Commun. Korean Math. Soc. **38** (2023), No. 4, pp. 1233–1247
https://doi.org/10.4134/CKMS.c230006
pISSN: 1225-1763 / eISSN: 2234-3024

COMPLETE LIFTS OF A SEMI-SYMMETRIC NON-METRIC CONNECTION FROM A RIEMANNIAN MANIFOLD TO ITS TANGENT BUNDLES

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Abstract. The aim of the present paper is to study complete lifts of a semi-symmetric non-metric connection from a Riemannian manifold to its tangent bundles. Some curvature properties of a Riemannian manifold to its tangent bundles with respect to such a connection have been investigated.

1. Introduction

Investigating lifts in connections and geometrical structures enables us to examine the manifold $\mathcal{M}$ on the tangent bundle $\mathcal{TM}$. Altunbas et al. ([3, 4]) studied lifts of metallic structures on tangent bundles of LP-Sasakian manifolds and established conditions for their parallelity. Lifts of various connections and geometric structures from a manifold to its tangent bundles have been studied by Akpinar [2], Das and Khan [10], Kazan and Karadag [18], Khan ([20,25,26]), Peyghan et al. [29]. For more contemporary research on lifts of connections, partial differential equations and geometric structures, see ([6, 11, 13–15, 22–24, 28, 33]) and a number of other references.

Semi-symmetric connection on a differentiable manifold was first proposed by Friedmann and Schouten [16] in 1924. If the torsion tensor T of a linear connection $\widetilde{\widetilde{\nabla}}$ on a differentiable manifold $\mathcal{M}$ fulfills

$$T(\mathfrak{X}_1, \mathfrak{X}_2) = \mathfrak{u}(\mathfrak{X}_2)\mathfrak{X}_1 - \mathfrak{u}(\mathfrak{X}_1)\mathfrak{X}_2, \tag{1}$$

where $\mathfrak{u}$ is a 1-form, for all vector fields $\mathfrak{X}_1 \in \chi(\mathcal{M})$, $\chi(\mathcal{M})$ is the set of all differentiable vector fields on $\mathcal{M}$, then such a connection is named semi-symmetric connection.

Received January 13, 2023; Accepted May 9, 2023.
2020 *Mathematics Subject Classification.* Primary 53C05, 53C25, 58A30.
Key words and phrases. Complete lift, tangent bundle, semi-symmetric non-metric connection, curvature tensor, Weyl curvature tensor, mathematical operators, Ricci tensor, Ricci-semisymmetric, partial differential equations.

Hayden [17] proposed semi-symmetric metric connections on a Riemannian manifold $(\mathcal{M}, g)$. A semi-symmetric connection $\widetilde{\nabla}$ is said to be

- a semi-symmetric metric connection if $\widetilde{\nabla} g = 0$.
- a semi-symmetric non metric connection (briefly, SSNMC) if $\widetilde{\nabla} g \neq 0$.

Singh and Pandey [31], Ozen et al. [39], Zhao et al. [40, 41], Velimirović et al. ([34,35]) and many others contributed to advancement of the study of semi-symmetric metric connection. After a long gap the study of a semi-symmetric connection $\widetilde{\nabla}$ satisfying

$$\widetilde{\nabla} g \neq 0 \tag{2}$$

was initiated by Prvanović [30] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [5].

Agashe and Chafle [1], De and Biswas [7], Liang [27], Smaranda and Andonie [32], Chaki [8], Yano et al. [36,37] and many others contributed to advancement of the study of SSNMC.

De et al. [12] introduced a linear connection $\bar{\nabla}$ given by

$$\bar{\nabla}_{\mathfrak{X}_1} \mathfrak{X}_2 = \nabla_{\mathfrak{X}_1} \mathfrak{X}_2 + a\omega(\mathfrak{X}_1)\mathfrak{X}_2 + b\omega(\mathfrak{X}_2)\mathfrak{X}_1, \tag{3}$$

$$\bar{T}(\mathfrak{X}_1, \mathfrak{X}_2) = (b-a)\omega(\mathfrak{X}_2)\mathfrak{X}_1 - (b-a)\omega(\mathfrak{X}_1)\mathfrak{X}_2 = \pi(\mathfrak{X}_2)\mathfrak{X}_1 - \pi(\mathfrak{X}_1)\mathfrak{X}_2, \tag{4}$$

$$\omega(\mathfrak{X}_1) = g(\mathfrak{X}_1, \rho), \tag{5}$$

where $a, b \neq 0$ (real numbers), $\mathfrak{X}_1 \in \chi(\mathcal{M})$ and $\bar{T}$ is the torsion tensor with respect to $\bar{\nabla}$ and $\pi(\mathfrak{X}_1) = (b-a)\omega(\mathfrak{X}_1)$ and ρ is a vector field.

Thus $\bar{\nabla}$ is a semi-symmetric connection.

In addition

$$\begin{aligned}\left(\bar{\nabla}_{\mathfrak{X}_1} g\right)(\mathfrak{X}_2, \mathfrak{X}_3) &= -2a\omega(\mathfrak{X}_1)g(\mathfrak{X}_2, \mathfrak{X}_3) - b\omega(\mathfrak{X}_2)g(\mathfrak{X}_1, \mathfrak{X}_3) - b\omega(\mathfrak{X}_3)g(\mathfrak{X}_1, \mathfrak{X}_2)\\ &\neq 0.\end{aligned}$$

Hence $\bar{\nabla}$ given by (3) is an SSNMC.

In the present paper, we investigate complete lifts of an SSNMC from a Riemannian manifold $\mathcal{M}$ to its tangent bundles and deduce some curvature tensors on $\mathcal{TM}$. The aim of this study is stated as follows:

- We have studied complete lifts of an SSNMC from $\mathcal{M}$ to $\mathcal{TM}$.
- We have developed the relationship of the curvature tensors between ∇ and $\bar{\nabla}$ from $\mathcal{M}$ to $\mathcal{TM}$.
- Weyl projective curvature tensor on $\mathcal{M}$ to $\mathcal{TM}$ endowed with an SS-NMC is studied.
- Some properties of Ricci-semisymmetric Riemannian manifolds endowed with an SSNMC on $\mathcal{TM}$ has been done.
- Applications of an SSNMC from $\mathcal{M}$ to $\mathcal{TM}$ has been shown.

2. Preliminaries

Let $\mathcal{TM}$ be the tangent bundle of a manifold $\mathcal{M}$ and let the function, a 1-form, a vector field and a tensor field (1,1) type be symbolized as f, η, $\mathfrak{X}_1$ and ϕ and ∇, respectively. The complete and vertical lifts of f, η, $\mathfrak{X}_1$ and ϕ are symbolized as f^C, η^C, ${\mathfrak{X}_1}^C$, ϕ^C and f^V, η^V, ${\mathfrak{X}_1}^V$, ϕ^V, respectively. Let $\Im_r^s(\mathcal{M})$ and $\Im_r^s(\mathcal{TM})$ be symbolised as the elements of $\mathcal{M}$ and $\mathcal{TM}$, respectively. The following operations on f, η, $\mathfrak{X}_1$ and ϕ are defined by [9,38]

$$(fX)^V = f^V X^V,\ (fX)^C = f^C X^V + f^V X^C, \tag{6}$$

$${\mathfrak{X}_1}^V f^V = 0,\ {\mathfrak{X}_1}^V f^C = {\mathfrak{X}_1}^C f^V = (Xf)^V,\ {\mathfrak{X}_1}^C f^C = (Xf)^C, \tag{7}$$

$$\eta^V(f^V) = 0,\ \eta^V({\mathfrak{X}_1}^C) = \eta^C({\mathfrak{X}_1}^V) = \eta(\mathfrak{X}_1)^V,\ \eta^C({\mathfrak{X}_1}^C) = \eta(\mathfrak{X}_1)^C, \tag{8}$$

$$\phi^V X^C = (\phi\mathfrak{X}_1)^V,\ \phi^C X^C = (\phi\mathfrak{X}_1)^C, \tag{9}$$

$$[\mathfrak{X}_1,\ \mathfrak{X}_2]^V = [{\mathfrak{X}_1}^C,\ {\mathfrak{X}_2}^V] = [{\mathfrak{X}_1}^V,\ {\mathfrak{X}_2}^C],\ [\mathfrak{X}_1,\ \mathfrak{X}_2]^C = [{\mathfrak{X}_1}^C,\ {\mathfrak{X}_2}^C], \tag{10}$$

$$\nabla^C_{{\mathfrak{X}_1}^C} {\mathfrak{X}_2}^C = (\nabla_{\mathfrak{X}_1}\mathfrak{X}_2)^C,\ \nabla^C_{{\mathfrak{X}_1}^C} {\mathfrak{X}_2}^V = (\nabla_{\mathfrak{X}_1}\mathfrak{X}_2)^V, \tag{11}$$

where ∇ being the Levi-Civita connection.

Applying complete lifts by mathematical operators on (1)-(4), we infer

$$\begin{aligned} T^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C) = {} & \mathfrak{u}^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \mathfrak{u}^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C - \mathfrak{u}^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V \\ & - \mathfrak{u}^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C, \end{aligned} \tag{12}$$

$$\mathfrak{u}^C({\mathfrak{X}_1}^C) = g^C\left({\mathfrak{X}_1}^C, \rho_1^C\right), \tag{13}$$

$$\tilde{\bar{\nabla}}^C g^C = 0, \tag{14}$$

$$\bar{\nabla}^C g^C \neq 0, \tag{15}$$

$$\begin{aligned} \bar{\nabla}^C_{{\mathfrak{X}_1}^C} {\mathfrak{X}_2}^C = {} & \nabla^C_{{\mathfrak{X}_1}^C} {\mathfrak{X}_2}^C + a(\omega^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \omega^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C) \\ & + b(\omega^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \omega^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C), \end{aligned} \tag{16}$$

$$\begin{aligned} \bar{T}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C) = {} & (b-a)(\omega^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \omega^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C) \\ & - (b-a)(\omega^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \omega^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C) \\ = {} & \pi^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \pi^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C \\ & - (\pi^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \pi^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C), \end{aligned} \tag{17}$$

where $\pi^C({\mathfrak{X}_1}^C) = (b-a)\omega^C({\mathfrak{X}_1}^C)$ and $\pi^V({\mathfrak{X}_1}^C) = (b-a)\omega^V({\mathfrak{X}_1}^C)$. Thus $\bar{\nabla}^C$ is a semi-symmetric connection.

In addition,

$$\begin{aligned} & \left(\bar{\nabla}^C_{{\mathfrak{X}_1}^C} g^C\right)({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C) \\ = {} & -2a(\omega^C({\mathfrak{X}_1}^C)g^C({\mathfrak{X}_2}^V, {\mathfrak{X}_3}^C) + \omega^V({\mathfrak{X}_1}^C)g^C({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C)) \end{aligned}$$

$$- b(\omega^C({\mathfrak{X}_2}^C)g^C({\mathfrak{X}_1}^V, {\mathfrak{X}_3}^C) + \omega^V({\mathfrak{X}_2}^C)g^C({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C))$$
$$- b(\omega^C({\mathfrak{X}_3}^C)g^C({\mathfrak{X}_1}^V, {\mathfrak{X}_2}^C) + \omega^V({\mathfrak{X}_3}^C)g^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)) \neq 0. \quad (18)$$

Hence $\bar{\nabla}^C$ defined by (16) is an SSNMC.

3. Existence of the complete lift of an SSNMC of a manifold to its tangent bundle

Let $\bar{\nabla}$ and ∇ be the Levi-Civita connection and the linear connection of $\mathcal{M}$, respectively. Then

$$\bar{\nabla}_{\mathfrak{X}_1}\mathfrak{X}_2 = \nabla_{\mathfrak{X}_1}\mathfrak{X}_2 + F(\mathfrak{X}_1, \mathfrak{X}_2), \quad (19)$$

where $F \in \Im_1^2(M)$, $\mathfrak{X}_1, \mathfrak{X}_2 \in \Im_0^1(M)$ [19,21].

For $\bar{\nabla}$ to be an SSNMC in $\mathcal{M}$, we have

$$F(\mathfrak{X}_1, \mathfrak{X}_2) = \frac{1}{2}[\bar{T}(\mathfrak{X}_1, \mathfrak{X}_2) - \dot{T}(\mathfrak{X}_1, \mathfrak{X}_2) + \dot{T}(\mathfrak{X}_2, \mathfrak{X}_1)]$$
$$+ a\omega(\mathfrak{X}_2)\mathfrak{X}_1 + b\omega(\mathfrak{X}_1)\mathfrak{X}_2, \quad (20)$$

where $g(\mathfrak{X}_1, \rho) = \omega(\mathfrak{X}_1)$ and $\dot{T} \in \Im_1^2(M)$ such that

$$g(\bar{T}(\mathfrak{X}_3, \mathfrak{X}_1), \mathfrak{X}_2) = g(\dot{T}(\mathfrak{X}_1, \mathfrak{X}_2), \mathfrak{X}_3). \quad (21)$$

Applying the complete lifts by mathematical operators on (19), (20) and (21), we infer

$$\bar{\nabla}^C_{{\mathfrak{X}_1}^C}{\mathfrak{X}_2}^C = \nabla^C_{{\mathfrak{X}_1}^C}{\mathfrak{X}_2}^C + F^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C), \quad (22)$$

$$F^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C) = \frac{1}{2}[\bar{T}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C) - \dot{T}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C) + \dot{T}^C({\mathfrak{X}_2}^C, {\mathfrak{X}_1}^C)]$$
$$+ a(\omega^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \omega^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C)$$
$$- b(\omega^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \omega^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C), \quad (23)$$

$$g^C(\bar{T}^C({\mathfrak{X}_3}^C, {\mathfrak{X}_1}^C), {\mathfrak{X}_2}^C) = g^C(\dot{T}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C), {\mathfrak{X}_3}^C). \quad (24)$$

Combining (17) and (24) implies that

$$\dot{T}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C) = \pi^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \pi^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C$$
$$- g^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\rho^V - g^C({\mathfrak{X}_1}^V, {\mathfrak{X}_2}^C)\rho^C, \quad (25)$$

where $\pi^C({\mathfrak{X}_1}^C) = (b-a)\omega^C({\mathfrak{X}_1}^C)$ and $\pi^V({\mathfrak{X}_1}^C) = (b-a)\omega^V({\mathfrak{X}_1}^C)$.

In view of (17), (23) and (25) yield

$$F(\mathfrak{X}_1, \mathfrak{X}_2) = a(\omega^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \omega^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C)$$
$$- b(\omega^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \omega^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C). \quad (26)$$

Therefore, the SSNMC on a Riemannian manifold is given by

$$\bar{\nabla}^C_{{\mathfrak{X}_1}^C}{\mathfrak{X}_2}^C = \nabla^C_{{\mathfrak{X}_1}^C}{\mathfrak{X}_2}^C + a(\omega^C({\mathfrak{X}_2}^C){\mathfrak{X}_1}^V + \omega^V({\mathfrak{X}_2}^C){\mathfrak{X}_1}^C)$$
$$- b(\omega^C({\mathfrak{X}_1}^C){\mathfrak{X}_2}^V + \omega^V({\mathfrak{X}_1}^C){\mathfrak{X}_2}^C). \quad (27)$$

In contrast, we demonstrate that $\bar{\nabla}^C$ such that

$$\begin{aligned}\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\mathfrak{X}_2{}^C &= \nabla^C_{\mathfrak{X}_1{}^C}\mathfrak{X}_2{}^C + a(\omega^C(\mathfrak{X}_2{}^C)\mathfrak{X}_1{}^V + \omega^V(\mathfrak{X}_2{}^C)\mathfrak{X}_1{}^C)\\ &\quad - b(\omega^C(\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^V + \omega^V(\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^C)\end{aligned}$$

is an SSNMC of $\mathcal{M}$ on $\mathcal{TM}$.

The torsion tensor $\bar{T}$ of the connection is given by

$$\begin{aligned}\bar{T}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C) &= (b-a)(\omega^C(\mathfrak{X}_2{}^C)\mathfrak{X}_1{}^V + \omega^V(\mathfrak{X}_2{}^C)\mathfrak{X}_1{}^C)\\ &\quad - (b-a)(\omega^C(\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^V + \omega^V(\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^C)\\ &= \pi^C(\mathfrak{X}_2{}^C)\mathfrak{X}_1{}^V + \pi^V(\mathfrak{X}_2{}^C)\mathfrak{X}_1{}^C\\ &\quad - (\pi^C(\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^V + \pi^V(\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^C).\end{aligned} \tag{28}$$

Thus from (28), $\bar{\nabla}^C$ is a semi-symmetric connection of $\mathcal{M}$ on $\mathcal{TM}$. In addition, we infer

$$\begin{aligned}&\left(\bar{\nabla}^C_{\mathfrak{X}_1{}^C} g^C\right)(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C)\\ =&\ -2a(\omega^C(\mathfrak{X}_1{}^C)g^C(\mathfrak{X}_2{}^V, \mathfrak{X}_3{}^C) + \omega^V(\mathfrak{X}_1{}^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C))\\ &\ - b(\omega^C(\mathfrak{X}_2{}^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{X}_3{}^C) + \omega^V(\mathfrak{X}_2{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{X}_3{}^C))\\ &\ - b(\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{X}_2{}^C) + \omega^V(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)) \neq 0.\end{aligned} \tag{29}$$

As a result, we can say that the connection $\bar{\nabla}$ is an SSNMC.

4. Some calculations on the curvature tensor of the SSNMC of a manifold to its tangent bundle

In [12], De et al. produced the formula for the curvature tensor $\bar{\mathcal{R}}$ of $\mathcal{M}$ with respect to the SSNMC $\bar{\nabla}$ as

$$\bar{\mathcal{R}}(\mathfrak{X}_1, \mathfrak{X}_2)\mathfrak{X}_3 = \bar{\nabla}_{\mathfrak{X}_1}\bar{\nabla}_{\mathfrak{X}_2}\mathfrak{X}_3 - \bar{\nabla}_{\mathfrak{X}_2}\bar{\nabla}_{\mathfrak{X}_1}\mathfrak{X}_3 - \bar{\nabla}_{[\mathfrak{X}_1, \mathfrak{X}_2]}\mathfrak{X}_3, \tag{30}$$

where $\forall \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3 \in \chi(\mathcal{M})$.

Applying the complete lifts by mathematical operators on (30), we infer

$$\begin{aligned}&\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C\\ &= \bar{\nabla}^C_{\mathfrak{X}_1{}^C}\bar{\nabla}^C_{\mathfrak{X}_2{}^C}\mathfrak{X}_3{}^C - \bar{\nabla}^C_{\mathfrak{X}_2{}^C}\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\mathfrak{X}_3{}^C - \bar{\nabla}^C_{[\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C]}\mathfrak{X}_3{}^C.\end{aligned} \tag{31}$$

Using (16) in (31), we infer

$$\begin{aligned}\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C &= \mathcal{R}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C\\ &\quad - a\{(\nabla_{\mathfrak{X}_2}\omega)^C(\mathfrak{X}_1{}^C)\mathfrak{X}_3{}^V + (\nabla_{\mathfrak{X}_2}\omega)^V(\mathfrak{X}_1{}^C)\mathfrak{X}_3{}^C\}\\ &\quad + a\{(\nabla_{\mathfrak{X}_1}\omega)^C(\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^V + (\nabla_{\mathfrak{X}_1}\omega)^V(\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C\}\\ &\quad - b\{(\nabla_{\mathfrak{X}_2}\omega)^C(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^V + (\nabla_{\mathfrak{X}_2}\omega)^V(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C\}\\ &\quad + b\{(\nabla_{\mathfrak{X}_1}\omega)^C(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^V + (\nabla_{\mathfrak{X}_1}\omega)^V(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C\}\\ &\quad + b^2\{\omega^C(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^V + \omega^C(\mathfrak{X}_2{}^C)\omega^V(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C\end{aligned}$$

$$+\,\omega^V(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C\} - b^2\{\omega^C(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^V + \omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C + \omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C\}. \tag{32}$$

From (17), we infer

$$\left(\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\mathcal{C}^1_1\bar{T}^C\right)(\mathfrak{X}_2{}^C) = (n-1)\pi^C(\mathfrak{X}_2{}^C) = (n-1)(b-a)\left(\bar{\nabla}_{\mathfrak{X}_1}\omega\right)^C(\mathfrak{X}_2{}^C), \tag{33}$$

where $\mathcal{C}^1_1$ symbolizes the contraction.

Suppose the torsion tensor $\bar{T}$ with respect to the SSNMC is pseudo symmetric, that is,

$$\begin{aligned}&\left(\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\bar{T}^C\right)(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)\\ &= \omega^C(\mathfrak{X}_1{}^C)\bar{T}^V(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C) + \omega^V(\mathfrak{X}_1{}^C)\bar{T}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)\\ &\quad + \omega^C(\mathfrak{X}_2{}^C)\bar{T}^V(\mathfrak{X}_1{}^C,\mathfrak{X}_3{}^C) + \omega^V(\mathfrak{X}_2{}^C)\bar{T}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_3{}^C)\\ &\quad + \omega^C(\mathfrak{X}_3{}^C)\bar{T}^V(\mathfrak{X}_2{}^C,\mathfrak{X}_1{}^C) + \omega^V(\mathfrak{X}_3{}^C)\bar{T}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_1{}^C)\\ &\quad + g^C(\bar{T}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C),\mathfrak{X}_1{}^C)\rho^V + g^C(\bar{T}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C),\mathfrak{X}_1{}^C)\rho^V,\end{aligned} \tag{34}$$

where $\omega^C(\mathfrak{X}_1{}^C) = g^C(\mathfrak{X}_1{}^C,\rho^C)$.

Contracting over $\mathfrak{X}_3$ in (34) and using (17), we infer

$$\begin{aligned}&\left(\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\mathcal{C}^1_1\bar{T}^C\right)(\mathfrak{X}_2{}^C)\\ &= 4(n-1)(b-a)\{\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_2{}^C) + \omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_2{}^C)\}\\ &\quad - (b-a)\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{X}_2{}^C) + \omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\}.\end{aligned} \tag{35}$$

Combining (33) and (35), we infer

$$\begin{aligned}&\left(\bar{\nabla}_{\mathfrak{X}_1}\omega\right)^C(\mathfrak{X}_2{}^C)\\ &= 4\{\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_2{}^C) + \omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_2{}^C)\}\\ &\quad - \frac{1}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{X}_2{}^C) + \omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\}.\end{aligned} \tag{36}$$

Therefore, from (55) and (36), it follows that

$$\begin{aligned}&\left(\bar{\nabla}_{\mathfrak{X}_1}\omega\right)^C(\mathfrak{X}_2{}^C)\\ &= (a+b+4)\{\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_2{}^C) + \omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_2{}^C)\}\\ &\quad - \frac{1}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{X}_2{}^C) + \omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\}.\end{aligned} \tag{37}$$

From (37), (32) becomes

$$\begin{aligned}&\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C\\ &= \mathcal{R}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C - b(a+4)\{\omega^C(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^V\\ &\quad + \omega^C(\mathfrak{X}_2{}^C)\omega^V(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C + \omega^V(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C\}\\ &\quad + b(a+4)\{\omega^C(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^V\end{aligned}$$

$$+\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C+\omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C\}$$
$$+\frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^V+\omega^C(\rho^C)g^C(\mathfrak{X}_2{}^V,\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C$$
$$+\omega^V(\rho^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C\}-\frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^V$$
$$+\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C+\omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_3{}^C)\mathfrak{X}_2{}^C\}. \tag{38}$$

From (38), we infer

$$\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C=-\bar{\mathcal{R}}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_1{}^C)\mathfrak{X}_3{}^C,$$

and

$$\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C+\bar{\mathcal{R}}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C+\bar{\mathcal{R}}^C(\mathfrak{X}_3{}^C,\mathfrak{X}_1{}^C)\mathfrak{X}_2{}^C=0. \tag{39}$$

The equation (39) represents the first Bianchi identity with respect to the SSNMC $\bar{\nabla}^C$.

Applying the inner product of (38) with $\mathfrak{u}$, we infer

$$'\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C,\mathfrak{u}^C)$$
$$='\mathcal{R}^C(\mathfrak{X}_1{}^C,\mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C-b(a+4)\{\omega^C(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{u}^C)$$
$$+\omega^C(\mathfrak{X}_2{}^C)\omega^V(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)+\omega^V(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)\}$$
$$+b(a+4)\{\omega^C(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^V,\mathfrak{u}^C)$$
$$+\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{u}^C)+\omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{u}^C)\}$$
$$+\frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{u}^C)$$
$$+\omega^C(\rho^C)g^C(\mathfrak{X}_2{}^V,\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)$$
$$+\omega^V(\rho^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)\}$$
$$-\frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^V,\mathfrak{u}^C)$$
$$+\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V,\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{u}^C)$$
$$+\omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C,\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{u}^C)\}, \tag{40}$$

where

$'\bar{\mathcal{R}}(\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{X}_3,\mathfrak{u})=g(\bar{\mathcal{R}}(\mathfrak{X}_1,\mathfrak{X}_2)\mathfrak{X}_3,\mathfrak{u})$ and $'\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{X}_3,\mathfrak{u})=g(\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2)\mathfrak{X}_3,\mathfrak{u})$.

Suppose that $\{e_1^C,\ldots,e_n^C\}$ is an orthonormal basis of $\mathcal{TM}$. Place $\mathfrak{X}_1=\mathfrak{u}=e_i$ in (40) and putting summation before i, $1\leq i\leq n$, we infer

$$\bar{\mathcal{S}}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)=\mathcal{S}^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)+b\{\omega^V(\rho^C)g^C(\mathfrak{X}_2{}^V,\mathfrak{X}_3{}^C)$$
$$+\omega^V(\rho^C)g^C(\mathfrak{X}_2{}^C,\mathfrak{X}_3{}^C)\}$$
$$-b(n-1)(a+4)\{\omega^C(\mathfrak{X}_2{}^C)\omega^V(\mathfrak{X}_3{}^C)$$
$$+\omega^V(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)\}, \tag{41}$$

where $\bar{\mathcal{S}}^C$ and $\mathcal{S}^C$ denote the complete lift of the Ricci tensors $\bar{\mathcal{S}}$ and $\mathcal{S}$.

The above discussions help us to conclude:

Theorem 4.1. *Let $\mathcal{TM}$ be the tangent bundle of a Riemannian manifold $\mathcal{M}$ endowed with an SSNMC $\bar{\nabla}^C$ whose torsion tensor is pseudo symmetric. Then*

(i) *The curvature tensor $\bar{\mathcal{R}}^C$ is given by* (40).
(ii) $\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C = -\bar{\mathcal{R}}^C(\mathfrak{X}_2{}^C, \mathfrak{X}_1{}^C)\mathfrak{X}_3{}^C$.
(iii) $\bar{\mathcal{R}}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C + \bar{\mathcal{R}}^C(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C)\mathfrak{X}_1{}^C + \bar{\mathcal{R}}^C(\mathfrak{X}_3{}^C, \mathfrak{X}_1{}^C)\mathfrak{X}_2{}^C = 0$.
(iv) *The Ricci tensor $\bar{\mathcal{S}}^C$ is given by* (41).
(v) *$\bar{\mathcal{S}}^C$ is symmetric.*

Let $\bar{\mathcal{R}}^C = 0$ and put it in (38), we deduce

$$\begin{aligned}
{}'\mathcal{R}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C = {}& b(a+4)\{\omega^C(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{u}^C) \\
& + \omega^C(\mathfrak{X}_2{}^C)\omega^V(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{u}^C) \\
& + \omega^V(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{u}^C)\} \\
& - b(a+4)\{\omega^C(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^V, \mathfrak{u}^C) \\
& + \omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{u}^C) \\
& + \omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{u}^C)\} \\
& - \frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{u}^C) \\
& + \omega^C(\rho^C)g^C(\mathfrak{X}_2{}^V, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{u}^C) \\
& + \omega^V(\rho^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{u}^C)\} \\
& + \frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^V, \mathfrak{u}^C) \\
& + \omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{u}^C) \\
& + \omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C U^C)\}.
\end{aligned} \tag{42}$$

Substituting $a = -4$ in (42), we infer

$$\begin{aligned}
{}'\mathcal{R}^C(\mathfrak{X}_1{}^C, \mathfrak{X}_2{}^C)\mathfrak{X}_3{}^C = {}& -\frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{u}^C) \\
& + \omega^C(\rho^C)g^C(\mathfrak{X}_2{}^V, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{u}^C) \\
& + \omega^V(\rho^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{u}^C)\} \\
& + \frac{b}{n-1}\{\omega^C(\rho^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^V, \mathfrak{u}^C) \\
& + \omega^C(\rho^C)g^C(\mathfrak{X}_1{}^V, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C, \mathfrak{u}^C) \\
& + \omega^V(\rho^C)g^C(\mathfrak{X}_1{}^C, \mathfrak{X}_3{}^C)g^C(\mathfrak{X}_2{}^C U^C)\}.
\end{aligned} \tag{43}$$

This outcome indicates that the manifold is of constant curvature.

Hence, we can make the following statement:

Theorem 4.2. *Let $\mathcal{TM}$ be the tangent bundle of a Riemannian manifold $\mathcal{M}$ endowed with an SSNMC $\bar{\nabla}^C$. If the curvature tensor vanishes, that is , $\bar{\mathcal{R}}^C = 0$ and the torsion tensor is pseudo symmetric, then the manifold $\mathcal{M}$ is of constant curvature with respect to ∇^C on $\mathcal{TM}$ subject to $a = -4$.*

5. Proposed theorem on Weyl projective curvature tensor on a Riemannian manifold to its tangent bundles endowed with the SSNMC

The Weyl projective curvature tensor $\bar{P}$ with respect to the SSNMC is given by

$$\bar{P}(\mathfrak{X}_1, \mathfrak{X}_2)\mathfrak{X}_3 = \bar{\mathcal{R}}(\mathfrak{X}_1, \mathfrak{X}_2)\mathfrak{X}_3 - \frac{1}{n-1}[\bar{\mathcal{S}}(\mathfrak{X}_2, \mathfrak{X}_3)\mathfrak{X}_1 - \bar{\mathcal{S}}(\mathfrak{X}_1, \mathfrak{X}_3)\mathfrak{X}_2]. \tag{44}$$

Operating the complete lift on (44), we infer

$$\begin{aligned} \bar{P}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C){\mathfrak{X}_3}^C = {} & \bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C){\mathfrak{X}_3}^C \\ & - \frac{1}{n-1}[\bar{\mathcal{S}}^C({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C){\mathfrak{X}_1}^V + \bar{\mathcal{S}}^V({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C){\mathfrak{X}_1}^C] \\ & - \frac{1}{n-1}[\bar{\mathcal{S}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C){\mathfrak{X}_2}^V + \bar{\mathcal{S}}^V({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C){\mathfrak{X}_2}^C]. \end{aligned} \tag{45}$$

From (45), it follows that

$$\begin{aligned} {}'\bar{P}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C) = {} & {}'\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C) \\ & - \frac{1}{n-1}[\bar{\mathcal{S}}^C({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_1}^V, \mathfrak{u}^C) \\ & + \bar{\mathcal{S}}^V({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_1}^C, \mathfrak{u}^C)] \\ & - \frac{1}{n-1}[\bar{\mathcal{S}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_2}^V, \mathfrak{u}^C) \\ & + \bar{\mathcal{S}}^V({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_2}^C, \mathfrak{u}^C)], \end{aligned} \tag{46}$$

where ${}'\bar{P}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C) = g^C(\bar{P}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C){\mathfrak{X}_3}^C, \mathfrak{u}^C)$ for all $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$, $\mathfrak{u} \in Im_0^1(\mathcal{M})$.

From (40) and (41) in (46), we get

$${}'\bar{P}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C) = {}'P^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C), \tag{47}$$

where

$$\begin{aligned} {}'P^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C) = {} & {}'\mathcal{R}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C, \mathfrak{u}^C) \\ & - \frac{1}{n-1}[\mathcal{S}^C({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_1}^V, \mathfrak{u}^C) \\ & + \mathcal{S}^V({\mathfrak{X}_2}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_1}^C, \mathfrak{u}^C)] \\ & - \frac{1}{n-1}[\mathcal{S}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_2}^V, \mathfrak{u}^C) \\ & + \mathcal{S}^V({\mathfrak{X}_1}^C, {\mathfrak{X}_3}^C)g^C({\mathfrak{X}_2}^C, \mathfrak{u}^C)]. \end{aligned} \tag{48}$$

Thus we have the following:

Theorem 5.1. *Let $\mathcal{TM}$ be the tangent bundle of a Riemannian manifold $\mathcal{M}$ endowed with an SSNMC $\bar{\nabla}^C$ whose torsion tensor is pseudo symmetric. Then the Weyl projective curvature tensors with respect to $\bar{\nabla}^C$ and ∇^C are equal.*

6. Proposed theorem on Ricci-semisymmetric manifolds on the tangent bundle

In [12], De et al. produced that a Riemannian manifold is said to Ricci-semisymmetric with respect to the $\bar{\nabla}$ if

$$(\bar{\mathcal{R}}(\mathfrak{X}_1, \mathfrak{X}_2) \cdot \bar{\mathcal{S}})(\mathfrak{u}, W) = 0,$$

where $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{u}, W \in \chi(\mathcal{M})$.

Applying the complete lift on the above equation, we infer

$$\begin{aligned}((\bar{\mathcal{R}}(\mathfrak{X}_1, \mathfrak{X}_2) \cdot \bar{\mathcal{S}})(\mathfrak{u}, W))^C = {} & \bar{\mathcal{S}}^C(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\mathfrak{u}^C, W^C) \\ & + \bar{\mathcal{S}}^C(\mathfrak{u}^C, \bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)W^C).\end{aligned} \tag{49}$$

From (41) in (49), we infer

$$\begin{aligned}((\bar{\mathcal{R}}(\mathfrak{X}_1, \mathfrak{X}_2) \cdot \bar{\mathcal{S}})(\mathfrak{u}, W))^C = {} & \mathcal{S}^C(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\mathfrak{u}^C, W^C) \\ & + \mathcal{S}^C(\mathfrak{u}^C, \bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)W^C) \\ & + b\{\omega^C(\rho^C) g^V(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\mathfrak{u}^C, W^C) \\ & + \omega^V(\rho^C) g^C(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\mathfrak{u}^C, W^C) \\ & + \omega^C(\rho^C) g^V(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)W^C, \mathfrak{u}^C) \\ & + \omega^V(\rho^C) g^C(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)W^C, \mathfrak{u}^C)\} \\ & - b(n-1)(a+4)[\omega^C(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\mathfrak{u}^C)\omega^V(W^C) \\ & + \omega^V(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)\mathfrak{u}^C)\omega^C(W^C) \\ & + \omega^C(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)W^C)\omega^V(\mathfrak{u}^C) \\ & + \omega^V(\bar{\mathcal{R}}^C({\mathfrak{X}_1}^C, {\mathfrak{X}_2}^C)W^C)\omega^C(\mathfrak{u}^C)].\end{aligned} \tag{50}$$

Using (38) and (50), we infer

$$\begin{aligned}((\bar{\mathcal{R}}(\mathfrak{X}_1, \mathfrak{X}_2) \cdot \bar{\mathcal{S}})(\mathfrak{u}, W))^C = {} & ((\mathcal{R}(\mathfrak{X}_1, \mathfrak{X}_2) \cdot \bar{\mathcal{S}})(\mathfrak{u}, W))^C \\ & + b\{\omega^C(\rho^C)('\mathcal{R}({\mathfrak{X}_1}^C, \mathfrak{X}_2, \mathfrak{u}, W))^V \\ & + \omega^V(\rho^C)('\mathcal{R}({\mathfrak{X}_1}^C, \mathfrak{X}_2, \mathfrak{u}, W))^C\} \\ & - \frac{b}{n-1}\{\omega^C(\rho^C)\mathcal{S}^C({\mathfrak{X}_2}^C, \mathfrak{u}^C) g^C({\mathfrak{X}_1}^V, W^C) \\ & + \omega^C(\rho^C)\mathcal{S}^V({\mathfrak{X}_2}^C, \mathfrak{u}^C) g^C({\mathfrak{X}_1}^C, W^C) \\ & + \omega^V(\rho^C)\mathcal{S}^C({\mathfrak{X}_2}^C, \mathfrak{u}^C) g^C({\mathfrak{X}_1}^C, W^C)\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{b}{n-1}\{\omega^C(\rho^C)\mathcal{S}^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)g^C(\mathfrak{X}_2{}^V,W^C)\\
&+\omega^C(\rho^C)\mathcal{S}^V(\mathfrak{X}_1{}^C,\mathfrak{u}^C)g^C(\mathfrak{X}_2{}^C,W^C)\\
&+\omega^V(\rho^C)\mathcal{S}^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)g^C(\mathfrak{X}_2{}^C,W^C)\}\\
&-b(n-1)(a+4)\{\omega^C(\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2)\mathfrak{u})^C\omega^V(W^C)\\
&+\omega^V(\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2)\mathfrak{u})^C\omega^C(W^C)\}\\
&-b(a+4)\{\omega^C(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{u}^C)\mathcal{S}^V(\mathfrak{X}_1{}^C,W^C)\\
&+\omega^C(\mathfrak{X}_2{}^C)\omega^V(\mathfrak{u}^C)\mathcal{S}^C(\mathfrak{X}_1{}^C,W^C)\\
&+\omega^V(\mathfrak{X}_2{}^C)\omega^C(\mathfrak{u}^C)\mathcal{S}^C(\mathfrak{X}_1{}^C,W^C)\}\\
&+b(a+4)\{\omega^C(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{u}^C)\mathcal{S}^V(\mathfrak{X}_2{}^C,W^C)\\
&+\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{u}^C)\mathcal{S}^C(\mathfrak{X}_2{}^C,W^C)\\
&+\omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{u}^C)\mathcal{S}^C(\mathfrak{X}_2{}^C,W^C)\}\\
&-b(n-1)(a+4)\{\omega^C(\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2)W)^C\omega^V(\mathfrak{u}^C)\\
&+\omega^V(\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2)W)^C\omega^C(\mathfrak{u}^C)\}\\
&-b(a+4)\{\omega^C(\mathfrak{X}_2{}^C)\omega^C(W^C)\mathcal{S}^V(\mathfrak{X}_1{}^C,\mathfrak{u}^C)\\
&+\omega^C(\mathfrak{X}_2{}^C)\omega^V(W^C)\mathcal{S}^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)\\
&+\omega^V(\mathfrak{X}_2{}^C)\omega^C(W^C)\mathcal{S}^C(\mathfrak{X}_1{}^C,\mathfrak{u}^C)\}\\
&+b(a+4)\{\omega^C(\mathfrak{X}_1{}^C)\omega^C(W^C)\mathcal{S}^V(\mathfrak{X}_2{}^C,\mathfrak{u}^C)\\
&+\omega^C(\mathfrak{X}_1{}^C)\omega^V(W^C)\mathcal{S}^C(\mathfrak{X}_2{}^C,\mathfrak{u}^C)\\
&+\omega^V(\mathfrak{X}_1{}^C)\omega^C(W^C)\mathcal{S}^C(\mathfrak{X}_2{}^C,\mathfrak{u}^C)\}. \qquad (51)
\end{aligned}
$$

Setting $a+4=0$ in (51) and from (5.4), we infer

$$
\begin{aligned}
&((\bar{\mathcal{R}}(\mathfrak{X}_1,\mathfrak{X}_2)\cdot\bar{\mathcal{S}})(\mathfrak{u},W))^C\\
&=((\mathcal{R}(\mathfrak{X}_1,\mathfrak{X}_2)\cdot\bar{\mathcal{S}})(\mathfrak{u},W))^C\\
&\quad+b\omega^C(\rho^C)[('P(\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{u},W))^V+('P(\mathfrak{X}_1,\mathfrak{X}_2,W,\mathfrak{u}))^V]\\
&\quad+b\omega^V(\rho^C)[('P(\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{u},W))^C+('P(\mathfrak{X}_1,\mathfrak{X}_2,W,\mathfrak{u}))^C]. \qquad (52)
\end{aligned}
$$

Thus we have the following:

Theorem 6.1. *Let $\mathcal{TM}$ be the tangent bundle of a Riemannian manifold $\mathcal{M}$ endowed with an SSNMC $\bar{\nabla}^C$. Then Ricci semi-symmetry of M on $\mathcal{TM}$ with respect to ∇^C and $\bar{\nabla}^C$ are equivalent, subject to $a+4=0$ and ρ^C is a null vector.*

7. Applications

In this section, we have discussed the applications of an irrotational field and geodesics with respect to the Levi-Civita connection and an SSNMC of $\mathcal{M}$ to $\mathcal{TM}$.

Let us recall the essentials of an irrotational vector field and geodesics.

The vector field ρ is irrotational if $g(\mathfrak{X}_2, \nabla_{\mathfrak{X}_1}\rho) = g(\mathfrak{X}_1, \nabla_{\mathfrak{X}_2}\rho)$ and the integral curves of the vector field ρ are geodesic if $\nabla_\rho \rho = 0$.

Definition. The 1-form ω is closed with respect to ∇ if

$$(\nabla_{\mathfrak{X}_1}\omega)(\mathfrak{X}_2) - (\nabla_{\mathfrak{X}_2}\omega)(\mathfrak{X}_1) = 0. \tag{53}$$

Theorem 7.1. *Let $\mathcal{TM}$ be the tangent bundle of a Riemannian manifold $\mathcal{M}$ endowed with a semi-symmetric non-metric connection. Then*

(i) *The 1-form ω^C is closed with respect to the Levi-Civita connection ∇ if and only if ω^C is closed with respect to the SSNMC $\bar{\nabla}^C$ on $\mathcal{TM}$.*

(ii) *The vector field ρ^C is irrotational with respect to ∇^C if and only if ρ^C is irrotational with respect to $\bar{\nabla}^C$ on $\mathcal{TM}$.*

(iii) *The integral curves of the unit vector field ρ^C are geodesic with respect to ∇^C if and only if the integral curves of the unit vector field ρ^C is geodesic with respect to $\bar{\nabla}^C$.*

Proof. Applying the complete lift on (53), we acquire

$$\left(\nabla^C_{\mathfrak{X}_1{}^C}\omega^C\right)(\mathfrak{X}_2{}^C) - \left(\nabla^C_{\mathfrak{X}_2{}^C}\omega^C\right)(\mathfrak{X}_1{}^C) = 0. \tag{54}$$

In view of (16), we deduce

$$\begin{aligned}\left(\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\omega^C\right)(\mathfrak{X}_2{}^C) = &\left(\nabla^C_{\mathfrak{X}_1{}^C}\omega^C\right)(\mathfrak{X}_2{}^C) - (a+b)\{\omega^C(\mathfrak{X}_1{}^C)\omega^V(\mathfrak{X}_2{}^C)\\ &+ \omega^V(\mathfrak{X}_1{}^C)\omega^C(\mathfrak{X}_2{}^C)\}.\end{aligned} \tag{55}$$

From (55), we deduce

$$\begin{aligned}&\left(\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\omega^C\right)(\mathfrak{X}_2{}^C) - \left(\bar{\nabla}^C_{\mathfrak{X}_2{}^C}\omega^C\right)(\mathfrak{X}_1{}^C)\\ &= \left(\nabla^C_{\mathfrak{X}_1{}^C}\omega^C\right)(\mathfrak{X}_2{}^C) - \left(\nabla^C_{\mathfrak{X}_2{}^C}\omega^C\right)(\mathfrak{X}_1{}^C).\end{aligned} \tag{56}$$

Thus the proof of (i) is completed.

Setting $\mathfrak{X}_2 = \rho$ in (16), we provide

$$\begin{aligned}\bar{\nabla}^C_{\mathfrak{X}_1{}^C}\rho^C = &\nabla^C_{\mathfrak{X}_1{}^C}\rho^C + a(\omega^C(\mathfrak{X}_1{}^C)\rho^V + \omega^V(\mathfrak{X}_1{}^C)\rho^C)\\ &+ b(\omega^C(\rho^C)\mathfrak{X}_1{}^V + \omega^V(\rho^C)\mathfrak{X}_1{}^C).\end{aligned} \tag{57}$$

The equation (57) yields

$$\begin{aligned}&g^C\left(\mathfrak{X}_2{}^C, \bar{\nabla}^C_{\mathfrak{X}_1{}^C}\rho^C\right) - g^C\left(\mathfrak{X}_1{}^C, \bar{\nabla}^C_{\mathfrak{X}_2{}^C}\rho^C\right)\\ &= g^C\left(\mathfrak{X}_2{}^C, \nabla^C_{\mathfrak{X}_1{}^C}\rho^C\right) - g^C\left(\mathfrak{X}_1{}^C, \nabla^C_{\mathfrak{X}_2{}^C}\rho^C\right).\end{aligned}$$

Thus the proof of (ii) is completed.

Setting $\mathfrak{X}_1 = \rho$ in (57), we deduce

$$\bar{\nabla}^C_{\rho^C}\rho^C = \nabla^C_{\rho^C}\rho^C + (a+b)(\omega^C(\rho^C)\rho^V + \omega^V(\rho^C)\rho^C). \tag{58}$$

If $a + b = 0$, then from (58), it follows that

$$\bar{\nabla}^C_{\rho^C}\rho^C = \nabla^C_{\rho^C}\rho^C.$$

Thus the proof of (iii) is completed. □

Acknowledgment. Researcher would like to thank the Deanship of Scientific Research, Qassim University, for funding publication of this project.

References

[1] N. S. Agashe and M. R. Chafle, *A semi-symmetric nonmetric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. **23** (1992), no. 6, 399–409.

[2] R. C. Akpinar, *Weyl connection on tangent bundle of hypersurface*, Int. J. Maps Math. **4** (2021), no. 1, 2–13.

[3] M. Altunbas, L. Bilen, and A. Gezer, *Remarks about the Kaluza-Klein metric on tangent bundle*, Int. J. Geom. Methods Mod. Phys. **16** (2019), no. 3, 1950040, 13 pp. https://doi.org/10.1142/S0219887819500403

[4] M. Altunbas and C. Şengül, *Metallic structures on tangent bundles of Lorentzian para-Sasakian manifolds*, Journal of Mahani Mathematical Research, **12** (2023), no. 1, 137–149.

[5] O. C. Andonie, *On semi-symmetric non-metric connection on a Riemannian manifold*, Ann. Fac. Sci. De Kinshasa, Zaire Sect. Math. Phys. **2** (1976).

[6] T. Q. Binh, *On semi-symmetric connections*, Period. Math. Hungar. **21** (1990), no. 2, 101–107. https://doi.org/10.1007/BF01946849

[7] S. C. Biswas and U. C. De, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Ganita **48** (1997), no. 2, 91–94.

[8] M. C. Chaki, *On pseudo symmetric manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **33** (1987), no. 1, 53–58.

[9] L. S. K. Das and M. N. I. Khan, *Almost r-contact structures on the tangent bundle*, Differ. Geom. Dyn. Syst. **7** (2005), 34–41.

[10] L. S. K. Das and M. N. I. Khan, *Symmetric and Ricci LP-Sasakian manifold*, Math. Sci. Research J. **17** (2013) no. 10, 263–268.

[11] L. S. K. Das, R. Nivas, and M. N. I. Khan, *On submanifolds of codimension* 2 *immersed in a Hsu-quaternion manifold*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **25** (2009), no. 1, 129–135.

[12] U. C. De, Y. Han, and P. Zhao, *A special type of semi-symmetric non-metric connection on a Riemannian manifold*, Facta Univ. Ser. Math. Inform. **31** (2016), no. 2, 529–541.

[13] M. H. Dida and F. Hathout, *Ricci soliton on the tangent bundle with semi-symmetric metric connection*, Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci. **1(63)** (2021), no. 2, 37–52.

[14] M. H. Dida, F. Hathout, and M. Djaa, *On the geometry of the second order tangent bundle with the diagonal lift metric*, Int. J. Math. Anal. (Ruse) **3** (2009), no. 9-12, 443–456.

[15] M. H. Dida and A. Ikemakhen, *A class of metrics on tangent bundles of pseudo-Riemannian manifolds*, Arch. Math. (Brno) **47** (2011), no. 4, 293–308.

[16] A. Friedmann and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragungen*, Math. Z. **21** (1924), no. 1, 211–223. https://doi.org/10.1007/BF01187468

[17] H. A. Hayden, *Sub-spaces of a space with torsion*, Proc. London Math. Soc. (2) **34** (1932), no. 1, 27–50. https://doi.org/10.1112/plms/s2-34.1.27
[18] A. Kazan and H. B. Karadağ, *Locally decomposable golden Riemannian tangent bundles with Cheeger-Gromoll metric*, Miskolc Math. Notes **17** (2016), no. 1, 399–411. https://doi.org/10.18514/MMN.2016.1534
[19] M. N. I. Khan, *Quarter-symmetric semi-metric connection on Sasakian manifold*, Tensor (N.S.) **68** (2007), no. 2, 154–157.
[20] M. N. I. Khan, *Lifts of hypersurfaces with quarter-symmetric semi-metric connection to tangent bundles*, Afr. Mat. **25** (2014), no. 2, 475–482. https://doi.org/10.1007/s13370-013-0150-x
[21] M. N. I. Khan, *Lift of semi-symmetric non-metric connection on a Kähler manifold*, Afr. Mat. **27** (2016), no. 3-4, 345–352. https://doi.org/10.1007/s13370-015-0350-7
[22] M. N. I. Khan, *Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold*, Chaos, Solitons & Fractals **146** (2021), 110872.
[23] M. N. I. Khan, *Proposed theorems for lifts of the extended almost complex structures on the complex manifold*, Asian-Eur. J. Math. **15** (2022), no. 11, Paper No. 2250200, 13 pp. https://doi.org/10.1142/S179355712250200X
[24] M. N. I. Khan and U. C. De, *Liftings of metallic structures to tangent bundles of order r*, AIMS Math. **7** (2022), no. 5, 7888–7897. https://doi.org/10.3934/math.2022441
[25] M. N. I. Khan and U. C. De, *Lifts of metallic structure on a cross-section*, Filomat **36** (2022), no. 18, 6369–6373.
[26] M. N. I. Khan, U. C. De, and L. S. Velimirovic, *Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle*, Mathematics **11** (2023), 53.
[27] Y. X. Liang, *On semi-symmetric recurrent-metric connection*, Tensor (N.S.) **55** (1994), no. 2, 107–112.
[28] P. Pandey and B. B. Chaturvedi, *On a Kähler manifold equipped with lift of quarter symmetric non-metric connection*, Facta Univ. Ser. Math. Inform. **33** (2018), no. 4, 539–546.
[29] E. Peyghan, F. Firuzi, and U. C. De, *Golden Riemannian structures on the tangent bundle with g-natural metrics*, Filomat **33** (2019), no. 8, 2543–2554.
[30] M. S. Prvanović, *On pseudo metric semi-symmetric connections*, Publ. Inst. Math. (Beograd) (N.S.) **18(32)** (1975), 157–164.
[31] R. N. Singh and G. Pandey, *On the W_2-curvature tensor of the semi-symmetric non-metric connection in a Kenmotsu manifold*, Novi Sad J. Math. **43** (2013), no. 2, 91–105.
[32] D. Smaranda and O. C. Andonie, *On semi-symmetric connection*, Ann. Fac. Sci. Univ. Nat. Zaï re (Kinshasa) Sect. Math.-Phys. **2** (1976), no. 2, 265–270.
[33] M. Tani, *Prolongations of hypersurfaces to tangent bundles*, Kodai Math. Sem. Rep. **21** (1969), 85–96. http://projecteuclid.org/euclid.kmj/1138845833
[34] L. S. Velimirović, S. M. Minčić, and M. Stanković, *Infinitesimal deformations of curvature tensors at non-symmetric affine connection space*, Mat. Vesnik **54** (2002), no. 3-4, 219–226.
[35] L. S. Velimirović, S. M. Minčić, and M. Stanković, *Infinitesimal deformations and Lie derivative of a non-symmetric affine connection space*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **42** (2003), 111–121.
[36] K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1579–1586.
[37] K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies, No. 32, Princeton Univ. Press, Princeton, NJ, 1953.
[38] K. Yano and S. Ishihara, *Tangent and cotangent bundles: differential geometry*, Pure and Applied Mathematics, No. 16, Marcel Dekker, Inc., New York, 1973.

[39] F. Ö. Zengin, S. A. Uysal, and S. A. Demirbağ, *On sectional curvature of a Riemannian manifold with semi-symmetric metric connection*, Ann. Polon. Math. **101** (2011), no. 2, 131–138. https://doi.org/10.4064/ap101-2-3

[40] P. Zhao, *Invariant of projective transformation of semisymmetric metric-recurrent connections and curvature tensor expressions*, J. Engineering Math. **17** (2000), no. 2, 105–108.

[41] P. Zhao, H. Z. Song, and X. P. Yang, *Some invariant properties of semi-symmetric metric recurrent connections and curvature tensor expressions*, Chinese Quart. J. Math. **19** (2004), no. 4, 355–361.

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