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NUMERICAL EVALUATION OF CAUCHY PRINCIPAL VALUE INTEGRALS USING A PARAMETRIC RATIONAL TRANSFORMATION

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Abstract. For numerical evaluation of Cauchy principal value integrals, we present a simple rational function with a parameter satisfying some reasonable conditions. The proposed rational function is employed in coordinate transformation for accelerating the accuracy of the Gauss quadrature rule. The efficiency of the proposed rational transformation method is demonstrated by the numerical result of a selected test example.

1. Introduction

In this work we consider the numerical evaluation of the Cauchy principal value (CPV) integral, which is very important for the implementation of numerical schemes such as boundary element method [1, 2, 3, 4, 5] for solving problems in many areas of engineering.

Among the well-known methods for numerical evaluation of CPV integrals, we may notice that the coordinate transformation techniques [6, 7, 8, 9, 10, 11, 12, 13] are prominent because of their ease of use in the adaptive approach. So, in this paper, we propose a rational function of a simple form in pursuit of improving the accuracy of the numerical integration method. Then we examine the usefulness of the proposed rational transformation method with a selected numerical example.

In the following section, for CPV integrals over an interval $[-1, 1]$, we define a rational function of type $(1, 2)$ including a parameter that shifts a singular point $-1 < s_0 < 1$ into the midpoint 0. An appropriate range of the parameter, with which the rational function is suitable for a coordinate transformation, is derived. The proposed rational function makes the singular part of the CPV integral disappear

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when an even number of integration points are used. Therefore, efficient evaluation of the CPV integral can be expected by the proposed rational function with a parameter selected in the appropriate range.

Based on the Gauss-Legendre quadrature rule, the numerical results of the proposed method for a test example are investigated and compared with those of some existing transformation methods.

2. Transformation Methods for CPV Integrals

In this section we consider the following Cauchy principal value integral.

$$K\phi(s_0) := \text{P.V.}\int_{-1}^{1}\frac{\phi(\xi)}{\xi - s_0}\,d\xi \;=\; \lim_{\epsilon\to 0^+}\left(\int_{-1}^{s_0-\epsilon} + \int_{s_0+\epsilon}^{1}\right)\frac{\phi(\xi)}{\xi - s_0}\,d\xi\,, \tag{2.1}$$

where $0 \le s_0 < 1$ and ϕ is a well-behaved function with $\phi(s_0) \ne 0$.

To evaluate the CPV integral efficiently, in general, we may use a transformation $h(x)$, $-1 \le x \le 1$, which is an increasing function satisfying

$$h(-1) = -1, \qquad h(0) = s_0, \qquad h(1) = 1\,. \tag{2.2}$$

By the change of variable $\xi = h(x)$ in the CPV integral (2.1) we have

$$\begin{aligned} K\phi(s_0) &= \int_{-1}^{1}\{\phi(h(x)) - \phi(s_0)\}\,\frac{h'(x)}{h(x) - s_0}\,dx \;+\; \phi(s_0)\text{P.V.}\int_{-1}^{1}\frac{h'(x)}{h(x)-s_0}\,dx \\ &= K_1 \,+\, K_2. \end{aligned} \tag{2.3}$$

We can see that K_1 is a regular integral and K_2 can be written by

$$K_2 = \phi(s_0)\left\{\text{P.V.}\int_{-1}^{1}\frac{c}{x}\,dx \,+\, \int_{-1}^{1} R(x)\,dx\right\}, \tag{2.4}$$

where c is the first non-vanishing derivative of $h(x)$ at $x = 0$ and $R(x)$ is a bounded function [11]. If we use a standard Gauss quadrature rule with any even number of integration points, then the CPV integral in the formula (2.4) vanishes and, consequently, the transformed CPV integral $K\phi(s_0)$ can be evaluated very accurately as mentioned in the literature [1, 2, 11].

2.1. Existing transformations Typical transformations that serve as the aforementioned function $h(x)$ and are known to perform well in practice are given below.

· Doblarè and Gracia transformation [6]:

$$h^{\mathrm{DG}}(x) \;=\; s_0\left(1 - x^4\right) \,+\, x^3, \qquad -1 \le x \le 1. \tag{2.5}$$

· Composite polynomial-sigmoidal transformation [12]:

$$h_r^{\mathrm{DG}}(x) \;=\; h^{\mathrm{DG}}\left(1-2\gamma_r\left(\frac{1-x}{2}\right)\right), \qquad -1 \le x \le 1\,, \tag{2.6}$$

where $\gamma_r(s)$, $0 \le s \le 1$, is a sigmoidal transformation of order $r \ge 1$.

· Composite rational-polynomial transformation [13]:

$$h_\eta^{\mathrm{Y}}(x) \;=\; f\left(\eta; g(\eta;x)\right), \qquad -1 \le x \le 1\,, \tag{2.7}$$

where $f(\eta;x) \;=\; s_0 + (1-s_0x)\,\frac{x-\eta}{1-\eta x}$ and $g(\eta;x) \;=\; \eta\left(1-x^2\right)+x$ for a parameter η such that $\eta \ge -1+2s_0$.

In the last part of this section, these transformations will be compared with the proposed rational transformation.

2.2. A parametric rational transformation For a given $0 \le s_0 < 1$ and for a parameter $\alpha > 0$ we set a simple rational function of type (1,2),

$$h_\alpha(x) = \frac{x+d}{ax^2+bx+c}, \qquad -1 \le x \le 1\,, \tag{2.8}$$

with a derivative

$$h_\alpha{}'(x) = -\frac{ax^2+2adx+(bd-c)}{\left(ax^2+bx+c\right)^2}. \tag{2.9}$$

The function h_α implicitly includes the parameter α because the coefficients a, b, c and d will be associated with α as we can see bellow.

Considering the conditions in (2.2) and an additional condition of the derivative at $x=0$,

$$h_\alpha{}'(0) \;=\; \alpha\,, \tag{2.10}$$

we have

$$a = 1-c, \qquad b = d = s_0c, \qquad c = \frac{1}{s_0^2+\alpha}\,. \tag{2.11}$$

Thus h_α and its derivative can be written by

$$h_\alpha(x) \;=\; \frac{\left(s_0^2+\alpha\right)x+s_0}{\left(s_0^2+\alpha-1\right)x^2+s_0x+1} \tag{2.12}$$

and

$$h_\alpha{}'(x) \;=\; \frac{\left(s_0^2+\alpha\right)\left(1-s_0^2-\alpha\right)x^2+2s_0\left(1-s_0^2-\alpha\right)x+\alpha}{\left\{\left(s_0^2+\alpha-1\right)x^2+s_0x+1\right\}^2}. \tag{2.13}$$

Lemma 2.1. *For* $0 < s_0 < 1$ *given, the function* $h_\alpha(x)$ *is increasing over the interval* $-1 \le x \le 1$ *for any parameter* α *satisfying*

$$s_0 - s_0^2 \le \alpha \le 2 - s_0 - s_0^2. \tag{2.14}$$

Proof. Rewrite the formula of $h_\alpha{}'$ in (2.13) as

$$h_\alpha{}'(x) = \frac{H(x)}{\left\{\left(s_0^2+\alpha-1\right)x^2+s_0x+1\right\}^2}.$$

Then

$$\begin{aligned} H(x) &= \left(s_0^2+\alpha\right)\left(1-s_0^2-\alpha\right)x^2+2s_0\left(1-s_0^2-\alpha\right)x+\alpha \\ &= \left(s_0^2+\alpha\right)\left(1-s_0^2-\alpha\right)\left\{x+\frac{s_0}{s_0^2+\alpha}\right\}^2+\frac{\left(\alpha-s_0+s_0^2\right)\left(\alpha+s_0+s_0^2\right)}{s_0^2+\alpha}. \end{aligned}$$

When $1 - s_0^2 - \alpha \ge 0$, we have $H(x) \ge 0$(that is, $h'_\alpha(x) \ge 0$) for all α satisfying $\alpha - s_0 + s_0^2 \ge 0$. That is, $h_\alpha(x)$ is increasing over the interval $[-1, 1]$ for all α such that

$$s_0 - s_0^2 \le \alpha \le 1 - s_0^2.$$

When $1 - s_0^2 - \alpha < 0$, the minimum of H is

$$\begin{aligned} H(1) &= \left(s_0^2+\alpha\right)\left(1-s_0^2-\alpha\right)+2s_0\left(1-s_0^2-\alpha\right)+\alpha \\ &= -\left\{\alpha^2-2\left(1-s_0-s_0^2\right)\alpha-\left(1-s_0^2\right)\left(2s_0+s_0^2\right)\right\} \\ &= -\left\{\alpha+s_0+s_0^2\right\}\left\{\alpha-2+s_0+s_0^2\right\}. \end{aligned}$$

Thus we have $h'_\alpha(x) \ge 0$ for all α satisfying $\alpha - 2 + s_0 + s_0^2 \le 0$. That is, $h_\alpha(x)$ is increasing over the interval $[-1, 1]$ for all α such that

$$1 - s_0^2 < \alpha \le 2 - s_0 - s_0^2.$$

As a result, we can say that the function $h_\alpha(x)$ is increasing over the interval $[-1, 1]$ for all α contained in the interval

$$s_0 - s_0^2 \le \alpha \le 2 - s_0 - s_0^2,$$

which completes the proof. □

This lemma shows the appropriate range of the parameter α with which $h_\alpha(x)$ becomes a bijective map from $[-1, 1]$ onto itself. That is, the rational function $h_\alpha(x)$ with the parameter α satisfying the condition (2.14) can be used for a suitable coordinate transformation $\xi = h_\alpha(x)$ in numerical evaluation of the CPV integral.

For a special case of $s_0 = 0$, if we set

$$h_{0,\alpha}(x) := \frac{\alpha x}{(\alpha-1)\,x^2+1} \tag{2.15}$$

from (2.12), then it can be seen that $h_{0,\alpha}(x)$ is strictly increasing over the interval $[-1,1]$ for any parameter $0 < \alpha \le 2$ and it satisfies

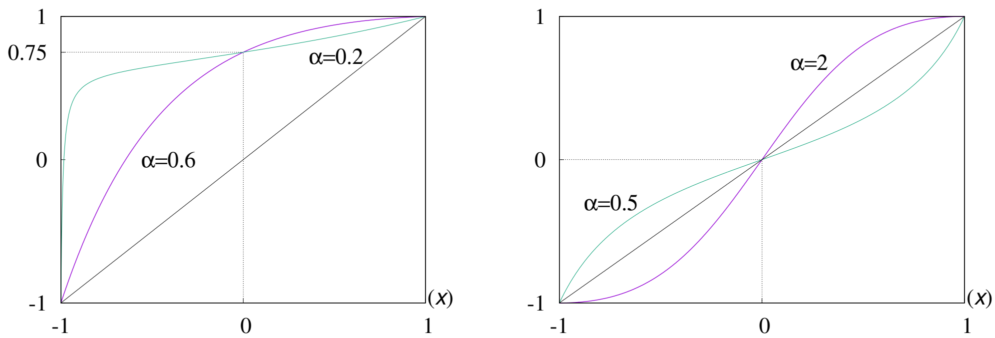
$$h_{0,\alpha}{}'(1) \;=\; h_{0,\alpha}{}'(-1) \;=\; \frac{2-\alpha}{\alpha}$$

and

$$h_{0,\alpha}(x) \;+\; h_{0,\alpha}(-x) \;=\; 0$$

for all $-1 \le x \le 1$. It should be noted that $h_{0,2}(x) = \frac{2x}{x^2+1}$ and $g(t) := \frac{1}{2}\left(h_{0,2}(2t-1)+1\right)$, $0 \le t \le 1$, becomes a sigmoidal transformation of order 2 based on the definition in the literature [7].

For comprehension, graphs of h_α in the case of $s_0 = 0.75$, for example, are given in Figure 1(a) and those of $h_{0,\alpha}$ are given in Figure 1(b).



(a) $s_0 = 0.75$ (b) $s_0 = 0$

Figure 1: Graphs of $h_\alpha(x)$ for $s_0 = 0.75$ with $\alpha = 0.2,\, 0.6$ in (a) and $s_0 = 0$ with $\alpha = 0.5,\, 2$ in (b).

3. A Numerical Example

We take a test example $K\phi\,(s_0)$ with $\phi(x) = 1 + x$, that is,

$$K\phi\,(s_0) \;=\; \text{P.V.} \int_{-1}^{1} \frac{1+\xi}{\xi - s_0}\, d\xi \;=\; 2 \;+\; (1+s_0)\log\frac{1-s_0}{1+s_0}\,.$$

Table 1 includes optimal value of α, denoted by α^*, for each s_0 fixed which results in the best error among the used values of α of the step-size 0.01 in numerical experiments using the N-point Gauss-Legendre quadrature rule with $N = 40$. In fact, we can identify that the optimal value α^* changes little with respect to the number of integration points, N. For the data of α^* given in Table 1, we have the least square approximation by the Marquardt-Levenberg algorithm as follows.

$$B(s) = 0.01558 + 1.31324(1-s)^{1/2} - 0.25039(1-s), \qquad 0 \le s < 1. \tag{3.1}$$

Table 1. Optimal values of α experimentally obtained for each s_0 ($N = 40$).

s_0	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	0.995	0.999
α^*	0.97	0.95	0.88	0.85	0.75	0.65	0.53	0.40	0.31	0.15	0.11	0.05

For $s_0 = 0.2, 0.4, 0.6, 0.8$, numerical results based on the presented transformation h_α are given in Table 2. Therein, the value of the parameter α is chosen by $\alpha = B(s_0)$ for each s_0. For comparison, the table also includes numerical results associated with Doblaré and Gracia transformation h^{DG} given in (2.5).

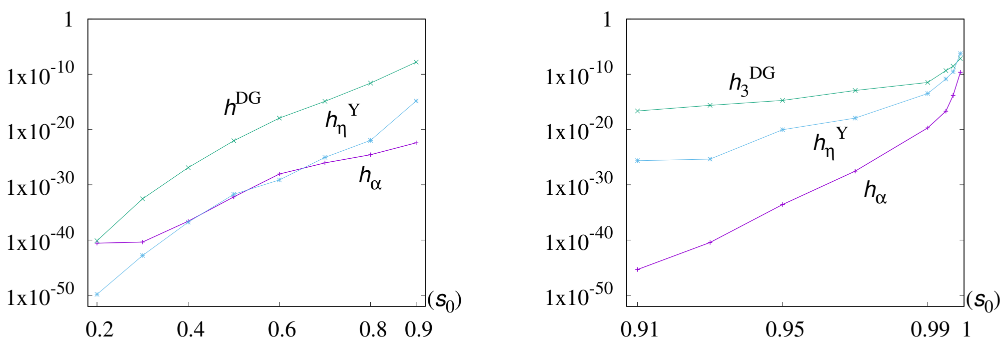
Table 2. Relative errors of the N-point Gauss-Legendre quadrature rule associated with the presented transformation h_α with $\alpha = B(s_0)$ and the Doblaré-Gracia transformation h^{DG} for the CPV integral $K\phi(s_0)$, $0 < s_0 < 1$.

		Existing transformation	Presented transformation	
s_0	N	h^{DG}	h_α	α
	4	5.2×10^{-9}	4.9×10^{-8}	
0.2	12	6.3×10^{-25}	1.6×10^{-24}	0.9899
	20	7.5×10^{-41}	2.7×10^{-41}	
	4	7.6×10^{-6}	1.4×10^{-6}	
0.4	12	1.0×10^{-16}	4.6×10^{-23}	0.8826
	20	1.3×10^{-27}	2.6×10^{-37}	
	4	2.2×10^{-3}	9.1×10^{-5}	
0.6	12	5.3×10^{-11}	1.4×10^{-16}	0.7460
	20	1.2×10^{-18}	8.9×10^{-29}	
	4	1.0×10^{-2}	1.7×10^{-4}	
0.8	12	1.7×10^{-7}	2.7×10^{-14}	0.5528
	20	2.6×10^{-12}	2.8×10^{-25}	

On the other hand, for $s_0 = 0.9, 0.95, 0.99, 0.995$ near the end-point $x = 1$, numerical results of h_α are given in Table 3. It also includes numerical results of

Table 3. Relative errors of the N-point Gauss-Legendre quadrature rule associated with the presented transformation h_α with $\alpha = B(s_0)$ and the composite transformation h_3^{DG} for the CPV integral $K\phi(s_0)$, with s_0's near the end-point $x = 1$.

		Existing transformation	Presented transformation	
s_0	N	h_3^{DG}	h_α	α
	20	1.4×10^{-8}	4.0×10^{-23}	
0.9	30	7.6×10^{-13}	3.8×10^{-35}	0.4058
	40	3.6×10^{-17}	2.1×10^{-47}	
	20	1.3×10^{-7}	7.3×10^{-17}	
0.95	30	2.1×10^{-11}	1.5×10^{-25}	0.2967
	40	1.9×10^{-15}	2.7×10^{-34}	
	20	6.5×10^{-6}	3.2×10^{-10}	
0.99	30	1.4×10^{-8}	2.8×10^{-15}	0.1444
	40	3.4×10^{-12}	2.0×10^{-20}	
	20	2.9×10^{-5}	7.2×10^{-9}	
0.995	30	1.2×10^{-7}	2.0×10^{-13}	0.1072
	40	4.7×10^{-10}	2.1×10^{-17}	



(a) $0.2 \leq s_0 \leq 0.9$ ($N = 20$)

(b) $0.9 < s_0 < 1$ ($N = 40$)

Figure 2: Relative errors of h_α, h^{DG} and h_η^{Y} for $0.2 \leq s_0 \leq 0.9$ in (a) and those of h_α, h_3^{DG} and h_η^{Y} for $0.9 < s_0 < 1$ in (b).

the existing composite transformation h_3^{DG}, given in (2.6), associated with the Sidi-sigmoidal transformation of order 3 [7]. In the numerical experiment we can find that h_3^{DG} results in better errors than h_r^{DG} of any order $r \neq 3$. Both Table 2 and

Table 3 show the superiority of the proposed transformation $h^{[\alpha]}$ with $\alpha = B(s_0)$ over the compared existing transformations.

Figure 2 illustrates the tendency of the relative errors of the proposed transformation h_α, compared with those of the existing transformations h^{DG}, h_3^{DG} and h_η^{Y} for $0.2 \leq s_0 \leq 0.9$ in (a) and $0.9 < s_0 < 1$ in (b). The parameter in h_η^{Y} was chosen as $\eta = \frac{1}{3}s_0$ in (a) and $\eta = 12$ in (b), referring to the literature [13]. From the figure we can see that the proposed transformation h_α is available for all $0 < s_0 < 1$ and the superiority over the compared existing transformations is evident over the range $0.9 < s_0 < 1$.

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