Commun. Korean Math. Soc. **38** (2023), No. 4, pp. 1299–1307
https://doi.org/10.4134/CKMS.c220349
pISSN: 1225-1763 / eISSN: 2234-3024

SPACE OF HOMEOMORPHISMS UNDER REGULAR TOPOLOGY

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Abstract. In this paper, we attempt to study several topological properties for the function space $H(X)$, space of self-homeomorphisms on a metric space endowed with the regular topology. We investigate its metrizability and countability and prove their coincidence at X compact. Furthermore, we prove that the space $H(X)$ endowed with the regular topology is a topological group when X is a metric, almost P-space. Moreover, we prove that the homeomorphism spaces of increasing and decreasing functions on $\mathbb{R}$ under regular topology are open subspaces of $H(\mathbb{R})$ and are homeomorphic.

1. Introduction

The notation $C(X,Y)$ indicates the space of continuous functions from a space X to a space Y. This space has been topologized in numerous ways and those topologies include the innate topologies such as point-open topology, compact-open topology and uniform topology. However, more stronger topologies than that of the uniform topology such as fine topology (also known as m-topology) and graph topology have also been studied.

Iberklied et al. in [5] introduced a more stronger topology than the fine topology on the space $C(X)$ and named it as regular topology or the r-topology. The basis elements for the regular topology on the space $C(X)$ are of the fashion: $R(f,r) = \{g \in C(X) \colon |f(x) - g(x)| < r(x),\ \forall x \in coz(r)\}$, where $f \in C(X)$, r is a positive regular element (non-zero divisor) of the ring $C(X)$ and $coz(r) = \{x \in X : r(x) \neq 0\}$.

Later, Jindal et al. [6] explored this regular topology on a more general space $C(X,Y)$, where X is a space and Y is a metric space with non-trivial path. They used the same idea as before to define the basis element for the regular topology on $C(X,Y)$ as: $R(f,r) = \{g \in C(X,Y) \colon d(f(x), g(x)) < r(x),\ \forall x \in coz(r)\}$, where $f \in C(X,Y)$, r is a positive regular element (non-zero divisor)

Received December 5, 2022; Accepted March 14, 2023.
2020 *Mathematics Subject Classification.* Primary 54C35, 54D99, 54D30, 54E35, 54H11.
Key words and phrases. Function space, regular topology, fine topology, metric space, homeomorphisms.

of the ring $C(X)$. Moreover, they studied various topological properties like metrizability, countability and several completeness properties.

Furthermore, several other topological properties such as submetrizability, various maps and some separation axioms are investigated in [1]. Afterwards, several compactness and cardinality properties for the function space $C_r(X, Y)$ are explored in [2].

We symbolize the space of self-homeomorphisms on a metric space (X, d) endowed with the regular topology by $H_r(X)$, which acts as the subspace topology inherited from the space $C_r(X, X)$. Therefore, the basis elements for the subspace $H_r(X)$ of $C_r(X, X)$ are of the type:

$$B_r(f, r) = \{h \in H(X) \colon d(f(x), h(x)) < r(x),\ \forall x \in X\},$$

where $g \in H(X)$ and r is the regular element of the ring $C(X)$. Based on the relation $C_p(X, Y) \leq C_k(X, Y) \leq C_d(X, Y) \leq C_f(X, Y) \leq C_r(X, Y)$, we get $H_p(X) \leq H_k(X) \leq H_d(X) \leq H_f(X) \leq H_r(X)$.

Arens [3] has proved that $H_k(X)$ is a topological group whenever X is compact, or locally connected and locally compact. However, $H_k(X)$ is not a topological group for a locally compact, separable metric space X [4]. One nice feature of $H_f(X)$, for every metric space X, is that it becomes a topological group [7] . Furthermore, $H_d(X)$ is always a metric space.

In this paper, we study some topological properties of the space $H(X)$ endowed with the regular topology. We first prove that the space $H_r(X)$ is a subspace of the function space $C_r(X, X)$. We further define an equivalence relation on the function space $H_r(X)$ and with the help of that, we investigate the connectedness and separability of $H(X)$. Then we investigate metrizability and first countability of the space $H_r(X)$. Afterwards, we show that the space $H_r(X)$ is a topological group. Moreover, we prove that the sets of increasing and decreasing homeomorphisms on $\mathbb{R}$ are open subspaces of $H(\mathbb{R})$ and are homeomorphic to one another.

The convictions that are used throughout this paper are: The notation $C(X, Y)$ is used to represent the space of continuous functions from a space X to a space Y. The symbol $C(X)$ represents the real valued continuous function space. The notation $C_r(X, Y)$ represents the space endowed with regular topology, yet other topologies such as graph topology, fine topology, uniform topology, compact-open topology and point-open topology are given the notations like $C_g(X, Y)$, $C_f(X, Y)$, $C_d(X, Y)$, $C_k(X, Y)$ and $C_p(X, Y)$, respectively, and similarly for the self-homeomorphism space $H(X)$. The set of all positive regular elements of the ring $C(X)$ is represented by $r^+(X)$. The space X is taken as a metric space throughout the paper. For more details about the notations, see [9].

2. Preliminaries

Here is a bunch of definitions that are being used while proving the results. These definitions have been taken from [9].

To study the properties of $H_r(X)$ for a metric space X, we ought to have sufficient number of homeomorphisms on X. For this purpose, we use a property called local homogeneity.

Definition. A space A is called locally homogeneous if for every $a \in A$ and neighborhood U of a, there exists a neighborhood V of a contained in U such that for each $b \in V$, there is a homeomorphism $h\colon A \to A$ with $h(a) = b$ and $h(c) = c$, $\forall c \in A \backslash V$.

Let us define an equivalence relation on $H(X)$ so that its equivalence classes can be used to investigate several properties of the space $H(X)$.

Definition. Define an equivalence relation $\sim$ on $H(X)$ by taking $g \sim f$ provided that

$$\sup\{d(f(x), g(x))\colon x \in X\} < \infty.$$

For each $h \in H(X)$, let $\xi(h)$ denotes the equivalence class of $\sim$ that contains h.

Definition. An element $h \in H(X)$ is a bounded (unbounded) member of $H_r(X)$ if $h \in \xi(e)$ ($h \notin \xi(e)$, respectively), where e is an identity element. Note that if $H_r(X)$ contains an unbounded member, then d is necessarily an unbounded metric on X.

Definition. A function $f \in H(\mathbb{R})$ is said to be increasing (decreasing) if $a_1 < a_2$ implies that $f(a_1) < f(a_2)$ ($f(a_2) > f(a_1)$), $\forall a_1, a_2 \in \mathbb{R}$.

3. Main results

In this section, we study various topological properties for the function space $H_r(X)$.

Theorem 3.1. *The space $H_r(X)$ is the subspace of $C_r(X, X)$.*

Proof. To prove the result, it is sufficient to show that every basis element of $H_r(X)$ can be written as the intersection of $H_r(X)$ with that of the basis element of $C_r(X, X)$. Consider a basis element of $C_r(X, X)$ as $B(f, r)$, $f \in C(X, X)$ and let $B(h, r)$, $h \in H(X)$ be a basis element of $H_r(X)$. We have two cases;

(1) Case-1: Since $h \in H(X) \subseteq C(X, X)$, so we can write $B(h, r) = H_r(X) \cap B(h, r)$.
(2) Case-2: For a basis element $B(f, r)$ in $C_r(X, X)$, we can choose a basis element $B(h, s)$ in $H_r(X)$ such that $s = r/2$. Which implies that $B(h, s) \subseteq B(f, r)$. Therefore, we can write $B(h, s) = H_r(X) \cap B(f, r)$.

□

Theorem 3.2. *For each h of $H(X)$, the equivalence class $\xi(h)$ is a clopen subset of $H_r(X)$.*

Proof. Consider an element $h \in H(X)$ and let r be a regular element of the ring $C(X)$, i.e., $r \in r^+(X)$. To prove that $B(g,r) \subseteq \xi(h)$ for every $g \in \xi(h)$, let $f \in B(g,r)$, then $f \sim g$ and $g \sim h$. Therefore, $f \sim h$, by the transitivity of $\sim$. This means that $f \in \xi(h)$ and justifies the claim that $\xi(h)$ is open subset of $H_r(X)$. To prove that $B(g,r) \subseteq H_r(X) \backslash \xi(h)$ for every $g \in H_r(X) \backslash \xi(h)$, let $f \in B(g,r)$ so that $f \sim g$. If it was so that $f \in \xi(h)$, then $h \sim f$ and $h \sim g$; which contradicts that $g \notin \xi(h)$. Thus, $f \notin \xi(h)$. Hence $\xi(h)$ is closed in $H_r(X)$. □

Corollary 3.3. *The component of $H_r(X)$ that contains an identity element e, is contained in $\xi(e)$.*

Proof. By Theorem 3.2, $\xi(e)$ is both closed and open in $H_r(X)$. As every clopen subset contains the components of its elements, then the component of $H_r(X)$ that contains e is contained in $\xi(e)$. □

Corollary 3.4. *If an unbounded member is contained in $H_r(X)$, then $H_r(X)$ is not connected.*

Since the regular topology is finer than the uniform topology, from [6], so we can prove the following result for regular topology on $H(X)$ as proved in [8]:

Theorem 3.5. *For a locally homogeneous metric space (X,d), that contains a closed copy of $\mathbb{R}$ on which d is unbounded, then $H_r(X)$ can be written as the topological sum of distinct members of the uncountable family $\{\varsigma(e) \colon h \in H(X)\}$.*

Corollary 3.6. *For a locally homogeneous metric space (X,d), that contains a closed copy of $\mathbb{R}$ on which d is unbounded, then $H_r(X)$ is neither separable nor connected.*

Since the regular topology is not always metrizable, so out of the countability properties, we can study the first countablity for the function spaces with regular topology. From [5], it is clear that $C_r(X)$ is first countable if and only if X is pseudocompact, almost P-space. Therefore, $C_r(X)$ is not first countable for non-compact metric spaces, and thus not metrizable. We explore this result for $H_r(X)$, for locally homogeneous space X, but at first we need the following result:

Lemma 3.7 ([9])**.** *If (Y,d) is a non-compact metric space, then it contains a discrete family $\{V_n \colon n \in \mathbb{N}\}$ of non-empty open sets.*

Theorem 3.8. *If (X,d) is a metric, almost P-space, locally homogeneous dense in itself (i.e., not having any isolated point), containing a non-trivial path, then the following are equivalent.*

(1) *$H_r(X)$ is first countable.*
(2) *$H_r(X)$ is metrizable.*

(3) *X is compact.*

Proof. (3) $\Rightarrow$ (1) Let X be compact. Then from [6], $C_d(X,X) = C_r(X,X)$. Consequently, $H_d(X) = H_r(X)$, so $H_r(X)$ is metrizable. Hence, $H_r(X)$ is first countable.

(1) $\Rightarrow$ (3) Let's prove the result contrapositively, so let's suppose X is not compact. Now consider a countable family of neighborhoods of e as $\mathcal{B} = \{B(e, r_n) \colon n \in \mathbb{N}\}$, where $r_n \in r^+(X) = C(X)$ (X is almost P-space). To prove that the collection $\mathcal{B}$ is not the base for e in $H_r(X)$.

From Lemma 3.7, X not compact implies that there exists a discrete family $\{V_n \colon n \in \mathbb{N}\}$ of non-empty open subsets in it. Let $x_n \in V_n$ for each n. As every r_n is continuous at x_n, there exists an open neighborhood U_n such that $U_n \subseteq V_n$, so that $r_n(x) > r_n(x_n)/2$, $\forall x \in U_n$. Define $Z_n = B(x_n, r_n(x_n)/5) \cap U_n$ for each n, so we get $\{Z_n \colon n \in \mathbb{N}\}$ as the disjoint family non-empty open subsets of X such that $\inf\{r_n(x) \colon x \in Z_n\}$ is greater than the diameter of each Z_n.

As X being dense in itself, so every Z_n contains distinct points x_n and y_n. Therefore, X being locally homogeneous implies that for each n there exists $t_n \in H_r(X)$ such that $t_n(x_n) = y_n$ and $t_n(x) = x$, $\forall x \in X \backslash Z_n$. As $\inf\{r_n(x) \colon x \in Z_n\}$ is greater than the diameter of Z_n, we get, each $t_n \in B(e, r_n)$.

Since all $t_n(x)$ are equal to x, $\forall x \in X$ except for one n. It means that for $\{t_n \colon n \in \mathbb{N}\}$, the only possible cluster point in $H_r(X)$ is e. However, there exists $r \in C(X)$ with $r(x_n) < d(t_n(x_n), x_n)$, $\forall n$, thus $B(e, r) \cap \{t_n \colon n \in \mathbb{N}\} = \phi$. Which implies that e is not the cluster point of $\{t_n \colon \mathbb{N}\}$ in $H_r(X)$, therefore, $\{t_n \colon \mathbb{N}\}$ is closed in $H_r(X)$. Now as $H_r(X) \backslash \{t_n \colon n \in \mathbb{N}\}$ is a neighborhood of e in $H_r(X)$ which contains no $B(e, r_n)$, this shows $\mathcal{B}$ is not a base for e in $H_r(X)$. □

Corollary 3.9. *If (X, d) is a metric, almost P-space, locally homogeneous dense in itself (that is, not having an isolated point), then $H_d(X) = H_r(X)$ if and only if X is compact.*

Theorem 3.10. *The set $H(X)$ is a group under the composition operator.*

Proof. Since composition of homeomorphisms is again a homeomorphism. Due to bijection, every inverse of a homeomorphism is again a homeomorphism as: Let $f \in H(X)$. Then $f \circ f^{-1} = e \in H(X)$, where e is an identity homeomorphism. Clearly, we have $f \circ e = f, f \in H(X)$. Hence, $H(X)$ is a group under the composition operator. □

Theorem 3.11. *For a metric, almost P-space (X, d), the space $H_r(X)$ is a topological group.*

Proof. The proof of the continuity of inversion in $H_r(X)$ can be proved using the graph topology. Let us first define an open set in $X \times Y$ as; $W = \cup\{\{x\} \times B(f(x), r(x)) \colon x \in X\}$, where $B(f(x), r(x))$ is the open ball in Y. Then we have $W^+ = \{f \in C(X, Y) \colon f \subseteq W\}$. Let $g \in H_r(X)$ and U be an open subset of $X \times X$ with $g^{-1} \in U^+$. Then $U^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in U\}$. It is clear that U^{-1} is open in $X \times X$ and that $g \in (U^{-1})^+$. But if $f \in (U^{-1})^+$, then $f^{-1} \in U^+$, which shows that the inverse operation in $H_r(X)$ is continuous.

To prove the continuity of composition, we use the metric structure of X. Let $f, g \in H_r(X)$ and $r \in r^+(X) = C^+(X)$. Note that $rf^{-1} \in C^+(X) = r^+(X)$, as X is an almost P-space. Now let us define $\lambda\colon X \to (0, \infty)$ by

$$\begin{aligned}\lambda(x) = \sup\{s \in (0,\infty)\colon &\text{ for some } t \in (0,\infty),\\ &g(B(x,s)) \subseteq B(g(x), rf^{-1}(x) - t) \text{ and}\\ &rf^{-1}(B(x,s)) \subseteq (t, 2rf^{-1}(x) - s)\},\ \forall x \in X.\end{aligned}$$

To show that λ is lower-semicontinuous, let $x \in X$ and $b \in (0, \infty)$. Then there exist $s, t \in (0, \infty)$ such that $r > \lambda(x) - b$, $g(B(x, s)) \subseteq B(g(x), rf^{-1}(x) - t)$, and $rf^{-1}(B(x, s)) \subseteq (t, 2rf^{-1}(x) - t)$. Let $u = (s - \lambda(x) + b)/2$. Finally, define

$$V = B(x,u) \cap g^{-1}(B(g(x), t/3)) \cap fr^{-1}((rf^{-1}(x) - t/3, rf^{-1}(x) + t/3))$$

which is a neighborhood of x in X.

We need to show that $\lambda(V) \subseteq (\lambda(x) - b, \infty)$. So let $\acute{x} \in V$. Now take $\acute{s} = s - t$, so that $\lambda(x) - b < \acute{s} < s$. Observe that $B(\acute{x}, \acute{s}) \subseteq B(x, s)$ because $\acute{x} \in B(x, u)$ and $\acute{s} + u = s$. So we have

$$g(B(\acute{x}, \acute{s})) \subseteq B(g(x), rf^{-1}(x) - t)$$

and

$$rf^{-1}(B(\acute{x}, \acute{s})) \subseteq (t, 2rf^{-1}(x) - t).$$

We also have

$$g(\acute{x}) \in B(g(x), t/3)$$

and

$$rf^{-1}(x^{-1}) \in (rf^{-1}(x) - t/3, rf^{-1}(x) + t/3).$$

Now we need to show that

$$B(g(x), rf^{-1}(x) - t) \subseteq B(g(\acute{x}), rf^{-1}(\acute{x}) - t/3).$$

So let $y \in B(g(x), rf^{-1}(x) - t)$. Then

$$\begin{aligned} d(y, g(\acute{x})) &\le d(y, g(x)) + d(g(x), g(\acute{x}))\\ &< rf^{-1}(x) - t + t/3\\ &< rf^{-1}(\acute{x} + t/3) - t + t/3\\ &= rf^{-1}(\acute{x}) - t/3. \end{aligned}$$

Therefore,

$$B(g(x), rf^{-1}(x) - t) \subseteq B(g(\acute{x}), rf^{-1}(\acute{x}) - t/3),$$

showing that

$$g(B(\acute{x}, \acute{s}) \subseteq B(g(\acute{x}), rf^{-1}(\acute{x}) - t/3)$$

with the same argument as above, we also get that

$$(t, rf^{-1}(x) - t) \subseteq (t/3, rf^{-1}(\acute{x}) - t/3)$$

which shows that

$$rf^{-1}(B(\acute{x},\acute{s})) \subseteq (t/3, rf^{-1}(\acute{x}) - t/3).$$

We can now conclude that $\acute{s} \leq \lambda(\acute{x})$ and have $\lambda(x) - b < \acute{s} \leq \lambda(\acute{x})$. This is true for all $\acute{x} \in V$, so that $\lambda(V) \subseteq (\lambda(x) - b, \infty)$, and thus λ is lower semi-continuous.

Since $0 < \lambda$, there is an $\alpha \in C^+(X) = r^+(X)$ such that $\alpha < \lambda$. Note that $\alpha f \in C^+(X) = r^+(X)$. Consider the neighborhoods $B(f, \alpha f)$ and $B(g, rf^{-1})$ of f and g in $H_r(X)$. We want to show that if $\acute{f} \in B(f, \alpha f)$ and $\acute{g} \in B(g, rf^{-1})$, then $\acute{x}g\acute{f} \in B(gf, 3r)$ (by using $r/3$ in defining λ and by taking $\acute{g}$ from $B(g, rf^{-1}/3)$, we can set $\acute{g}\acute{f} \in B(gf, r)$). So to show that $\acute{g}\acute{f} \in B(gf, 3r)$, let $x \in X$. Then $\acute{f}(x) \in B(f(x), \alpha f(x))$. Now, $\alpha f(x) < \lambda(f(x))$ so that

$$\begin{aligned} g(B(f(x), \alpha(f(x)))) &\subseteq B(g(f(x), rf^{-1}(f(x)))) \\ &= B(gf(x), r(x)). \end{aligned}$$

Therefore, $g\acute{f}(x) \in B(gf(x), r(x))$. Also, since $\acute{g} \in B(g, rf^{-1})$, we have

$$\acute{g}\acute{f}(x) \in B(g\acute{f}(x), rf^{-1}(\acute{f}(x))).$$

But

$$rf^{-1}(B(f(x), \alpha f(x))) \subseteq (0, 2r(x)),$$

so that $rf^{-1}(\acute{f}(x)) \in (0, 2r(x))$; that is $rf^{-1}(\acute{f}(x))) < 2r(x)$. So $\acute{g}\acute{f}(x) \in B(g\acute{f}(x), 2r(x))$, and thus $\acute{g}\acute{f}(x) \in B(gf(x), 3r(x))$, as needed. This completes the argument that composition operation in $H_r(X)$ is continuous. □

Example 3.12. As an example, we can consider $\mathbb{R}$ with discrete metric that generates a discrete topology on it. Since $\mathbb{R}$ is a P-space and thus an almost P-space. Therefore, the space $H_r(\mathbb{R})$ becomes a topological group.

Now, we restrict our study on the function space $H(\mathbb{R})$ and study its behaviour in the following results.

Proposition 3.13 ([9, Prop. 6.9])**.** *Every $f \in H(\mathbb{R})$ is either increasing or decreasing .*

The above result declares that $H(\mathbb{R}) = H_r^-(\mathbb{R}) \cup H_r^+(\mathbb{R})$, where $H_r^-(\mathbb{R})$ and $H_r^+(\mathbb{R})$ are set of all decreasing and increasing homeomorphisms, respectively.

Theorem 3.14. *The space $H_r^-(\mathbb{R})$ is homeomorphic to $H_r^+(\mathbb{R})$.*

Proof. Let's prove this result considering that $\mathbb{R}$ is carrying the usual metric. So let's define a map $\psi\colon H_r^+(\mathbb{R}) \to H_r^-(\mathbb{R})$ by $\psi(h)(x) = -h(x)$, $\forall h \in H_r^+(\mathbb{R})$ and $\forall\, x \in \mathbb{R}$. Clearly, ψ is one-one and onto. Now, we show ψ is continuous, so consider a basic neighborhood $B(\psi(h), r)$ of $\psi(h) = -h$ in $H_r^-(\mathbb{R})$, where $r \in r^+(\mathbb{R})$. Then $B(h, r)$ is a basic neighborhood of h in $H_r^+(\mathbb{R})$. Let $f \in B(h, r)$. Then

$$\sup\{d(f(x), h(x)) < r(x),\ \forall x \in coz(r)\}$$

$$
\begin{aligned}
&= \sup\{|f(x) - h(x)| < r(x),\ \forall x \in coz(r)\}\\
&= \sup\{|(-f(x)) - (-h(x))| < r(x),\ \forall x \in coz(r)\}\\
&= \sup\{d((-f(x)), (-h(x))) < r(x),\ \forall x \in coz(r)\}.
\end{aligned}
$$

Thus, we have $\psi(B(h,r)) \subseteq B(\psi(h), r)$. Similarly, we can prove the continuity of $\psi^{-1}\colon H_r^-(\mathbb{R}) \to H_r^+(\mathbb{R})$. Hence, ψ is a homeomorphism. □

Theorem 3.15. *The spaces $H_r^-(\mathbb{R})$ and $H_r^+(\mathbb{R})$ are open subspaces of $H_r(\mathbb{R})$.*

Proof. Let's first proceed with proving the result for $H_r^+(\mathbb{R})$. Let $g \in H_r^+(\mathbb{R})$, and consider $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$. Consider a basic neighborhood $B(g,r)$ of g in $H_r(\mathbb{R})$, where $r \in r_+(\mathbb{R})$. Let $f \in B(g,r) \Rightarrow |g(x) - f(x)| < r(x),\ \forall x \in coz(r)$. We shall prove that $f \in H^+(\mathbb{R})$, let's assume this is not true. That is $f \notin H^+(\mathbb{R})$, which implies $f(x_2) < f(x_1)$. Thus we have $f(x_2) < f(x_1) < r(x_1) + g(x_1) < g(x_2) + r(x_2) < f(x_2)$. This implies $f(x_2) - g(x_2) > r(x_2)$, which is a contradiction. Therefore, our assumption is wrong. Thus, $f \in H^+(\mathbb{R})$ and hence $B(g,r) \subseteq H^+(\mathbb{R})$ implies $H_r^+(\mathbb{R})$ is open in $H_r(\mathbb{R})$. We can similarly prove the result for $H^-(\mathbb{R})$. □

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