subseteq (r-s)\cap I$. Thus, $r+I = s+I$. (2) $f(r+J\oplus s+J) = f(r+s+J) = r+s+I = r+I\oplus s+I = f(r+J)\oplus f(s+J)$. (3) $f(r+J\odot s+J) = f(r\cdot s+J) = r\cdot s+I = r+I\odot s+I = f(r+J)\odot f(s+J)$. (4) It is clear that f is surjective. $Ker(f) = \{r+J \in R/J : r+I = I\} = I/J$. Since $Ker(f)$ is a hyperideal of R/J (by Lemma 3.13), it follows by using the first isomorphism theorem that $(R/J)/(I/J) \cong R/I$. □

4. Chinese Remainder Theorem in Hypernear Rings

In this section, we derive Chinese Remainder Theorem in hypernear rings. First, we prove the existence of a solution when the system consists of two equations. Then we generalize it to n equations. Moreover, we apply Chinese Remainder Theorem to near rings and to rings (that are not necessary commutative).

Let $(R, +, \cdot)$ be hypernear ring, I be a hyperideal of R, and $x, y \in R$. We write $x \equiv y \pmod I$ if and only if $(x-y)\cap I \neq \emptyset$.

Lemma 4.1. *Let $(R, +, \cdot)$ be a hypernear ring, I be a hyperideal of R, and $x, y, z \in R$. If $x \equiv y$ (mod I) and $y \equiv z$ (mod I) then $x \equiv z$ (mod I).*

Proof. Let $x \equiv y \pmod I$ and $y \equiv z \pmod I$. Then $x - y \cap I \neq \emptyset$ and $y - z \cap I \neq \emptyset$. The latter implies that there exist $a, b \in I$ such that $a \in x - y$ and $b \in y - z$. We get now that $x \in a+y$ and $-z \in -y+b$. The latter implies that $-z+x \subseteq -y+b+a+y$. Since, $b+a \subseteq I$, it follows $-y+b+a+y \subseteq I$. Thus, $(-z+x) \cap I \neq \emptyset$ and hence, $x \equiv z \pmod I$. □

Lemma 4.2. *Let $(R,+,\cdot)$ be a hypernear ring, I be a hyperideal of R, and $x,y,z \in R$. Then the relation "$\equiv$" defined as:*

$$x \equiv y \ (mod\ I) \text{ if and only if } (x-y) \cap I \neq \emptyset$$

is an equivalence relation on R.

Theorem 4.3. *Let $(R,+,\cdot)$ be a unitary hypernear ring, I,J be hyperideals of R with $I+J=R$, and $a_1,a_2 \in R$. Then the system*

$$x \equiv a_1 \ (mod\ I),$$
$$x \equiv a_2 \ (mod\ J).$$

has a unique solution (mod I) and (mod J).

Proof. Since $1 \in R = I+J$, it follows that there exist $c_2 \in I, c_1 \in J$ such that $1 \in c_2 + c_1$. We get that $c_2 \in 1 - c_1$ and $c_1 \in -c_2 + 1$. The latter implies that $1 - c_1 \cap I \neq \emptyset$ and $-c_2 + 1 \cap J \neq \emptyset$. Let $a \in a_1 \cdot c_1 + a_2 \cdot c_2$. Then $a - a_2 \subseteq a_1 \cdot c_1 + a_2 \cdot c_2 - a_2 = a_1 \cdot c_1 + a_2 \cdot (c_2 - 1)$. Since $a_1 \cdot c_1 \in J$ and $a_2 \cdot (c_2-1) \cap J \neq \emptyset$ (as $-c_2 + 1 \cap J \neq \emptyset$), it follows that $a - a_2 \cap J \neq \emptyset$ and hence, $a \equiv a_2 \pmod J$. Moreover, $-a_1 + a \subseteq -a_1 + a_1 \cdot c_1 + a_2 \cdot c_2 = a_1 \cdot (-1+c_1) + a_2 \cdot c_2$. Since $a_2 \cdot c_2 \in I$ and $a_1 \cdot (-1+c_1) \cap I \neq \emptyset$ (as $(1-c_1) \cap I \neq \emptyset$), it follows that $(-a_1+a) \cap I \neq \emptyset$ and hence, $a \equiv a_1 \pmod I$. Thus, a is a solution of the system for every $a \in a_1 \cdot c_1 + a_2 \cdot c_2$. Let b be also a solution for the system. Then

$$b \equiv a_1 \pmod I,$$
$$b \equiv a_2 \pmod J.$$

Having

$$a \equiv a_1 \pmod I,$$
$$a \equiv a_2 \pmod J,$$

implies by applying Lemma 4.1 that

$$b \equiv a \pmod I, b \equiv a \pmod J. \qquad \square$$

Corollary 4.4. *Let $(R,+,\cdot)$ be a unitary hypernear ring and I,J be hyperideals of R with $I+J=R$. Then the system*

$$x \equiv 0 \ (mod\ I),$$
$$x \equiv 0 \ (mod\ J),$$

has a unique solution (mod $I \cap J$).

Proof. Theorem 4.3 asserts that the system has a solution. Let a, b be two solutions of the system. Then

$$a \equiv 0 \text{ (mod } I),$$
$$a \equiv 0 \text{ (mod } J),$$

and

$$b \equiv 0 \text{ (mod } I),$$
$$b \equiv 0 \text{ (mod } J).$$

It is easy to see that $a, b \in I \cap J$. Having $I \cap J$ a hyperideal of R implies that $a - b \subseteq I \cap J$. Thus, $a \equiv b$ (mod $I \cap J$). □

Corollary 4.5. *Let $(R, +, \cdot)$ be a unitary Krasner hyperring, I, J be hyperideals of R with $I + J = R$, and $a_1, a_2 \in R$. Then the system*

$$x \equiv a_1 \text{ (mod } I),$$
$$x \equiv a_2 \text{ (mod } J),$$

has a unique solution (mod I) and (mod J).

Proof. The proof follows from Theorem 4.3 and the fact that every Krasner hyperring is a hypernear ring. □

Corollary 4.6. *Let $(R, +, \cdot)$ be a unitary near ring, I, J be ideals of R with $I + J = R$, and $a_1, a_2 \in R$. Then the system*

$$x \equiv a_1 \text{ (mod } I),$$
$$x \equiv a_2 \text{ (mod } J).$$

has a unique solution (mod $I \cap J$).

Proof. Having that $(R, +, \cdot)$ is a unitary near ring which implies that it is a unitary hypernear ring. Theorem 4.3 asserts that the system has a solution. Let a, b be two solutions of the system. Then

$$a \equiv a_1 \text{ (mod } I),$$
$$a \equiv a_2 \text{ (mod } J),$$

and

$$b \equiv a_1 \text{ (mod } I),$$
$$b \equiv a_2 \text{ (mod } J).$$

We get that $a - a_1, b - a_1 \in I$ and $a - a_2, b - a_2 \in J$. Since I, J are ideals of R, it follows that $a - b = (a - a_1) - (b - a_1) \in I$ and $a - b = (a - a_2) - (b - a_2) \in J$. Thus, $a - b \in I \cap J$. Therefore, $a \equiv b$ (mod $I \cap J$). □

Corollary 4.7. *Let $(R, +, \cdot)$ be a unitary ring (not necessary commutative), I, J be ideals of R with $I + J = R$, and $a_1, a_2 \in R$. Then the system*

$$x \equiv a_1 \text{ (mod } I),$$
$$x \equiv a_2 \text{ (mod } J),$$

has a unique solution (mod $I \cap J$).

Proof. The proof follows from Corollary 4.6 and having every ring a near ring. □

We present next an application of Chinese Remainder Theorem and First Isomorphism Theorem in hypernear rings.

Theorem 4.8. *Let $(R, +, \cdot)$ be a unitary hypernear ring, I, J be hyperideals of R with $I + J = R$. Then $R/(I \cap J) \cong R/I \times R/J$.*

Proof. Let $f : R \to R/I \times R/J$ be defined as $f(r) = (r + I, r + J)$ for all $r \in R$. We prove that f is surjective hypernear ring homomorphism. (1) f is well defined. Let $r = s$. Then $0 \in r - s \cap I$ and $0 \in r - s \cap J$. The latter implies that $r + I = s + I$ and $r + J = s + J$. Thus, $f(r) = f(s)$. (2) $f(r + s) = (r + s + I, r + s + J) = (r+I \oplus s+I, r+J \oplus s+J) = f(r) \oplus f(s)$. (3) $f(r \cdot s) = (r \cdot s+I, r \cdot s+J) = ((r+I) \odot (s+I), (r+J) \odot (s+J)) = f(r) \odot f(s)$. (4) f is surjective. Let $(r+I, s+J) \in R/I \times R/J$. Theorem 4.3 asserts that there exists $a \in R$ such that $(a+I, a+J) = (r+I, s+J)$. Thus, $f(a) = (r + I, s + J)$. We have $Ker(f) = \{r \in R : (r + I, r + J) = (I, J)\}$. The latter implies that $r \equiv 0$ (mod I) and $r \equiv 0$ (mod J). Corollary 4.4 asserts that $r \in I \cap J$ and hence, $Ker(f) \subseteq I \cap J$. If $x \in I \cap J$ then $x + I = I, x + J = J$. Thus, $x \in Ker(f)$ and hence, $I \cap J \subseteq Ker(f)$. Since $Ker(f) = I \cap J$ is a hyperideal of R, it follows by the first isomorphism theorem that $R/(I \cap J) \cong R/I \times R/J$. □

Theorem 4.9. *Let $(R, +, \cdot)$ be a unitary hypernear ring with $(R, +)$ commutative, $a_k \in R$, I_k be hyperideal of R with $I_kR \subseteq I_k$ and $I_k + I_l = R$ for all $1 \leq k \neq l \leq n$. Then the system*

$$x \equiv a_k \text{ (mod } I_k)$$

has a unique solution (mod I_k).

Proof. Having $1 \in R = I_1 + I_k$ for every $k = 2, \ldots, n$ implies that there exist $b_k \in I_1, d_k \in I_k$ such that $1 \in b_k + d_k$ for every $k = 1, \ldots, n$. Since $1 = 1 \ldots 1$, it follows that $1 \in (b_2+d_2)(b_3+d_3) \ldots (b_n+d_n) = [(b_2+d_2)b_3+(b_2+d_2)d_3] \ldots (b_n+d_n)$. Having $(R, +)$ commutative implies that $(b_2 + d_2)b_3 + (b_2 + d_2)d_3 \subseteq (b_2 + d_2)b_3 + (d_2 + b_2)d_3 - d_2d_3 + d_2d_3 \subseteq I_1 + d_2d_3$. We get now that there exists $i_1 \in I_1$ such that $1 \in (i + d_2d_3)\dots(b_n + d_n)$. Continuing on this pattern and using the given that $I_1R \subseteq I_1$, we get that $1 \in I_1 + d_2d_3\dots d_n$. By setting $c_1 = d_2d_3\dots d_n \in I_k$ for all $k = 2,\dots,n$, we get that $c_1 \equiv 1 \pmod{I_1}$ and $c_1 \equiv 0 \pmod{I_k}$ for $k = 2,\dots,n$. In a similar manner, we can get c_i with $c_i \equiv 1 \pmod{I_i}$ and $c_i \equiv 0 \pmod{I_k}$ for $k = 1,\dots,i-1,i+1,\dots,n$. Let $a \in a_1 \cdot c_1 + \dots + a_n \cdot c_n$. Since $(R,+)$ is commutative, it follows that $-a_i + a \subseteq a_i(-1+c_i) + a_1 \cdot c_1 + \dots + a_{i-1}c_{i-1} + a_{i+1}c_{i+1} + \dots + a_nc_n$. It is clear that $(-a_i + a) \cap I_i \neq \emptyset$ for all $i = 1,\dots,n$. Thus, $a \equiv a_k \pmod{I_k}$ for all $k = 1,\dots,n$.

Let b be another solution for the system. Then $a \equiv a_k \pmod{I_k}$ and $b \equiv a_k \pmod{I_k}$. Using Lemma 4.2, we get that $b \equiv a \pmod{I_k}$ for all $k = 1,\dots,n$. □

Corollary 4.10. *Let $(R,+,\cdot)$ be a unitary hypernear ring with $(R,+)$ commutative, I_k be a hyperideal of R with $I_kR \subseteq I_k$ and $I_k + I_l = R$ for all $1 \leq k \neq l \leq n$, and $a_k \in R$. Then the system*

$$x \equiv 0 \pmod{I_k}$$

has a unique solution (mod $\bigcap_{k=1}^{n} I_k$).

Proof. Theorem 4.9 asserts that the system has a solution. Let a, b be two solutions of the system. Then

$$a \equiv 0 \pmod{I_k}$$

and

$$b \equiv 0 \pmod{I_k}.$$

It is easy to see that $a, b \in \bigcap_{k=1}^{n} I_k$. Having $\bigcap_{k=1}^{n} I_k$ a hyperideal of R implies that $a - b \subseteq \bigcap_{k=1}^{n} I_k$. Thus, $a \equiv b \pmod{\bigcap_{k=1}^{n} I_k}$. □

Corollary 4.11. *Let $(R,+,\cdot)$ be a unitary Krasner hyperring, I_k be a hyperideal of R and $I_k + I_l = R$ for all $1 \leq k \neq l \leq n$, and $a_k \in R$. Then the system*

$$x \equiv a_k \pmod{I_k}$$

has a unique solution (mod I_k).

Proof. The proof follows for Theorem 4.9 and the fact that every Krasner hyperring is a hypernear ring. □

Corollary 4.12. *Let $(R,+,\cdot)$ be a unitary near ring with $(R,+)$ commutative, I_k be an ideal of R with $I_kR \subseteq I_k$ and $I_k + I_l = R$ for all $1 \leq k \neq l \leq n$, and $a_k \in R$. Then the system*

$$x \equiv a_k \pmod{I_k}$$

has a unique solution (mod $\bigcap_{k=1}^{n} I_k$).

Proof. We have that $(R, +, \cdot)$ is a unitary near ring which implies that it is a unitary hypernear ring. Theorem 4.9 asserts that the system has a solution. Let a, b be two solutions of the system. Then

$$a \equiv a_k \text{ (mod } I_k)$$

and

$$b \equiv a_k \text{ (mod } I_k)$$

We get that $a - a_k, b - a_k \in I_k$ for all $k = 1, \ldots, n$. Since I_k is an ideal of R, it follows that $a - b = (a - a_k) - (b - a_k) \in I_k$. Thus, $a - b \in \bigcap_{k=1}^{n} I_k$. Therefore, $a \equiv b$ (mod $\bigcap_{k=1}^{n} I_k$). □

Corollary 4.13. *Let $(R, +, \cdot)$ be a unitary ring, I_k be an ideal of R and $I_k + I_l = R$ for all $1 \leq k \neq l \leq n$, and $a_k \in R$. Then the system*

$$x \equiv a_k \text{ (mod } I_k)$$

has a unique solution (mod $\bigcap_{k=1}^{n} I_k$).

Proof. Since every ring is a near ring, the proof follows from Corollary 4.12. □

Theorem 4.14. *Let $(R, +, \cdot)$ be a unitary hypernear ring with $(R, +)$ commutative, I_k be hyperideal of R with $I_k R \subseteq I_k$ and $I_k + I_l = R$ for all $1 \leq k \neq l \leq n$. Then*

$$R/\bigcap_{k=1}^{n} I_k \cong R/I_1 \times \ldots \times R/I_n.$$

Proof. The proof is similar to that of Theorem 4.8. □

5. Conclusion

Hypernear rings, as a generalization of near rings, grabbed the interest of many researchers in the field of hyperstructure theory. This paper studied hypernear rings from theoretical point of view. Several results related to properties of hypernear rings have been discussed. And main theorems in this subject have been derived.

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