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GEOMETRY OF GENERALIZED BERGER-TYPE DEFORMED METRIC ON B-MANIFOLD

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Abstract. Let (M^{2m}, φ, g) be a B-manifold. In this paper, we introduce a new class of metric on (M^{2m}, φ, g), obtained by a non-conformal deformation of the metric g, called a generalized Berger-type deformed metric. First we investigate the Levi-Civita connection of this metric. Secondly we characterize the Riemannian curvature, the sectional curvature and the scalar curvature. Finally, we study the proper biharmonicity of the identity map and of a curve on M with respect to a generalized Berger-type deformed metric.

1. Introduction

Let (M^m, g) be an m-dimensional Riemannian manifold, ∇ be the Levi-Civita connection of g. This connection is characterized by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) \\ &\quad + g(Y,[Z,X]) - g(X,[Y,Z]) \end{aligned} \tag{1}$$

for all vector fields X, Y and Z on M.

Let f be a smooth function on M, the gradient of f, denoted by $gradf$, is defined by

$$g(gradf, X) = X(f)$$

and also, the Hessian of f, denoted by $Hess_f$, is defined by

$$Hess_f(X,Y) = g(\nabla_X gradf, Y) = X(Y(f)) - (\nabla_X Y)(f)$$

for all vector fields X and Y on M.

A vector field X on M is said to be a Killing vector field if

$$g(\nabla_Y X, Z) + g(Y, \nabla_X Z) = 0$$

for all vector fields Y and Z on M. We say that a smooth function f on M is a Killing potential if $gradf$ is a Killing vector field. Then $gradf$ is called a

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Killing gradient [10]. Moreover f is a Killing potential if and only if $Hess_f = 0$ or equivalently $\nabla_X grad f = 0$ for all vector field X on M.

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. The map ϕ is said to be harmonic if it is a critical point of the energy functional

$$E(\phi, K) = \frac{1}{2}\int_K |d(\phi)|^2 \, dv^g$$

for any compact domain $K \subseteq M$. Here $|d(\phi)|$ is the Hilbert-Schmitd norm of $d(\phi)$ and v^g is the Riemannian volume form on M. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations given by the following formula:

$$\tau(\phi) = trace_g \nabla d\phi. \tag{2}$$

Here $\tau(\phi)$ is the tension field of ϕ. For more details see [14–16]. In recent years, this theme has been widely developed even on the tangent bundle and on the cotangent bundle has been done by many authors [5, 9, 12, 13, 19–21].

As a natural generalization of harmonic maps, biharmonic maps are defined similarly, as follows. A map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional

$$E_2(\phi, K) = \frac{1}{2}\int_K |\tau(\phi)|^2 \, dv^g$$

over any compact domain K. Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_2(\phi) = -trace_g R^N(\tau(\phi), d\phi)d\phi - trace_g(\nabla^\phi)^2\tau(\phi) = 0. \tag{3}$$

Here

$$(\nabla^\phi)^2\tau(\phi) = \nabla^\phi\nabla^\phi\tau(\phi) - \nabla^\phi_{\nabla^M}\tau(\phi),$$

$\tau_2(\phi)$ is the bitension field of ϕ (see [3, 4, 6–8, 11, 12]). It is obvious to see that any harmonic map is biharmonic, therefore it is interesting to construct proper biharmonic maps (non-harmonic biharmonic maps).

In previous work [13], we introduced a new class of metric on Riemannian manifold (M^m, g) defined by

$$G(X, Y) = fg(X, Y) + g(\xi, X)g(\xi, Y)$$

for all vector fields X, Y and ξ on M such that $g(\xi, \xi) = 1$ and $\xi(f) = 0$, where f is a strictly positive smooth function on M. We searched some properties of the Riemannian manifold with this metric.

Motivated by the above studies, we introduce a new class of metric on B-manifold (M^{2m}, φ, g), denoted by ${}^{GB}g$, given by

$${}^{GB}g(X, Y) = g(X, Y) + fg(X, \varphi\xi)g(Y, \varphi\xi)$$

for all vector fields X, Y and ξ on M such that $g(\xi, \xi) = 1$ and $\varphi\xi(f) = 0$, where f is a positive smooth function on M. We call ${}^{GB}g$ a generalized Berger-type deformed metric.

The Berger-type deformation idea with respect to a $(1,1)$-tensor field such as a complex or a paracomplex structure was introduced in [18] and developed in [1] and [2].

In this paper, we introduce a generalized Berger-type deformed metric on B-manifold (M^{2m}, φ, g) by non-conformal deformation of g and study its geometry. Firstly, we establish the Levi-Civita connection of this metric (Theorem 2.2). Secondly, we study the curvature tensor (Theorem 3.1 and Corollary 3.2) and characterize the sectional curvature (Theorem 3.3, Corollary 3.4), the Ricci tensor (Theorem 3.6 and Corollary 3.7), the Ricci curvature (Theorem 3.8 and Corollary 3.9) and the scalar curvature (Theorem 3.10, Corollary 3.11 and Corollary 3.12). In the last section, we study the proper biharmonicity of the identity map between Riemannian manifolds (Theorem 4.4 and Theorem 4.8), where one manifold is equipped with a generalized Berger-type deformed metric. Finally, we study the proper biharmonicity of a curve in Riemannian manifold equipped with a generalized Berger-type deformed metric (Theorem 5.4).

2. Generalized Berger-type deformed metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, $\varphi \neq \pm id$, such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

In order that an almost paracomplex structure φ be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection ∇ such that $\nabla\varphi = 0$. On the other hand, the integrability of an almost paracomplex structure φ is equivalent to the vanishing of the Nijenhuis tensor,

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y]$$

for all vector fields X and Y on M.

Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is a B-metric if

$$g(\varphi X, \varphi Y) = g(X,Y) \tag{4}$$

for any vector fields X, Y on M [17], or equivalently (purity condition)

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{5}$$

for all vector fields X and Y on M. If (M^{2m}, φ) is an almost paracomplex manifold with B-metric g, we say that (M^{2m}, φ, g) is an almost B-manifold. If φ is integrable, we say that (M^{2m}, φ, g) is a B-manifold [17].

The purity conditions for a $(0,q)$-tensor field ω with respect to the almost paracomplex structure φ given by

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \cdots = \omega(X_1, X_2, \ldots, \varphi X_q)$$

for any vector fields $X_1, X_2, \ldots, X_q$ on M [17].

It is well known that if (M^{2m},φ,g) is a B-manifold, the Riemannian curvature tensor R is pure [17]. Moreover R has the following properties:

$$\begin{cases} R(\varphi Y,Z)=R(Y,\varphi Z)=R(Y,Z)\varphi=\varphi R(Y,Z), \\ R(\varphi Y,\varphi Z)=R(Y,Z) \end{cases} \tag{6}$$

for any vector fields Y and Z on M.

Definition. Let (M^{2m},φ,g) be an almost B-manifold and $f : M \to [0,+\infty[$ be a positive smooth function on M. We define a generalized Berger-type deformed metric of g on M, say ${}^{GB}g$, by

$$ {}^{GB}g(X,Y)=g(X,Y)+fg(X,\varphi\xi)g(Y,\varphi\xi) \tag{7}$$

for all vector fields X, Y and ξ on M such that

$$\begin{cases} g(\xi,\xi)=1, \\ (\varphi\xi)(f)=0, \\ \nabla\xi=0, \end{cases} \tag{8}$$

where ∇ denote the Levi-Civita connection of (M^{2m},φ,g).

Note that we have

$$\begin{cases} g(\varphi\xi,\varphi\xi)=1, \\ g(X,\varphi\xi)=\dfrac{1}{1+f}{}^{GB}g(X,\varphi\xi), \\ X(f)={}^{GB}g(X,grad f), \\ Hess_f(X,\varphi\xi)=0 \end{cases} \tag{9}$$

for any vector field X on M.

Lemma 2.1. *Let (M^{2m},φ,g) be a B-manifold. Then we have*

$$ X\,{}^{GB}g(Y,Z)={}^{GB}g(\nabla_XY,Z)+{}^{GB}g(Y,\nabla_XZ)+X(f)g(Y,\varphi\xi)g(Z,\varphi\xi) \tag{10}$$

for any vector fields X, Y and Z on M.

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of $(M^{2m},{}^{GB}g)$ as follows.

Theorem 2.2. *Let (M^{2m},φ,g) be a B-manifold. Then the Levi-Civita connection $\widetilde{\nabla}$ of $(M^{2m},{}^{GB}g)$ is given by*

$$\begin{aligned} \widetilde{\nabla}_XY &= \nabla_XY+\frac{1}{2(1+f)}\big(X(f)g(Y,\varphi\xi)+Y(f)g(X,\varphi\xi)\big)\varphi\xi \\ &\quad -\frac{1}{2}g(X,\varphi\xi)g(Y,\varphi\xi)grad f \end{aligned} \tag{11}$$

for any vector fields X and Y on M.

Proof. From Kozul formula (1) and (10), we have

$$\begin{aligned} 2\,{}^{GB}g(\widetilde{\nabla}_XY,Z) &= X\,{}^{GB}g(Y,Z)+Y\,{}^{GB}g(Z,X)-Z\,{}^{GB}g(X,Y)+{}^{GB}g(Z,[X,Y]) \\ &\quad +{}^{GB}g(Y,[Z,X])-{}^{GB}g(X,[Y,Z]) \\ &= {}^{GB}g(\nabla_XY,Z)+{}^{GB}g(Y,\nabla_XZ)+X(f)g(Y,\varphi\xi)g(Z,\varphi\xi) \end{aligned}$$

$$\begin{aligned}
&+ {}^{GB}g(\nabla_Y Z, X) + {}^{GB}g(Z, \nabla_X Y) + Y(f)g(Z,\varphi\xi)g(X,\varphi\xi)\\
&- {}^{GB}g(\nabla_Z X, Y) - {}^{GB}g(X, \nabla_Z Y) - Z(f)g(X,\varphi\xi)g(Y,\varphi\xi)\\
&+ {}^{GB}g(Z, \nabla_X Y) - {}^{GB}g(Z, \nabla_Y X) + {}^{GB}g(Y, \nabla_Z X)\\
&- {}^{GB}g(Y, \nabla_X Z) - {}^{GB}g(X, \nabla_Y Z) - {}^{GB}g(X, \nabla_Z Y)\\
= {}& 2{}^{GB}g\big(\nabla_X Y, Z) + X(f)g(Y,\varphi\xi)g(Z,\varphi\xi)\\
&+ Y(f)g(Z,\varphi\xi)g(X,\varphi\xi) - Z(f)g(X,\varphi\xi)g(Y,\varphi\xi).
\end{aligned}$$

Using (9), we get

$$\begin{aligned}
2{}^{GB}g(\widetilde{\nabla}_X Y, Z) = {}& 2{}^{GB}g\big(\nabla_X Y, Z) + \frac{X(f)}{1+f} g(Y,\varphi\xi){}^{GB}g(Z,\varphi\xi)\\
&+ \frac{Y(f)}{1+f} g(X,\varphi\xi){}^{GB}g(Z,\varphi\xi) - g(X,\varphi\xi)g(Y,\varphi\xi){}^{GB}g(Z, grad f).
\end{aligned}$$

This completes the proof. □

Using (9) and (11), we obtain the following:

$$\widetilde{\nabla}_X grad f = \nabla_X grad f + \frac{|grad\, f|^2}{2(1+f)} g(X,\varphi\xi)\varphi\xi, \tag{12}$$

$$\widetilde{\nabla}_X(\varphi\xi) = \frac{X(f)}{2(1+f)}\varphi\xi - \frac{1}{2} g(X,\varphi\xi) grad f \tag{13}$$

for any vector field X on M.

3. Curvatures of generalized Berger-type deformed metric

We shall calculate the Riemannian curvature tensor $\widetilde{R}$ of $(M^{2m}, {}^{GB}g)$ as follows.

Theorem 3.1. *Let (M^{2m}, φ, g) be a B-manifold. Then the Riemannian curvature tensor $\widetilde{R}$ of $(M^{2m}, {}^{GB}g)$ is given by*

$$\begin{aligned}
\widetilde{R}(X,Y)Z = {}& R(X,Y)Z + \frac{1}{2} g(Z,\varphi\xi)\big(g(X,\varphi\xi)\nabla_Y grad f - g(Y,\varphi\xi)\nabla_X grad f\big)\\
&+ \frac{1}{4(1+f)} g(Z,\varphi\xi)\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big) grad f\\
&+ \frac{1}{2(1+f)}\big(Hess_f(X,Z)g(Y,\varphi\xi) - Hess_f(Y,Z)g(X,\varphi\xi)\big)\varphi\xi\\
&- \frac{Z(f)}{4(1+f)^2}\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big)\varphi\xi
\end{aligned}\tag{14}$$

for any vector fields X, Y and Z on M, where ∇ and R denotes, respectively, the Levi-Civita connection and the curvature tensor of (M^{2m}, φ, g).

Proof. For any vector fields X, Y and Z on M, we have

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{15}$$

Using (11), (12) and (13), we obtain

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_Y Z &= \nabla_X\nabla_Y Z + \frac{X(f)}{2(1+f)}g(\nabla_Y Z,\varphi\xi)\varphi\xi + \frac{(\nabla_Y Z)(f)}{2(1+f)}g(X,\varphi\xi)\varphi\xi\\ &- \frac{1}{2}g(X,\varphi\xi)g(\nabla_Y Z,\varphi\xi)grad f - \frac{1}{2}g(Y,\varphi\xi)g(Z,\varphi\xi)\nabla_X grad f\\ &+ \big(\frac{XY(f)}{2(1+f)} - \frac{X(f)Y(f)}{4(1+f)^2}\big)g(Z,\varphi\xi)\varphi\xi + \frac{Z(f)}{2(1+f)}g(\nabla_X Y,\varphi\xi)\varphi\xi\\ &- \big(\frac{Y(f)}{4(1+f)}g(Z,\varphi\xi) + \frac{Z(f)}{4(1+f)}g(Y,\varphi\xi)\big)g(X,\varphi\xi)grad f\\ &+ \big(\frac{XZ(f)}{2(1+f)} - \frac{X(f)Z(f)}{4(1+f)^2}\big)g(Y,\varphi\xi)\varphi\xi + \frac{Y(f)}{2(1+f)}g(\nabla_X Z,\varphi\xi)\varphi\xi\\ &- \frac{1}{2}\big(g(\nabla_X Y,\varphi\xi)g(Z,\varphi\xi) + g(Y,\varphi\xi)g(\nabla_X Z,\varphi\xi)\big)grad f\\ &- \frac{|grad f|^2}{4(1+f)}g(X,\varphi\xi)g(Y,\varphi\xi)g(Z,\varphi\xi)\varphi\xi. \qquad (16)\end{aligned}$$

In fact, by substituting X by Y into the $\widetilde{\nabla}_X\widetilde{\nabla}_Y Z$, we get

$$\begin{aligned}\widetilde{\nabla}_Y\widetilde{\nabla}_X Z &= \nabla_Y\nabla_X Z + \frac{Y(f)}{2(1+f)}g(\nabla_X Z,\varphi\xi)\varphi\xi + \frac{(\nabla_X Z)(f)}{2(1+f)}g(Y,\varphi\xi)\varphi\xi\\ &- \frac{1}{2}g(Y,\varphi\xi)g(\nabla_X Z,\varphi\xi)grad f - \frac{1}{2}g(X,\varphi\xi)g(Z,\varphi\xi)\nabla_Y grad f\\ &+ \big(\frac{YX(f)}{2(1+f)} - \frac{Y(f)X(f)}{4(1+f)^2}\big)g(Z,\varphi\xi)\varphi\xi + \frac{Z(f)}{2(1+f)}g(\nabla_Y X,\varphi\xi)\varphi\xi\\ &- \big(\frac{X(f)}{4(1+f)}g(Z,\varphi\xi) - \frac{Z(f)}{4(1+f)}g(X,\varphi\xi)\big)g(Y,\varphi\xi)grad f\\ &+ \big(\frac{YZ(f)}{2(1+f)} - \frac{Y(f)Z(f)}{4(1+f)^2}\big)g(X,\varphi\xi)\varphi\xi + \frac{X(f)}{2(1+f)}g(\nabla_Y Z,\varphi\xi)\varphi\xi\\ &- \frac{1}{2}\big(g(\nabla_Y X,\varphi\xi)g(Z,\varphi\xi) + g(X,\varphi\xi)g(\nabla_Y Z,\varphi\xi)\big)grad f\\ &- \frac{|grad f|^2}{4(1+f)}g(Y,\varphi\xi)g(X,\varphi\xi)g(Z,\varphi\xi)\varphi\xi. \qquad (17)\end{aligned}$$

We also find

$$\begin{aligned}\widetilde{\nabla}_{[X,Y]}Z &= \nabla_{[X,Y]}Z + \frac{1}{2(1+f)}\big([X,Y](f)g(Z,\varphi\xi) + Z(f)g([X,Y],\varphi\xi)\big)\varphi\xi\\ &- \frac{1}{2}g([X,Y],\varphi\xi)g(Z,\varphi\xi)grad f. \qquad (18)\end{aligned}$$

Substituting (16), (17) and (18) into (15) we find (14). □

Corollary 3.2. *Let (M^{2m},φ,g) be a B-manifold. If f is a Killing potential, then the Riemannian curvature tensor $\widetilde{R}$ of $(M^{2m}, {}^{GB}g)$ is given by*

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - \frac{Z(f)}{4(1+f)^2}\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big)\varphi\xi + \frac{1}{4(1+f)}g(Z,\varphi\xi)\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big)grad f$$

for any vector fields X, Y and Z on M.

Theorem 3.3. *Let (M^{2m},φ,g) be a B-manifold. If K (resp., $\widetilde{K}$) denote the sectional curvature of (M^{2m},φ,g) (resp., $(M^{2m},{}^{GB}g)$), then we have*

$$\widetilde{K}(X,Y) = \frac{1}{1+f(g(X,\varphi\xi)^2+g(Y,\varphi\xi)^2)}\Big(K(X,Y) - \frac{1}{2}g(Y,\varphi\xi)^2 Hess_f(X,X) - \frac{1}{2}g(X,\varphi\xi)^2 Hess_f(Y,Y) + g(X,\varphi\xi)g(Y,\varphi\xi)Hess_f(X,Y) + \frac{1}{4(1+f)}\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big)^2\Big) \tag{19}$$

for any X and Y two vector fields orthonormal with respect to g.

Proof. For any vector fields V, W on M, $x \in M$ such that V_x and W_x are linearly independent, the sectional curvature of the plane spanned by V_x and W_x is given by

$$\widetilde{K}(V,W) = \frac{{}^{GB}g(\widetilde{R}(V,W)W,V)}{{}^{GB}g(V,V)\,{}^{GB}g(W,W) - {}^{GB}g(V,W)^2}. \tag{20}$$

First we calculate,

$${}^{GB}g(\widetilde{R}(X,Y)Y,X) = g(\widetilde{R}(X,Y)Y,X) + fg(\widetilde{R}(X,Y)Y,\varphi\xi)g(X,\varphi\xi). \tag{21}$$

From (9) and (14) with direct computation, we get

$$g(\widetilde{R}(X,Y)Y,X) = g(R(X,Y)Y,X) + \frac{2+f}{2(1+f)}g(X,\varphi\xi)g(Y,\varphi\xi)Hess_f(X,Y) - \frac{1}{2}g(Y,\varphi\xi)^2 Hess_f(X,X) - \frac{1}{2}g(X,\varphi\xi)^2 Hess_f(Y,Y) + \frac{X(f)^2}{4(1+f)}g(Y,\varphi\xi)^2 + \frac{Y(f)^2}{4(1+f)}g(X,\varphi\xi)^2 + \frac{(2+f)X(f)Y(f)}{4(1+f)^2}g(X,\varphi\xi)g(Y,\varphi\xi), \tag{22}$$

$$fg(\widetilde{R}(X,Y)Y,\varphi\xi)g(X,\varphi\xi) = \frac{f}{2(1+f)}g(X,\varphi\xi)g(Y,\varphi\xi)Hess_f(X,Y) - \frac{f}{2(1+f)}g(X,\varphi\xi)^2 Hess_f(Y,Y) + \frac{fY(f)^2}{4(1+f)^2}g(X,\varphi\xi)^2 - \frac{fX(f)Y(f)}{4(1+f)^2}g(X,\varphi\xi)g(Y,\varphi\xi). \tag{23}$$

On the other hand, we have

$${}^{GB}g(X,X)\,{}^{GB}g(Y,Y) - {}^{GB}g(X,Y)^2 = 1 + f(g(X,\varphi\xi)^2 + g(Y,\varphi\xi)^2). \tag{24}$$

Substituting (22) and (23) into (21), we find

$$
\begin{aligned}
g(\widetilde{R}(X,Y)Y,X) = g(R(X,Y)Y,X) &- \frac{1}{2}g(Y,\varphi\xi)^2 Hess_f(X,X)\\
&- \frac{1}{2}g(X,\varphi\xi)^2 Hess_f(Y,Y) + g(X,\varphi\xi)g(Y,\varphi\xi)Hess_f(X,Y)\\
&+ \frac{1}{4(1+f)}\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big)^2. \qquad (25)
\end{aligned}
$$

Finally, substituting (24) and (25) into (20) we find (19). □

Corollary 3.4. *Let (M^{2m},φ,g) be a B-manifold. If f is a Killing potential, then the sectional curvature $\widetilde{K}$ of $(M^{2m}, {}^{GB}g)$ is given by*

$$
\begin{aligned}
\widetilde{K}(X,Y) = \frac{1}{1+f(g(X,\varphi\xi)^2+g(Y,\varphi\xi)^2)}\Big(&K(X,Y)\\
&+ \frac{1}{4(1+f)}\big(X(f)g(Y,\varphi\xi) - Y(f)g(X,\varphi\xi)\big)^2\Big)
\end{aligned}
$$

for any X, Y two vector fields orthonormal with respect to g.

Remark 3.5. Let $\{E_i\}_{i=\overline{1,m}}$ such that $E_1 = \varphi\xi$ be a local orthonormal frame on (M^{2m},φ,g). We define the orthonormal vector fields

$$
\widetilde{E}_1 = \frac{1}{\sqrt{1+f}}\varphi\xi, \widetilde{E}_i = E_i,\ i = \overline{2,2m}. \qquad (26)
$$

Then $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ is a local orthonormal frame on $(M^{2m}, {}^{GB}g)$.

Theorem 3.6. *Let (M^{2m},φ,g) be a B-manifold. If Ricci (resp. $\widetilde{Ricci}$) denote the Ricci tensor of (M^{2m},φ,g) (resp., $(M^{2m}, {}^{GB}g)$), then we have*

$$
\begin{aligned}
\widetilde{Ricci}(X) = Ricci(X) &- \frac{1}{2(1+f)}\nabla_X grad f + \frac{1}{2(1+f)}g(X,\varphi\xi)\nabla_{\varphi\xi}grad f\\
&+ \frac{X(f)}{4(1+f)^2}grad f + \big(\frac{|grad f|^2}{4(1+f)^2} - \frac{\Delta(f)}{2(1+f)}\big)g(X,\varphi\xi)\varphi\xi \qquad (27)
\end{aligned}
$$

for any vector field X on M.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{GB}g)$ defined by (26). By the definition of Ricci tensor, we have

$$
\begin{aligned}
\widetilde{Ricci}(X) &= \sum_{i=1}^{2m}\widetilde{R}(X,\widetilde{E}_i)\widetilde{E}_i\\
&= \frac{1}{1+f}\widetilde{R}(X,\varphi\xi)\varphi\xi + \sum_{i=2}^{2m}\widetilde{R}(X,E_i)E_i. \qquad (28)
\end{aligned}
$$

From (6), (9) and (14) with direct computation, we get

$$
\frac{1}{1+f}\widetilde{R}(X,\varphi\xi)\varphi\xi = \frac{1}{2(1+f)}g(X,\varphi\xi)\nabla_{\varphi\xi}grad f - \frac{1}{2(1+f)}\nabla_X grad f
$$

$$+\frac{X(f)}{4(1+f)}grad f, \tag{29}$$

and

$$\sum_{i=2}^{2m}\widetilde{R}(X,E_i)E_i = Ricci(X)-\frac{\Delta(f)}{2(1+f)}g(X,\varphi\xi)\varphi\xi$$
$$+\frac{|grad\, f|^2}{4(1+f)^2}g(X,\varphi\xi)\varphi\xi. \tag{30}$$

Substituting (29) and (30) into (28), we find

$$\widetilde{Ricci}(X) = Ricci(X)-\frac{1}{2(1+f)}\nabla_X grad f+\frac{1}{2(1+f)}g(X,\varphi\xi)\nabla_{\varphi\xi}grad f$$
$$+\frac{X(f)}{4(1+f)^2}grad f+\Big(\frac{|grad\, f|^2}{4(1+f)^2}-\frac{\Delta(f)}{2(1+f)}\Big)g(X,\varphi\xi)\varphi\xi. \qquad \square$$

Corollary 3.7. *Let (M^{2m},φ,g) be a B-manifold. If f is a Killing potential, then the Ricci tensor $\widetilde{Ricci}$ of $(M^{2m},{}^{GB}g)$ is given by*

$$\widetilde{Ricci}(X) = Ricci(X)+\frac{X(f)}{4(1+f)^2}grad f+\frac{|grad\, f|^2}{4(1+f)^2}g(X,\varphi\xi)\varphi\xi$$

for any vector field X on M.

Theorem 3.8. *Let (M^{2m},φ,g) be a B-manifold. If Ric (resp. $\widetilde{Ric}$) denote the Ricci curvature of (M^{2m},φ,g) (resp., $(M^{2m},{}^{GB}g)$), then we have*

$$\widetilde{Ric}(X,Y) = Ric(X,Y)-\frac{1}{2(1+f)}Hess_f(X,Y)+\frac{X(f)Y(f)}{4(1+f)^2}$$
$$+\Big(\frac{|grad\, f|^2}{4(1+f)}-\frac{\Delta(f)}{2}\Big)g(X,\varphi\xi)g(Y,\varphi\xi) \tag{31}$$

for any vector fields X and Y on M.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{GB}g)$ defined by (26). By the definition of Ricci tensor, we have

$$\widetilde{Ric}(X,Y) = {}^{GB}g(\widetilde{Ricci}(X),Y)$$
$$= g(\widetilde{Ricci}(X),Y)+fg(\widetilde{Ricci}(X),\varphi\xi)g(Y,\varphi\xi). \tag{32}$$

From the formula (27) and direct computation, we get

$$g(\widetilde{Ricci}(X),Y) = Ric(X,Y)-\frac{1}{2(1+f)}Hess_f(X,Y)+\frac{X(f)Y(f)}{4(1+f)^2}$$
$$+\Big(\frac{|grad\, f|^2}{4(1+f)^2}-\frac{\Delta(f)}{2(1+f)}\Big)g(X,\varphi\xi)g(Y,\varphi\xi), \tag{33}$$

and

$$fg(\widetilde{Ricci}(X),\varphi\xi)g(Y,\varphi\xi) = f\Big(\frac{|grad\, f|^2}{4(1+f)^2}-\frac{\Delta(f)}{2(1+f)}\Big)g(X,\varphi\xi)g(Y,\varphi\xi). \tag{34}$$

Substituting (33) and (34) into (32), we find

$$\begin{aligned}\widetilde{Ric}(X,Y) &= Ric(X,Y) - \frac{1}{2(1+f)}Hess_f(X,Y) + \frac{X(f)Y(f)}{4(1+f)^2}\\ &\quad + (\frac{|grad\, f|^2}{4(1+f)} - \frac{\Delta(f)}{2})g(X,\varphi\xi)g(Y,\varphi\xi).\end{aligned}$$

□

Corollary 3.9. *Let (M^{2m},φ,g) be a B-manifold. If f is a Killing potential, then the Ricci curvature $\widetilde{Ric}$ of $(M^{2m},{}^{GB}g)$ is given by*

$$\widetilde{Ric}(X,Y) = Ric(X,Y) + \frac{X(f)Y(f)}{4(1+f)^2} + \frac{|grad\, f|^2}{4(1+f)}g(X,\varphi\xi)g(Y,\varphi\xi)$$

for any vector field X on M.

Theorem 3.10. *Let (M^{2m},φ,g) be a B-manifold. If σ (resp., $\widetilde{\sigma}$) denotes the scalar curvature of (M^{2m},φ,g) (resp., $(M^{2m},{}^{GB}g)$), then we have*

$$\widetilde{\sigma} = \sigma + \frac{|grad\, f|^2}{2(1+f)^2} - \frac{\Delta(f)}{1+f}. \tag{35}$$

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{GB}g)$ defined by (26). By the definition of the scalar curvature, we have

$$\begin{aligned}\widetilde{\sigma} &= \sum_{i=1}^{2m}\widetilde{Ric}(\widetilde{E}_i,\widetilde{E}_i)\\ &= \frac{1}{1+f}\widetilde{Ric}(\varphi\xi,\varphi\xi) + \sum_{i=2}^{2m}\widetilde{Ric}(E_i,E_i).\end{aligned}$$

From the formula (31) and direct computation, we get

$$\begin{aligned}\widetilde{\sigma} &= \frac{1}{1+f}\Big(Ric(\varphi\xi,\varphi\xi) - \frac{1}{2(1+f)}Hess_f(\varphi\xi,\varphi\xi) + \frac{(\varphi\xi(f))^2}{4(1+f)^2}\\ &\quad + (\frac{|grad\, f|^2}{4(1+f)} - \frac{\Delta(f)}{2})g(\varphi\xi,\varphi\xi)^2\Big)\\ &\quad + \sum_{i=2}^{2m}\Big(Ric(E_i,E_i) - \frac{1}{2(1+f)}Hess_f(E_i,E_i) + \frac{E_i(f)^2}{4(1+f)^2}\\ &\quad + (\frac{|grad\, f|^2}{4(1+f)} - \frac{\Delta(f)}{2})g(E_i,\varphi\xi)^2\Big)\\ &= \frac{|grad\, f|^2}{4(1+f)^2} - \frac{\Delta(f)}{2(1+f)} + \sigma - \frac{\Delta(f)}{2(1+f)} + \frac{|grad\, f|^2}{4(1+f)^2}\\ &= \sigma + \frac{|grad\, f|^2}{2(1+f)^2} - \frac{\Delta(f)}{1+f}.\end{aligned}$$

□

Corollary 3.11. *Let (M^{2m}, φ, g) be a B-manifold. If f is a Killing potential, then the scalar curvature $\widetilde{\sigma}$ of $(M^{2m}, {}^{GB}g)$ is given by*

$$\widetilde{\sigma} = \sigma + \frac{|grad\, f|^2}{2(1+f)^2}.$$

Corollary 3.12. *Let (M^{2m}, φ, g) be a flat B-manifold. Then $(M^{2m}, {}^{GB}g)$ has zero scalar curvature if and only if f is a solution of the following differential equation*:

$$\Delta(f) = \frac{|grad\, f|^2}{2(1+f)}.$$

4. The proper biharmonicity of the identity map

Lemma 4.1. *The tension field of the identity map $I : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{GB}g)$ is given by*

$$\widetilde{\tau}(I) = -\frac{1}{2} grad f. \tag{36}$$

Proof. Let $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M. We compute the tension field $\widetilde{\tau}(I)$ of the identity map $I : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{GB}g)$:

$$\begin{aligned}
\widetilde{\tau}(I) &= trace_g \widetilde{\nabla} dI \\
&= \sum_{i=1}^{2m} \big(\widetilde{\nabla}^I_{E_i} dI(E_i) - dI(\nabla_{E_i} E_i)\big) \\
&= \sum_{i=1}^{2m} \big(\widetilde{\nabla}_{dI(E_i)} dI(E_i) - \nabla_{E_i} E_i\big) \\
&= \sum_{i=1}^{2m} \big(\widetilde{\nabla}_{E_i} E_i - \nabla_{E_i} E_i\big),
\end{aligned}$$

from Theorem 2.2, we have

$$\begin{aligned}
\widetilde{\tau}(I) &= \sum_{i=1}^{2m} \Big(\nabla_{E_i} E_i + \frac{E_i(f)}{1+f} g(E_i, \varphi\xi)\varphi\xi - \frac{1}{2} g(E_i, \varphi\xi)^2 grad f - \nabla_{E_i} E_i\Big) \\
&= -\frac{1}{2} grad f.
\end{aligned}$$

□

From Lemma 4.1 we obtain:

Corollary 4.2. *The identity map $I : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{GB}g)$ is harmonic if and only if f is constant.*

Theorem 4.3. *The bitension field of the identity map $I : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{GB}g)$ is given by*

$$\widetilde{\tau}_2(I) = \frac{1}{2} trace_g \nabla^2 grad f + \frac{1}{2} Ricci(grad f) - \frac{1}{4} \nabla_{grad f} grad f. \tag{37}$$

Proof. Let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on (M^{2m},φ,g). The bitension field of the identity map $I:(M^{2m},\varphi,g)\to(M^{2m},{}^{GB}g)$ is given by

$$\widetilde{\tau}_2(I)=-trace_g\widetilde{R}(\widetilde{\tau}(I),dI)dI-trace_g(\widetilde{\nabla}^I)^2\widetilde{\tau}(I). \tag{38}$$

From Lemma 4.1, we have

$$\widetilde{\tau}(I)=-\frac{1}{2}gradf.$$

Now, we compute the first term of (38), from Theorem 3.1, we have

$$\begin{aligned}-trace_g\widetilde{R}(\widetilde{\tau}(I),dI)dI&=\frac{1}{2}\sum_{i=1}^{2m}\widetilde{R}(gradf,dI(E_i))dI(E_i)\\&=\frac{1}{2}\sum_{i=1}^{2m}\widetilde{R}(gradf,E_i)E_i\\&=\frac{1}{2}Ricci(gradf)-\frac{1}{4}\nabla_{gradf}gradf+\frac{|gradf|^2}{8(1+f)}gradf.\end{aligned} \tag{39}$$

We also compute the second term of (38), from Theorem 2.2, we have

$$\begin{aligned}-trace_g(\widetilde{\nabla}^I)^2\widetilde{\tau}(I)&=\frac{1}{2}\sum_{i=1}^{2m}\big(\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}gradf-\widetilde{\nabla}_{\nabla_{E_i}E_i}gradf\big)\\&=\frac{1}{2}trace_g\nabla^2gradf-\frac{|gradf|^2}{8(1+f)}gradf.\end{aligned} \tag{40}$$

Substituting (39) and (40) into (38), we find (37). □

Theorem 4.4. *The identity map $I:(M^{2m},\varphi,g)\to(M^{2m},{}^{GB}g)$ is proper biharmonic if and only if the function f is non constant on M, and satisfying the following*

$$trace_g\nabla^2gradf+Ricci(gradf)=\frac{1}{2}\nabla_{gradf}gradf. \tag{41}$$

Example 4.5. Let $M=]0,+\infty[\times]0,\pi[$ be a B-manifold, equipped with the almost paracomplex structure φ and the B-metric g in the polar coordinate defined by

$$\begin{aligned}g&=dr^2+r^2d\theta^2,\\\varphi\partial_r&=\sin2\theta\partial_r+\frac{1}{r}\cos2\theta\partial_\theta,\\\varphi\partial_\theta&=r\cos2\theta\partial_r-\sin2\theta\partial_\theta.\end{aligned}$$

Let $f(r,\theta)=r\sin\theta$ and $\xi=\sin\theta\partial_r+\frac{1}{r}\cos\theta\partial_\theta$. Then we have

$$\varphi\xi=\cos\theta\partial_r-\frac{1}{r}\sin\theta\partial_\theta\,,\ g(\xi,\xi)=1\,,\ (\varphi\xi)(f)=0\,,\ \nabla\xi=0.$$

From this we find

$$grad\, f = \sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta, \quad |grad\, f| = 1,$$

from Lemma 4.1, the identity map $I : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{GB}g)$ is non harmonic. By direct computations we obtain

$$Ricci(grad f) = \nabla_{grad f} grad f = trace_g \nabla^2 grad f = 0.$$

So, thus the identity map $I : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{GB}g)$ is proper biharmonic, where

$${}^{GB}g = g + r\sin\theta(\cos\theta dr - r\sin\theta d\theta)^2.$$

Lemma 4.6. *The tension field of the identity map* $I : (M^{2m}, {}^{GB}g) \to (M^{2m}, \varphi, g)$ *is given by*

$$\widetilde{\tau}(I) = \frac{1}{2(1+f)} grad f. \tag{42}$$

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{GB}g)$ defined by (26). The tension field $\widetilde{\tau}(I)$ of the identity map $I : (M^{2m}, {}^{GB}g) \to (M^{2m}, \varphi, g)$ is given by

$$\begin{aligned}\widetilde{\tau}(I) &= trace_{{}^{GB}g} \nabla dI \\ &= \sum_{i=1}^{2m} \big(\nabla_{\widetilde{E}_i}\widetilde{E}_i - \widetilde{\nabla}_{\widetilde{E}_i}\widetilde{E}_i\big),\end{aligned}$$

from Theorem 2.2, we have

$$\begin{aligned}\widetilde{\tau}(I) &= \sum_{i=1}^{2m}\Big(\nabla_{\widetilde{E}_i}\widetilde{E}_i - \nabla_{\widetilde{E}_i}\widetilde{E}_i - \frac{\widetilde{E}_i(f)}{1+f} g(\widetilde{E}_i, \varphi\xi)\varphi\xi + \frac{1}{2} g(\widetilde{E}_i, \varphi\xi)^2 grad f\Big) \\ &= \frac{1}{2(1+f)} grad f.\end{aligned}$$

□

Theorem 4.7. *The bitension field of the identity map* $I : (M^{2m}, {}^{GB}g) \to (M^{2m}, \varphi, g)$ *is given by*

$$\begin{aligned}\widetilde{\tau}_2(I) =& -\frac{1}{2(1+f)} Ricci(grad f) - \frac{1}{2(1+f)} trace_g \nabla^2 grad f \\ &+ \frac{3}{4(1+f)^2}\nabla_{grad f} grad f + \frac{f}{2(1+f)^2}\nabla_{\varphi\xi}\nabla_{\varphi\xi} grad f \\ &+ (\frac{\Delta(f)}{2(1+f)^2} - \frac{3|grad f|^2}{4(1+f)^3}) grad f.\end{aligned} \tag{43}$$

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{GB}g)$ defined by (26). The bitension field of the identity map $I : (M^{2m}, {}^{GB}g) \to (M^{2m}, \varphi, g)$ is given by

$$\widetilde{\tau}_2(I) = -trace_{{}^{GB}g} R(\widetilde{\tau}(I), dI)dI - trace_{{}^{GB}g} (\nabla^I)^2 \widetilde{\tau}(I). \tag{44}$$

From Theorem 4.6, we have

$$\widetilde{\tau}(I) = \frac{1}{2(1+f)} grad f.$$

Using Theorem 3.1, we find that the first term of (44) is given by

$$\begin{aligned}
-trace_{{}^{GB}g} R(\widetilde{\tau}(I), dI)dI &= -\frac{1}{2(1+f)} trace_g R(grad f, dI)dI \\
&= -\frac{1}{2(1+f)} \sum_{i=1}^{2m} R(grad f, \widetilde{E}_i)\widetilde{E}_i \\
&= -\frac{1}{2(1+f)} Ricci(grad f). \qquad (45)
\end{aligned}$$

To calculate the second term of (44) we use Theorem 2.2, we find

$$\begin{aligned}
-trace_{{}^{GB}g}(\nabla^I)^2\widetilde{\tau}(I) = & -\frac{1}{2}\sum_{i=1}^{2m}\big(\nabla_{\widetilde{E}_i}\nabla_{\widetilde{E}_i}(\frac{1}{1+f}grad f) - \nabla_{\widetilde{\nabla}_{\widetilde{E}_i}\widetilde{E}_i}(\frac{1}{1+f}grad f)\big) \\
= & \frac{3}{4(1+f)^2}\nabla_{grad f} grad f + \frac{f}{2(1+f)^2}\nabla_{\varphi\xi}\nabla_{\varphi\xi} grad f \\
& + (\frac{\Delta(f)}{2(1+f)^2} - \frac{3|grad f|^2}{4(1+f)^3}) grad f \\
& - \frac{1}{2(1+f)} trace_g \nabla^2 grad f. \qquad (46)
\end{aligned}$$

Substituting (45) and (46) into (44), we find (43). □

Theorem 4.8. *The identity map $I : (M^{2m}, {}^{GB}g) \to (M^{2m}, \varphi, g)$ is proper biharmonic if and only if the function f is non constant on M, and satisfies the following*

$$\begin{aligned}
& Ricci(grad f) + trace_g \nabla^2 grad f \\
= & \frac{3}{2(1+f)}\nabla_{grad f} grad f + \frac{f}{1+f}\nabla_{\varphi\xi}\nabla_{\varphi\xi} grad f \\
& + (\frac{\Delta(f)}{1+f} - \frac{3|grad f|^2}{2(1+f)^2}) grad f. \qquad (47)
\end{aligned}$$

5. The proper biharmonicity of curve in $(M, \widetilde{g})$

Let $\gamma : J \to M$, $t \mapsto \gamma(t)$ be a differentiable curve in M, where J is an open interval of $\mathbb{R}$.

Lemma 5.1. *The curve $\gamma : J \to (M^{2m}, \widetilde{g})$ is harmonic if and only if*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = -g(\dot{\gamma}, \varphi\xi)\big(\frac{\dot{\gamma}(f)}{1+f}\varphi\xi - \frac{1}{2}g(\dot{\gamma}, \varphi\xi) grad f\big) \circ \gamma. \qquad (48)$$

Proof. The tension field of the curve γ is given by

$$
\begin{aligned}
\widetilde{\tau}(\gamma) &= \widetilde{\nabla}^{\gamma}_{\frac{d}{dt}} d\gamma(\frac{d}{dt}) - d\gamma(\nabla^{J}_{\frac{d}{dt}}\frac{d}{dt}) \\
&= \widetilde{\nabla}_{\dot\gamma}\dot\gamma \\
&= \nabla_{\dot\gamma}\dot\gamma + \frac{\dot\gamma(f)}{1+f} g(\dot\gamma,\varphi\xi)\varphi\xi\circ\gamma - \frac{1}{2} g(\dot\gamma,\varphi\xi)^2 grad\, f\circ\gamma \\
&= \nabla_{\dot\gamma}\dot\gamma + g(\dot\gamma,\varphi\xi)\big(\frac{\dot\gamma(f)}{1+f}\varphi\xi - \frac{1}{2} g(\dot\gamma,\varphi\xi) grad\, f\big)\circ\gamma. \qquad (49)
\end{aligned}
$$

Then, the curve γ is harmonic if and only if $\widetilde{\tau}(\gamma)=0$, i.e.,

$$\nabla_{\dot\gamma}\dot\gamma = -g(\dot\gamma,\varphi\xi)\big(\frac{\dot\gamma(f)}{1+f}\varphi\xi - \frac{1}{2} g(\dot\gamma,\varphi\xi) grad\, f\big)\circ\gamma. \qquad \square$$

Corollary 5.2. *If f is a constant, then γ is harmonic in $(M^{2m}, {}^{GB}g)$ if and only if γ is geodesic.*

Corollary 5.3. *If $\dot\gamma$ is orthogonal to $\varphi\xi$, then γ is harmonic in $(M^{2m}, {}^{GB}g)$ if and only if γ is geodesic.*

Now we suppose that $\dot\gamma$ is orthogonal to $\varphi\xi$, i.e., $g_{\gamma(t)}(\dot\gamma,\varphi\xi)=0$. Then

$$0 = \dot\gamma g(\dot\gamma,\varphi\xi) = g(\nabla_{\dot\gamma}\dot\gamma,\varphi\xi) + g(\dot\gamma,\nabla_{\dot\gamma}\varphi\xi)$$

from (8), we get $g(\nabla_{\dot\gamma}\dot\gamma,\varphi\xi)=0$, then there is a function $\lambda : J\to\mathbb{R}$ such that

$$\nabla_{\dot\gamma}\dot\gamma = \lambda\dot\gamma. \qquad (50)$$

We have the following theorem.

Theorem 5.4. *The curve $\gamma : J \to (M^{2m}, {}^{GB}g)$ is proper biharmonic if and only if*

$$\nabla_{\dot\gamma}\dot\gamma = \frac{2at+b}{at^2+bt+c}\dot\gamma, \qquad (51)$$

where $a,b,c\in\mathbb{R}$ and $(a,b)\neq(0,0)$ such that $at^2+bt+c\neq 0$ for all $t\in J$.

Proof. Using (49) and (50), we get $\widetilde{\tau}(\gamma)=\widetilde{\nabla}_{\dot\gamma}\dot\gamma=\lambda\dot\gamma$. We compute the bitension field $\widetilde{\tau}_2(\gamma)$ of $\gamma : J\to(M^{2m},\widetilde{g})$:

$$
\begin{aligned}
\widetilde{\tau}_2(\gamma) &= -trace\widetilde{R}(\widetilde{\tau}(I), d\gamma)d\gamma - trace(\widetilde{\nabla}^{\gamma})^2\widetilde{\tau}(\gamma) \\
&= -\widetilde{R}(\widetilde{\tau}(\gamma), d\gamma(\frac{d}{dt}))d\gamma(\frac{d}{dt}) - \big(\widetilde{\nabla}^{\gamma}_{\frac{d}{dt}}\widetilde{\nabla}^{\gamma}_{\frac{d}{dt}}\widetilde{\tau}(\gamma) - \widetilde{\nabla}^{\gamma}_{\nabla_{\frac{d}{dt}}\frac{d}{dt}}\widetilde{\tau}(\gamma)\big) \\
&= -\widetilde{R}(\lambda\dot\gamma,\dot\gamma)\dot\gamma - \widetilde{\nabla}^{\gamma}_{\frac{d}{dt}}\widetilde{\nabla}^{\gamma}_{\frac{d}{dt}}(\lambda\dot\gamma) \\
&= -\widetilde{\nabla}^{\gamma}_{\frac{d}{dt}}(\lambda'\dot\gamma + \lambda\widetilde{\nabla}_{\dot\gamma}\dot\gamma) \\
&= -\widetilde{\nabla}^{\gamma}_{\frac{d}{dt}}(\lambda'+\lambda^2)\dot\gamma \\
&= -(\lambda''+3\lambda'\lambda+\lambda^3)\dot\gamma.
\end{aligned}
$$

Now, the curve $\gamma : J \to (M^{2m}, \widetilde{g})$ is biharmonic if and only if the function λ satisfies the differential equation $\lambda'' + 3\lambda'\lambda + \lambda^3 = 0$, i.e.,

$$\lambda(t) = \frac{2at + b}{at^2 + bt + c}, \tag{52}$$

where $a, b, c \in \mathbb{R}$ such that $at^2 + bt + c \neq 0$ for all $t \in J$. Furthermore, if the function λ is non-null on J, i.e., $(a, b) \neq (0, 0)$, then the curve γ is proper biharmonic in $(M^{2m}, {}^{GB}g)$. □

Example 5.5. Let $M = \mathbb{R}^* \times \mathbb{R}^*$ be a B-manifold, equipped with the almost paracomplex structure φ and the B-metric g in the Cartesian coordinates defined by

$$g = x^2 dx^2 + y^2 dy^2,$$
$$\varphi\partial_x = \frac{x}{y}\partial_y, \ \varphi\partial_y = \frac{y}{x}\partial_x.$$

Let $\xi = \frac{1}{\sqrt{2}x}\partial_x - \frac{1}{\sqrt{2}y}\partial_y$ and $f(x, y) = x^2 + y^2$. Then we have

$$\varphi\xi = -\frac{1}{\sqrt{2}x}\partial_x + \frac{1}{\sqrt{2}y}\partial_y = \xi\,,\ g(\xi, \xi) = 1\,,\ (\varphi\xi)(f) = 0\,,\ \nabla\xi = 0.$$

Consider the curve γ on (M, φ, g) defined by

$$\gamma(t) = (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}).$$

Then we have

$$\begin{aligned}
&\dot\gamma = \frac{1}{\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_y,\\
&g_{\gamma(t)}(\dot\gamma, \varphi\xi) = 0,\\
&\nabla_{\dot\gamma}\dot\gamma = \frac{1}{\sqrt{2}t}\partial_x + \frac{1}{\sqrt{2}t}\partial_y = \frac{1}{t}\dot\gamma.
\end{aligned}$$

So, thus the curve $\gamma : J \to (M^{2m}, {}^{GB}g)$ is proper biharmonic, where

$${}^{GB}g = g + \frac{x^2 + y^2}{2}(xdx - ydy)^2.$$

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References

[1] M. Altunbas, R. Simsek, and A. Gezer, *A study concerning Berger type deformed Sasaki metric on the tangent bundle*, J. Math. Phys. Anal. Geom. **15** (2019), no. 4, 435–447. https://doi.org/10.15407/mag15.04.435

[2] M. Altunbas, R. Simsek, and A. Gezer, *Some harmonic problems on the tangent bundle with a Berger-type deformed Sasaki metric*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **82** (2020), no. 2, 37–42.

[3] P. Baird, A. Fardoun, and S. Ouakkas, *Conformal and semi-conformal biharmonic maps*, Ann. Global Anal. Geom. **34** (2008), no. 4, 403–414. https://doi.org/10.1007/s10455-008-9118-8

[4] P. Baird and D. Kamissoko, *On constructing biharmonic maps and metrics*, Ann. Global Anal. Geom. **23** (2003), no. 1, 65–75. https://doi.org/10.1023/A:1021213930520

[5] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, 29, The Clarendon Press, Oxford University Press, Oxford, 2003. https://doi.org/10.1093/acprof:oso/9780198503620.001.0001

[6] A. Balmuş, *Biharmonic properties and conformal changes*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **50** (2004), no. 2, 361–372.

[7] A. Benkartab and A. M. Cherif, *New methods of construction for biharmonic maps*, Kyungpook Math. J. **59** (2019), no. 1, 135–147. https://doi.org/10.5666/KMJ.2019.59.1.135

[8] A. Benkartab and A. M. Cherif, *Deformations of metrics and biharmonic maps*, Commun. Math. **28** (2020), no. 3, 263–275. https://doi.org/10.2478/cm-2020-0022

[9] G. Chen, Y. Liu, and J. Wei, *Nondegeneracy of harmonic maps from* $\mathbb{R}^2$ *to* $\mathbb{S}^2$, Discrete Contin. Dyn. Syst. **40** (2020), no. 6, 3215–3233. https://doi.org/10.3934/dcds.2019228

[10] M. Crâşmăreanu, *Killing potentials*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **45** (1999), no. 1, 169–176.

[11] N. E. Djaa, F. Latti, and A. Zagane, *Proper biharmonic maps on tangent bundle*, Commun. Math. **31** (2023), no. 1, 137–154. https://doi.org/10.46298/cm.10305

[12] N. E. Djaa and A. Zagane, *Harmonicity of Mus-gradient metric*, Int. J. Maps Math. **5** (2022), no. 1, 61–77.

[13] N. E. Djaa and A. Zagane, *Some results on the geometry of a non-conformal deformation of a metric*, Commun. Korean Math. Soc. **37** (2022), no. 3, 865–879. https://doi.org/10.4134/CKMS.c210207

[14] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), no. 5, 385–524. https://doi.org/10.1112/blms/20.5.385

[15] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. https://doi.org/10.2307/2373037

[16] J. J. Konderak, *On harmonic vector fields*, Publ. Mat. **36** (1992), no. 1, 217–228. https://doi.org/10.5565/PUBLMAT_36192_17

[17] A. A. Salimov, M. İşcan, and F. Etayo, *Paraholomorphic B-manifold and its properties*, Topology Appl. **154** (2007), no. 4, 925–933. https://doi.org/10.1016/j.topol.2006.10.003

[18] A. Yampolsky, *On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kählerian manifold*, J. Math. Phys. Anal. Geom. **8** (2012), no. 2, 177–189.

[19] A. Zagane, *A study of harmonic sections of tangent bundles with vertically rescaled Berger-type deformed Sasaki metric*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **47** (2021), no. 2, 270–285. https://doi.org/10.30546/2409-4994.47.2.270

[20] A. Zagane and N. E. Djaa, *Notes about a harmonicity on the tangent bundle with vertical rescaled metric*, Int. Electron. J. Geom. **15** (2022), no. 1, 83–95. https://doi.org/10.36890/iejg.1033998

[21] A. Zagane and A. Gezer, *Vertical rescaled Cheeger-Gromoll metric and harmonicity on the cotangent bundle*, Adv. Stud. Euro-Tbil. Math. J. **15** (2022), no. 3, 11–29. https://doi.org/10.32513/asetmj/19322008221

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