Commun. Korean Math. Soc. **38** (2023), No. 4, pp. 1271–1280
https://doi.org/10.4134/CKMS.c230032
pISSN: 1225-1763 / eISSN: 2234-3024

ON WEAKLY CYCLIC GENERALIZED B-SYMMETRIC MANIFOLDS

Mohabbat Ali, Aziz Ullah Khan, Quddus Khan, and Mohd Vasiulla

Abstract. The object of the present paper is to introduce a type of non-flat Riemannian manifold, called a weakly cyclic generalized B-symmetric manifold $(WCGBS)_n$. We obtain a sufficient condition for a weakly cyclic generalized B-symmetric manifold to be a generalized quasi Einstein manifold. Next we consider conformally flat weakly cyclic generalized B-symmetric manifolds. Then we study Einstein $(WCGBS)_n$ $(n > 2)$. Finally, it is shown that the semi-symmetry and Weyl semi-symmetry are equivalent in such a manifold.

1. Introduction

Let the manifold (M^n, g) ($\dim M = n \geq 3$) be connected and semi-Riemannian, and the endowed metric g of signature $(s, n-s)$, $0 \leq s \leq n$. If $s = n$ or 0 (resp., $s = n-1$ or 1), then (M^n, g) is a Riemannian or semi-Riemannian (resp., Lorentzian) manifold. A Riemannian manifold has mainly three notions of curvature, namely, the Riemann-Christoffel curvature tensor K (simply called curvature tensor), the Ricci tensor Ric and the scalar curvature r. Let ∇ be the Levi-Civita connection on the manifold (M^n, g), which is the unique torsion free metric connection.

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [3], who, in particular, obtained a classification of those spaces. A Riemannian manifold is called locally symmetric [3] if $\nabla K = 0$, where K is the Riemannian curvature tensor of (M^n, g). The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

A Riemannian manifold (M^n, g) $(n \geq 3)$ is called semi-symmetric if

$$K.\, K = 0, \tag{1}$$

Received February 11, 2023; Accepted April 26, 2023.
2020 *Mathematics Subject Classification.* Primary 53C15, 53C25.
Key words and phrases. Generalized B-tensor, generalized quasi-Einstein manifold, Weyl semi-symmetry, conformally-flat manifold, generalized quasi-constant curvature, Einstein manifold.

holds on M^n. It is well known that the class of semi-symmetric manifolds includes the set of locally symmetric manifolds ($\nabla K = 0$) as a proper subset. Semi-symmetric Riemannian manifolds were studied by Cartan [3] and Lichnerowich [8]. A fundamental study on such Riemannian manifolds was made by Szabó [12] and Kowalski [7].

A Riemannian manifold (M^n, g) $(n \geq 3)$ is called Ricci semi-symmetric [12] if

$$K.Ric = 0, \tag{2}$$

where Ric is the Ricci tensor. The class of Ricci semi-symmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla Ric = 0$) as a proper subset.

A Riemannian manifold (M^n, g) $(n \geq 3)$ is called Weyl semi-symmetric [7] if

$$K.C = 0, \tag{3}$$

where C is the Weyl conformal curvature tensor and is defined [1] as

$$\begin{aligned} C(U,X,Y) = K(U,X,Y) &- \frac{1}{n-2}\Big[g(X,Y)R(U) - g(U,Y)R(X) \\ &+ Ric(X,Y)U - Ric(U,Y)X\Big] \\ &+ \frac{r}{(n-1)(n-2)}\Big[g(X,Y)U - g(U,Y)X\Big], \end{aligned} \tag{4}$$

where R is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor Ric, that is,

$$g(R(U), X) = Ric(U, X).$$

The notion of weakly symmetric manifolds introduced and studied by Tamássy and Binh [13]. The existence of weakly symmetric manifolds was proved by Prvanović [10] and a concrete example is given by De and Bandyopadhyay [2].

A Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric [14] if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and satisfies the condition:

$$(\nabla_U Ric)(Y, V) = A(U)Ric(Y, V) + D(Y)Ric(V, U) + E(V)Ric(U, Y), \tag{5}$$

where A, D and E are 1-forms which are not simultaneously zero and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such an n-dimensional Riemannian manifold is denoted by $(WRS)_n$.

Generalizing the notion of quasi-Einstein manifold, in [5], De and Ghosh introduced the notion of generalized quasi-Einstein manifolds and studied its geometrical properties with the existence of such notion by several nontrivial examples. A Riemannian manifold is said to be generalized quasi-Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and satisfies the condition:

$$Ric(Y, V) = \alpha g(Y, V) + \beta\xi(Y)\xi(V) + \gamma\eta(Y)\eta(V), \tag{6}$$

where α, β, γ are smooth functions of which $\beta \neq 0$, $\gamma \neq 0$ and ξ, η are two non-zero 1-forms such that

$$g(Y,P) = \xi(Y), \quad g(Y,Q) = \eta(Y)$$

for every vector field Y. Such as an n-dimensional manifold is denoted by $G(QE)_n$.

In the subsequent paper [11] authors introduced a $(0,2)$ symmetric tensor B as follows:

$$B(Y,V) = aRic(Y,V) + brg(Y,V),$$

where a and b are non-zero arbitrary scalar functions and r is the scalar curvature.

In this paper, we define a generalized B-tensor as follows:

A $(0,2)$-type symmetric tensor $\mathcal{B}$ is called a generalized B-tensor if

$$\mathcal{B}(Y,V) = aRic(Y,V) + brg(Y,V) + \psi\omega(Y)\omega(V), \tag{7}$$

where ψ is an arbitrary scalar function and ω is a 1-form defined by

$$\omega(Y) = g(Y,\rho)$$

for a unit vector field ρ. The scalar $\tilde{\mathcal{B}}$ is obtained by contracting (7) over Y and V as follows:

$$\tilde{\mathcal{B}} = (a+nb)r + \psi. \tag{8}$$

Pseudo B-symmetric manifolds, weakly cyclic Z-symmetric manifolds and weakly Z-symmetric manifolds have been studied in ([11], [6] and [9]), respectively.

Inspired by these works, we introduce a new type of Riemannian manifold called weakly cyclic generalized B-symmetric manifolds. A manifold is called weakly cyclic generalized B-symmetric if the generalized B tensor of type $(0,2)$ is non-zero and satisfies the condition:

$$\begin{aligned} &(\nabla_U\mathcal{B})(Y,V) + (\nabla_Y\mathcal{B})(V,U) + (\nabla_V\mathcal{B})(U,Y) \\ &= A(U)\mathcal{B}(Y,V) + D(Y)\mathcal{B}(V,U) + E(V)\mathcal{B}(U,Y), \end{aligned} \tag{9}$$

where A, D and E are 1-forms simultaneously non-zero. We denoted such a manifold by $(WCGBS)_n$.

A conformally-flat Riemannian manifold (M^n, g) $(n > 3)$ is said to be of generalized quasi constant curvature [5] if its curvature tensor $\tilde{K}$ of type $(0,4)$ satisfies the condition:

$$\begin{aligned} \tilde{K}(U,X,Y,V) = {} & p[g(X,Y)g(U,V) - g(U,Y)g(X,V)] \\ & + q[g(U,V)H(X)H(Y) + g(X,Y)H(U)H(V) \\ & - g(U,Y)H(X)H(V) - g(X,V)H(U)H(Y)] \\ & + s[g(U,V)H(X)H(Y) - g(X,V)H(U)H(Y) \\ & + g(X,Y)H(U)H(V) - g(U,Y)H(X)H(V)], \end{aligned} \tag{10}$$

where

$$\tilde{K}(U,X,Y,V) = g(K(U,X,Y),V),$$

and p, q, s are scalar functions of which $q \neq 0$ and H is a non-zero 1-form defined by

$$g(U,\mu) = H(U)$$

for all U, μ being a unit vector field. In such a case p, q and s are called associated scalars, H is called the associated 1-form and μ is called the generator of the manifold.

In 1956, S. S. Chern [4] studied a type of Riemannian manifold whose curvature tensor $\tilde{K}$ of type $(0,4)$ satisfies the condition:

$$\tilde{K}(U,X,Y,V) = I(X,Y)I(U,V) - I(U,Y)I(X,V), \tag{11}$$

where I is a symmetric tensor of type $(0,2)$. Such an n-dimensional manifold was called a special manifold with the associated symmetric tensor I and was denoted by $\psi(I)_n$. Such a manifold is important for the following reasons:

Firstly, for possessing some important properties relating to curvature and characteristic classes and secondly, for containing a manifold of quasi-constant curvature as a subclass.

The object of the present paper is to study $(WCGBS)_n$. The paper is presented as follows:

After introduction in Section 2, we obtain a sufficient condition for a $(WCGBS)_n$ to be a generalized quasi-Einstein manifold $G(QE)_n$. In Section 3, we have proved that the conformally flat $(WCGBS)_n$ is a manifold of generalized quasi-constant curvature. Also, it is shown that a $(WCGBS)_n$, which is a generalized quasi-constant curvature, is a manifold of $\psi(I)_n$. Section 4, deals with Einstein $(WCGBS)_n$ $(n > 2)$. In the last section, we have proved that in a $(WCGBS)_n$ $(n > 2)$, semi-symmetry and Weyl semi-symmetry are equivalent.

2. Sufficient condition for a $(WCGBS)_n$ to be a generalized quasi Einstein manifold

In this section, we study a weakly cyclic generalized B-symmetric manifold $(WCGBS)_n$. Here, a sufficient condition is discussed for a $(WCGBS)_n$ to be a generalized quasi-Einstein manifold $G(QE)_n$.

Interchanging Y and V in (9), we get

$$\begin{aligned}(\nabla_U\mathcal{B})(V,Y) + (\nabla_V\mathcal{B})(Y,U) + (\nabla_Y\mathcal{B})(U,V)\\ = A(U)\mathcal{B}(V,Y) + D(V)\mathcal{B}(Y,U) + E(Y)\mathcal{B}(U,V).\end{aligned} \tag{12}$$

Subtracting (12) from (9), we have

$$[D(Y) - E(Y)]\mathcal{B}(V,U) = [D(V) - E(V)]\mathcal{B}(U,Y). \tag{13}$$

If possible, let

$$J(U) = g(U,\mu) = D(U) - E(U) \tag{14}$$

for all vector field U, where μ is a unit vector field associated with 1-form J such that $J(\rho) = g(\rho, \mu) = 0$. Then the relation (13) becomes

$$J(Y)\mathcal{B}(V,U) = J(V)\mathcal{B}(U,Y). \tag{15}$$

Contracting (15) over U and V and using (8), we get

$$J(R(Y)) = \frac{J(Y)}{a}\Big[\{a + (n-1)b\}r + \psi\Big]. \tag{16}$$

Setting $V = \mu$ in (15), we get

$$J(Y)\mathcal{B}(\mu,U) = \mathcal{B}(U,Y), \tag{17}$$

which in view of (7) and (16), the relation (17) reduces to

$$Ric(U,Y) = \frac{b}{a}r\, g(U,Y) + \Big\{\frac{a + nbr + \psi}{a}\Big\}J(U)J(Y) - \frac{\psi}{a}\omega(U)\omega(Y). \tag{18}$$

The relation (18) takes the form

$$Ric(U,Y) = \alpha g(U,Y) + \beta J(U)J(Y) + \gamma\omega(U)\omega(Y),$$

where $\alpha = \frac{b}{a}r$, $\beta = \Big\{\frac{a+nbr+\psi}{a}\Big\}$ and $\gamma = -\frac{\psi}{a}$. Hence the manifold under consideration is a generalized quasi-Einstein manifold.

Thus, we have the following result:

Theorem 2.1. *If the vector field μ defined by* (14) *is a unit vector field, then $(WCGBS)_n$ is a generalized quasi-Einstein manifold .*

3. Conformally flat $(WCGBS)_n$ $(n > 3)$

In this section, we consider the conformally flat $(WCGBS)_n (n > 3)$. Since, in a conformally flat Riemannian manifold, the curvature tensor $\tilde{K}$ of type $(0,4)$ is of the form

$$\begin{aligned}\tilde{K}(U,X,Y,V) = {} & \frac{1}{(n-2)}\Big[Ric(X,Y)g(U,V) - Ric(U,Y)g(X,V)\\ & + Ric(U,V)g(X,Y) - Ric(X,V)g(U,Y)\Big]\\ & - \frac{r}{(n-1)(n-2)}\Big[g(X,Y)g(U,V) - g(U,Y)g(X,V)\Big].\end{aligned} \tag{19}$$

Using (18) in (19) and simplifying, we get

$$\begin{aligned}\tilde{K}(U,X,Y,V) = {} & p\Big[g(X,Y)g(U,V) - g(U,Y)g(X,V)\Big]\\ & + q\Big[g(U,V)J(X)J(Y) - g(X,V)J(U)J(Y)\\ & + g(X,Y)J(U)J(V) - g(U,Y)J(X)J(V)\Big]\\ & + s\Big[g(U,V)\omega(X)\omega(Y) - g(X,V)\omega(U)\omega(Y)\\ & + g(X,Y)\omega(U)\omega(V) - g(U,Y)\omega(X)\omega(V)\Big],\end{aligned}$$

where $p = -\frac{1}{(n-2)}\left[\frac{2b}{a}r + \frac{r}{(n-1)}\right]$, $q = \frac{(a+nbr+\psi)}{a(n-2)}$ and $s = -\frac{\psi}{a(n-2)}$, which is a form of generalized quasi-constant curvature.

Hence we can state the following:

Theorem 3.1. *If the vector field μ defined by* (14) *is a unit vector field, then the conformally flat $(WCGBS)_n$ is a manifold of generalized quasi-constant curvature.*

Let us consider a $(0, 2)$ type symmetric tensor field defined by

$$I(U, X) = ag(U, X) + bN(U)N(X) + c\omega(U)\omega(X), \tag{20}$$

where N and ω are 1-forms defined by $N(U) = g(U, \xi)$ and $\omega(U) = g(U, \rho)$ such that $N(\rho) = \omega(\xi) = g(\rho, \xi) = 0$ then, from (11) the Riemannian curvature tensor for $\psi(I)_n$ leads to

$$\begin{aligned}\tilde{K}(U, X, Y, V) = p\Big[&g(X, Y)g(U, V) - g(U, Y)g(X, V)\Big] \\ &+ q\Big[g(X, Y)N(U)N(V) + g(U, V)N(X)N(Y) \\ &- g(U, Y)N(X)N(V) - g(X, V)N(U)N(Y)\Big] \\ &+ s\Big[g(X, Y)\omega(U)\omega(V) + g(U, V)\omega(X)\omega(Y) \\ &- g(U, Y)\omega(X)\omega(V) - g(X, V)\omega(U)\omega(Y)\Big],\end{aligned} \tag{21}$$

where $p = a^2$, $q = ab$ and $s = ac$ are scalars, which represents a form of generalized quasi-constant curvature, provided that N and ω commute to each other, that is $N(U)\omega(X) = N(X)\omega(U)$.

Hence we can state the following:

Theorem 3.2. *If $(WCGBS)_n$ is a generalized quasi-constant curvature and the two* 1*-forms N, ω satisfy the law of commutative, then it is a manifold of $\psi(I)_n$.*

4. Einstein $(WCGBS)_n$ $(n > 2)$

This section deals with Einstein $(WCGBS)_n$ $(n > 2)$. The Ricci tensor of an Einstein manifold is given by

$$Ric(Y, V) = \frac{r}{n}g(Y, V), \tag{22}$$

which implies

$$(\nabla_U Ric)(Y, V) = 0. \tag{23}$$

Let ρ be a parallel unit vector field such that

$$A(\rho) = D(\rho) = E(\rho) = 0.$$

Then we have

$$(\nabla_U \omega)(Y, V) = 0 \quad \text{and} \quad \nabla_U \rho = 0. \tag{24}$$

Now, taking covariant derivative of (7) along U, we get

$$(\nabla_U\mathcal{B})(Y,V) = a(\nabla_U Ric)(Y,V) + bU(r)g(Y,V) + U(\psi)\omega(Y)\omega(V) + \psi\Big[(\nabla_U\omega)(Y)\omega(V) + \omega(Y)(\nabla_U\omega)(V)\Big]. \tag{25}$$

By virtue of (24), the foregoing equation takes the form

$$(\nabla_U\mathcal{B})(Y,V) = U(\psi)\omega(Y)\omega(V). \tag{26}$$

Using (7) and (26), then the relation (9) reduces to

$$\begin{aligned} &U(\psi)\omega(Y)\omega(V) + Y(\psi)\omega(V)\omega(U) + V(\psi)\omega(U)\omega(Y) \\ &= \Big(\frac{a}{n}r + br\Big)\Big[A(U)g(Y,V) + D(Y)g(V,U) + E(V)g(U,Y)\Big] \\ &\quad + \psi\Big[A(U)\omega(Y)\omega(V) + D(Y)\omega(U)\omega(V) + E(V)\omega(U)\omega(Y)\Big]. \end{aligned} \tag{27}$$

Contracting equation (27) over Y and V, we get

$$U(\psi)\Big[2\rho(\psi)+1\Big] = \Big(\frac{a}{n}r + br\Big)\Big[nA(U) + D(U) + E(U)\Big] + \psi A(U). \tag{28}$$

Similarly, we can find by contraction over V and U, and U and Y

$$U(\psi)\Big[2\rho(\psi)+1\Big] = \Big(\frac{a}{n}r + br\Big)\Big[A(U) + nD(U) + E(U)\Big] + \psi D(U) \tag{29}$$

and

$$U(\psi)\Big[2\rho(\psi)+1\Big] = \Big(\frac{a}{n}r + br\Big)\Big[A(U) + D(U) + nE(U)\Big] + \psi E(U). \tag{30}$$

Adding equations (28), (29) and (30), we have

$$3U(\psi)\Big[2\rho(\psi)+1\Big] = \Big[(n+2)\Big(\frac{a}{n}r + br\Big) + \psi\Big]\Big[A(U) + D(U) + E(U)\Big]. \tag{31}$$

Let the scalar function ψ be constant. Then (31) reduces to

$$\Big[(n+2)\Big(\frac{a}{n}r + br\Big) + \psi\Big]\Big[A(U) + D(U) + E(U)\Big] = 0,$$

which gives

$$\Big[A(U) + D(U) + E(U)\Big] = 0 \quad \text{if} \quad \Big[(n+2)\Big(\frac{a}{n}r + br\Big) + \psi\Big] \neq 0.$$

Hence, we can state the following:

Theorem 4.1. *If in an Einstein $(WCGBS)_n$, the vector field ρ is a unit parallel vector field, then the sum of 1-forms vanishes if and only if the scalar function ψ is constant and $\psi \neq -(n+2)\Big(\frac{a}{n}r + br\Big)$.*

Replacing Y and V by U in (9), we have

$$(\nabla_U\mathcal{B})(U,U) = \frac{1}{3}\Big[A(U) + D(U) + E(U)\Big]\mathcal{B}(U,U). \tag{32}$$

In view of Theorem 4.1, the above equation gives

$$(\nabla_U\mathcal{B})(U,U) = 0, \tag{33}$$

which implies that the generalized B-tensor is covariantly constant in the direction of U.

Hence, we can state the following:

Theorem 4.2. *If in an Einstein $(WCGBS)_n$, the vector field ρ is a unit parallel vector field, then the generalized B tensor is covariantly constant in the direction of U if and only if ψ is constant, provided that $\psi \neq -(n+2)\Big(\frac{a}{n}r+br\Big)$.*

5. Equivalence of semi-symmetry and Weyl semi-symmetry in a $(WCGBS)_n$ $(n > 2)$

Now, we consider that the unit vector field μ defined in (14) is parallel in $(WCGBS)_n$ $(n > 2)$. The condition of Ricci semi-symmetry yields

$$(K(U,X)Ric)(Y,V) = -Ric(K(U,X)Y,V) - Ric(Y,K(U,X)V). \tag{34}$$

Using (18) in the equation (34), we get

$$\begin{aligned}&(K(U,X)Ric)(Y,V)\\&= \frac{\psi}{a}\Big[\omega(K(U,X)Y)\omega(V) + \omega(Y)\omega(K(U,X)V)\Big]\\&\quad - \Big\{a + nbr + \psi\Big\}\Big[J(K(U,X)Y)J(V) + J(Y)J(K(U,X)V)\Big].\end{aligned}$$

Let us suppose, if possible $K \cdot Ric = 0$, then we have

$$\begin{aligned}&\frac{\psi}{a}\Big[\omega(K(U,X)Y)\omega(V) + \omega(Y)\omega(K(U,X)V)\Big]\\&- \Big\{a + nbr + \psi\Big\}\Big[J(K(U,X)Y)J(V) + J(Y)J(K(U,X)V)\Big] = 0.\end{aligned} \tag{35}$$

Setting $U = \mu$ in (35), we get

$$\Big\{a + nbr + \psi\Big\}J(K(U,X)V) = 0,$$

which implies

$$\text{Either} \quad J(K(U,X)V) = 0 \quad \text{or} \quad a + nbr + \psi = 0.$$

Let if possible, $a + nbr + \psi \neq 0$, then we have

$$J(K(U,X)V) = 0.$$

Hence, we can conclude the following:

Theorem 5.1. *$(WCGBS)_n$ is a semi-symmetric manifold if and only if $J(K(U,X)V) = 0$, provided $a + nbr + \psi \neq 0$.*

Now, we consider that μ is a parallel unit vector field then we have $\nabla_U\mu = 0$. Applying Ricci identity we have $J(K(U,X)V) = 0$. From Theorem 5.1, we have $(WCGBS)_n$ is a Ricci semi-symmetric manifold, that is, $K.Ric = 0$, provided that $a + nbr + \psi \neq 0$.

Now the Weyl conformal curvature tensor is given by

$$C = K - \frac{1}{(n-2)} g \wedge Ric + \frac{1}{(n-1)(n-2)} G, \tag{36}$$

where $\wedge$ denotes the Kulkarni-Nomizu product defined by

$$\begin{aligned} g \wedge Ric(U,X)Y = {} & Ric(X,Y)U - Ric(U,Y)X \\ & + g(X,Y)R(U) - g(U,Y)R(X), \end{aligned} \tag{37}$$

and

$$G(U,X,Y) = g(X,Y)U - g(U,Y)X,$$

which implies

$$K.C = K.K \quad (\text{since } K.Ric = 0).$$

Thus the following theorem holds:

Theorem 5.2. *In $(WCGBS)_n$ $(n > 2)$, semi-symmetry and Weyl semi-symmetry are equivalent, provided $a + nbr + \psi \neq 0$.*

Acknowledgments. We gratefully acknowledge the constructive comments from the editor and the anonymous referees.

References

[1] M. Ali and M. Vasiulla, *Almost pseudo Ricci symmetric manifold admitting Schouten tensor*, J. Dyn. Syst. Geom. Theor. 19 (2021), no. 2, 217–225. https://doi.org/10.1080/1726037X.2021.2020422

[2] S. Bandyopadhyay and U. C. De, *On weakly symmetric spaces*, Acta Math. Hungar. **87** (2000), no. 3, 205–212. https://doi.org/10.1023/A:1006759519940

[3] E. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264.

[4] S. S. Chern, *On curvature and characteristic classes of a Riemann manifold*, Abh. Math. Sem. Univ. Hamburg **20** (1955), 117–126. https://doi.org/10.1007/BF02960745

[5] U. C. De and G. C. Ghosh, *On generalized quasi Einstein manifolds*, Kyungpook Math. J. **44** (2004), no. 4, 607–615.

[6] U. C. De, C. A. Mantica, and Y. J. Suh, *On weakly cyclic Z symmetric manifolds*, Acta Math. Hungar. **146** (2015), no. 1, 153–167. https://doi.org/10.1007/s10474-014-0462-9

[7] O. Kowalski, *An explicit classification of* 3*-dimensional Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$, Czechoslovak Math. J. **46(121)** (1996), no. 3, 427–474.

[8] A. Lichnerowicz, *Courbure, nombres de Betti, et espaces symétriques*, in Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, 216–223, Amer. Math. Soc., Providence, RI, 1952.

[9] C. A. Mantica and L. G. Molinari, *Weakly Z-symmetric manifolds*, Acta Math. Hungar. **135** (2012), no. 1-2, 80–96. https://doi.org/10.1007/s10474-011-0166-3

[10] M. S. Prvanović, *On weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen **46** (1995), no. 1-2, 19–25. https://doi.org/10.5486/pmd.1995.1476

[11] Y. J. Suh, C. A. Mantica, U. C. De, and P. Pal, *Pseudo B-symmetric manifolds*, Int. J. Geom. Methods Mod. Phys. **14** (2017), no. 9, 1750119, 14 pp. https://doi.org/10.1142/S0219887817501195

[12] Z. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$. *I. The local version*, J. Differential Geometry **17** (1982), no. 4, 531–582 (1983). http://projecteuclid.org/euclid.jdg/1214437486

[13] L. Tamássy and T. Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, in Differential geometry and its applications (Eger, 1989), 663–670, Colloq. Math. Soc. János Bolyai, 56, North-Holland, Amsterdam, 1992.

[14] L. Tamássy and T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N.S.) **53** (1993), Commemoration Volume I, 140–148.

Mohabbat Ali
Department of Applied Sciences & Humanities
Jamia Millia Islamia
New Delhi 110025, India
Email address: mohabbat188923@st.jmi.ac.in

Aziz Ullah Khan
Department of Applied Sciences & Humanities
Jamia Millia Islamia
New Delhi 110025, India
Email address: aukhan@jmi.ac.in

Quddus Khan
Department of Applied Sciences & Humanities
Jamia Millia Islamia
New Delhi 110025, India
Email address: qkhan@jmi.ac.in

Mohd Vasiulla
Department of Applied Sciences & Humanities
Jamia Millia Islamia
New Delhi 110025, India
Email address: vsmlk45@gmail.com