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LAPLACE TRANSFORM AND HYERS-ULAM STABILITY OF DIFFERENTIAL EQUATION FOR LOGISTIC GROWTH IN A POPULATION MODEL

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Abstract. In this paper, we prove the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of a differential equation of Logistic growth in a population by applying Laplace transforms method.

1. Introduction

The stability problem for various forms of a functional equations arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was initiated in [31] by a great Mathematician Ulam and which was answered in [9] by Hyers in the year 1941. And then it was generalized by various authors in [3, 8, 27, 28] for additive mappings and linear mappings, respectively. Since then several stability problems for different functional equations have been investigated in [4, 6, 18, 24].

Let Y be a normed space and let I be an open interval. Assume that for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(t)y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) + h(t) \right\| \leq \epsilon$$

for all $t \in I$ and for some $\epsilon > 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(t)y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) + h(t) = 0$$

such that $\|f(t) - f_0(t)\| \leq K(\epsilon)$ for any $x \in I$, where $K(\epsilon)$ is an expression of ϵ only. Then, we say that the above differential equation has the Hyers-Ulam stability.

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If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [25, 26]. Thereafter, in 1998, Alsina and Ger [2] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in [2] the following theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of Alsina and Ger [2] has been generalized by Takahasi [30]. They proved in [30] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [10–13, 17]. In 2014, Alqifiary and Jung [1] proved the generalized Hyers-Ulam stability of linear differential equation by using the Laplace transform method (see also [29]).

Now a days, the Hyers-Ulam stability of differential equations are investigated by number of authors in [5, 7, 14, 15, 19–23, 32] and the Hyers-Ulam stability of differential equations has been given attention.

We may apply these terminologies for other differential equations also. In this paper, by applying Laplace transforms method, we are interested in proving the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of a differential equation of logistic growth in a population of the form

$$\frac{du}{dt} = u - g(u) \tag{1}$$

with initial condition

$$u(0) = u_0, \tag{2}$$

where g is a nonlinear function of u.

2. Preliminaries

In this section, we introduce some notations, definitions and preliminaries which are used throughout this paper.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ is of exponential order if there exists a constant $M(> 0) \in \mathbb{R}$ such that $|f(t)| \leq Me^{at}$ for all $t > 0$. For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Laplace transform of f by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

The Laplace transform of f is sometimes denoted by $\mathcal{L}(f)$. It is also well-known that $\mathcal{L}$ is linear and one-to-one. Then, at points of continuity of f, we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}\ ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\alpha+iy} F(\alpha+iy)\ dy, \end{aligned}$$

this is called the inverse Laplace transform.

Definition (Convolution)**.** Given two functions f and g, both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\ g(x-t)\ dt$$

exists. This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Theorem 2.1. *The Laplace transform of the convolution of $f(x)$ and $g(x)$ is the product of the Laplace transform of $f(x)$ and $g(x)$. That is,*

$$\mathcal{L}\{f(x) * g(x)\} = \mathcal{L}\{f(x)\}\ \mathcal{L}\{g(x)\} = F(s)\ G(s)$$

or

$$\mathcal{L}\left\{\int_{0}^{\infty} f(t)\ g(x-t)\ dt\right\} = \mathcal{L}(f)\mathcal{L}(g) = F(s)\ G(s),$$

where $F(s)$ and $G(s)$ are the Laplace transforms of $f(x)$ and $g(x)$, respectively.

Definition ([16])**.** The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} z^k,$$

where $z, \alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Definition ([16])**.** The generalization of $E_\alpha(z)$ is defined as a function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} z^k,$$

where $z, \alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

3. Main results

Firstly, we prove the Hyers-Ulam stability of the differential equation (1) with initial condition (2).

Theorem 3.1. *The logistic growth in a population differential equation* (1) *has the Hyers-Ulam stability.*

Proof. For every $\epsilon > 0$, and assume that $u(t) \in C(I)$ satisfies the differential inequality

$$|u'(t) - u(t) + g(u)| \leq \epsilon \tag{3}$$

for all $t \in I$. We have to show that there is a real number $K > 0$ which is independent of ϵ and $u(t)$ such that $|u(t) - v(t)| \leq K\epsilon$ for some $v \in C(I)$ which satisfies the differential equation $v'(t) = v(t) - g(v)$ for all $t \in I$.

Let us consider a function $p : (0, \infty) \to \mathbb{R}$ such that $p(t) =: u'(t) - u(t) + g(u)$ for all $t > 0$. In view of (3), we have $|p(t)| \leq \epsilon$. Taking Laplace transform to $p(t)$, we have

$$\mathcal{L}\{p\} = (s-1)\mathcal{L}\{u\} - u(0) - \mathcal{L}\{g(u)\}, \tag{4}$$

$$P(s) = (s-1)\ U(s) - u_0 - G(s) \tag{5}$$

and thus

$$\mathcal{L}\{u\} = U(s) = \frac{U(s) + u(0) - G(s)}{s-1}, \tag{6}$$

where $U(s) = \mathcal{L}\{u(t)\}$, $P(s) = \mathcal{L}\{p(t)\}$ and $G(s) = \mathcal{L}\{g(u)\}$ are the Laplace transforms of the functions $u(t)$, $p(t)$ and $g(u(t))$, respectively. In view of the (4), a function $v_0 : (0, \infty) \to \mathbb{R}$ is a solution of (1) with (2) if and only if

$$(s-1)\mathcal{L}\{v_0\} = v_0(0) + \mathcal{L}(g).$$

Let us define a solution $v(t) = u_0\ e^t + (e^t * g)$, then we have $v(0) = u(0)$. Then applying the Laplace transform to $v(t)$, we obtain

$$\mathcal{L}\{v\} = V(s) = \frac{u_0 + G(s)}{(s-1)}. \tag{7}$$

On the other hand, we have

$$\mathcal{L}\{v'(t) - v(t) + g(v)\} = (s-1)V(s) - v(0) - \mathcal{L}(g).$$

Using (7), we get $\mathcal{L}\{v'(t) - v(t) + g(v)\} = 0$. Since $\mathcal{L}$ is a one-to-one operator and linear, then we get $v'(t) = v(t) - g(v)$. This means that $v(t)$ is a solution of (1) with (2). It follows from the equations (6) and (7) that

$$\mathcal{L}\{u(t)\} - \mathcal{L}\{v(t)\} = U(s) - V(s) = \frac{P(s) - G(s) + u_0}{(s-1)} - \frac{u_0 + G(s)}{(s-1)} = \frac{\mathcal{L}\{p\}}{(s-1)}$$
$$\Rightarrow\ \mathcal{L}\{u(t) - v(t)\} = \mathcal{L}\left\{p(t) * e^t\right\}.$$

The above equalities gives that $u(t)-v(t)=p(t)*e^t$. Taking modulus on both sides and using $|p(t)|\le\epsilon$, we get

$$\begin{aligned}|u(t)-v(t)| = \left|p(t)*e^t\right| &\le \left|\int_0^t p(t)\ e^{(t-x)}\ dx\right| \\ &\le \epsilon \int_0^t \left|e^{(t-x)}\right|\ dx \le \epsilon\ e^t \int_0^t e^{-x}\ dx = K\epsilon\end{aligned}$$

for all $t>0$. Hence, $|u(t)-v(t)|\le K\epsilon$. Therefore, the linear differential equation (1) with (2) has the Hyers-Ulam stability. This completes the proof. □

Similar to Theorem 3.1, we will prove the Hyers-Ulam-Rassias stability for the differential equation (1) with (2). For the sake of the completeness of this paper, we provide some part of the proof.

Theorem 3.2. *Let $\epsilon>0$, $u(t)$ be a continuously differentiable function and $\phi:(0,\infty)\to(0,\infty)$ satisfies the inequality*

$$|u'(t)-u(t)+g(u)|\le\phi(t)\epsilon \tag{8}$$

for all $t\in I$. Then there exists a real number $K>0$ which is independent of ϵ and u such that

$$|u(t)-v(t)|\le K\phi(t)\epsilon$$

for some $v\in C(I)$ satisfies the differential equation $v'(t)-v(t)+g(v)=0$ for all $t\in I$.

Proof. Consider a function $p:(0,\infty)\to\mathbb{R}$ such that $p(t)=:u'(t)-u(t)+g(u)$ for all $t>0$. In view of (8), we have $|p(t)|\le\phi(t)\epsilon$.

By applying the same procedure which is used in the proof of Theorem 3.1, we can reach that

$$u(t)-v(t)=p(t)*e^t.$$

Taking modulus on both sides and using $|p(t)|\le\phi(t)\epsilon$, we get

$$\begin{aligned}|u(t)-v(t)| = \left|p(t)*e^t\right| &\le \left|\int_0^t p(t)\ e^{(t-x)}\ dx\right| \\ &\le \epsilon\phi(t) \int_0^t \left|e^{(t-x)}\right|\ dx \le K\phi(t)\epsilon\end{aligned}$$

for all $t>0$. Then the linear differential equation (1) with (2) has the Hyers-Ulam-Rassias stability. Hence the proof. □

Finally, we shall investigate the Mittag-Leffler-Hyers-Ulam stability of the equation (1) with initial condition (2).

Theorem 3.3. *The logistic growth in a population differential equation* (1) *is Mittag-Leffler-Hyers-Ulam stable.*

Proof. Given $\epsilon > 0$. Suppose that $u(t) \in C(I)$ satisfying the differential inequality

$$|u'(t) - u(t) + g(u)| \leq \epsilon E_\alpha(t) \tag{9}$$

for all $t \in I$, where $E_\alpha(t)$ is the Mittag-Leffler function. We wish to prove that there exists a real number $K > 0$ which is independent of ϵ and $u(t)$ such that $|u(t) - v(t)| \leq K\epsilon E_\alpha(t)$ for some $v \in C(I)$ satisfies $v'(t) = v(t) - g(v)$ for all $t \in I$.

Define a function $p : (0, \infty) \to \mathbb{R}$ such that $p(t) =: u'(t) - u(t) + g(u)$ for all $t > 0$. In view of (9), we have $|p(t)| \leq \epsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, we have

$$\mathcal{L}\{p\} = (s-1)\mathcal{L}\{u\} - \ u(0) - \mathcal{L}\{g(u)\} \tag{10}$$

and thus

$$\mathcal{L}\{u\} = U(s) = \frac{U(s) + u(0) - G(s)}{s-1}. \tag{11}$$

In view of the (10), a function $v_0 : (0, \infty) \to \mathbb{R}$ is a solution of (1) with (2) if and only if

$$(s-1)\mathcal{L}\{v_0\} = v_0(0) + \mathcal{L}(g).$$

Set $v(t) = u_0 e^t + (e^t * g)$, then we have $v(0) = u(0)$. Taking Laplace transform to $v(t)$, we obtain

$$\mathcal{L}\{v\} = V(s) = \frac{u_0 + G(s)}{(s-1)}. \tag{12}$$

On the other hand,

$$\mathcal{L}\{v'(t) - v(t) + g(v)\} = (s-1)V(s) - v(0) - \mathcal{L}(g).$$

Using (12), we get $\mathcal{L}\{v'(t) - v(t) + g(v)\} = 0$. Since $\mathcal{L}$ is a one-to-one operator and linear, then we get $v'(t) = v(t) - g(v)$. This means that $v(t)$ is a solution of (1). It follows from (11) and (12) that

$$\begin{aligned}
U(s) - V(s) &= \frac{P(s) - G(s) + u_0}{(s-1)} - \frac{u_0 + G(s)}{(s-1)} = \frac{\mathcal{L}\{p\}}{(s-1)} \\
\Rightarrow\ \mathcal{L}\{u(t) - v(t)\} &= \mathcal{L}\left\{p(t) * e^t\right\}.
\end{aligned}$$

The above equalities show that $u(t) - v(t) = p(t) * e^t$. Taking modulus on both sides and using $|p(t)| \leq \epsilon E_\alpha(t)$, we get

$$\begin{aligned}
|u(t) - v(t)| = \left|p(t) * e^t\right| &\leq \left| \int_0^t p(t)\ e^{(t-x)}\ dx \right| \\
&\leq |p(t)|\ \left| \int_0^t e^{(t-x)}\ dx \right| \\
&\leq \epsilon E_\alpha(t)\ \left| \int_0^t e^{(t-x)}\ dx \right|
\end{aligned}$$

for all $t > 0$, where $K = \left|\int_0^t e^{(t-x)}\ dx\right|$ exists. Hence, $|u(t) - v(t)| \leq K\epsilon E_\alpha(t)$. Then the linear differential equation (1) with (2) has the Mittag-Leffler-Hyers-Ulam stability. $\square$

In analogous to Theorem 3.3, we have the following result which shows the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (1). The method of proof is similar, but we still state it for the sake of completeness.

Theorem 3.4. *The logistic growth in a population equation* (1) *with* (2) *has the Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Given $\epsilon > 0$. Suppose that $u(t) \in C(I)$ and $\phi(t) : (0,\infty) \to (0,\infty)$ satisfying

$$|u'(t) - u(t) + g(u)| \leq \phi(t)\epsilon E_\alpha(t) \tag{13}$$

for all $t \in I$. We wish to prove that there exists a real number $K > 0$ which is independent of ϵ and u such that

$$|u(t) - v(t)| \leq K\phi(t)\epsilon E_\alpha(t)$$

for some $v \in C(I)$ satisfies $v'(t) - v(t) + g(v) = 0$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbb{R}$ such that $p(t) =: u'(t) - u(t) + g(u)$ for all $t > 0$.

In view of (13), we have

$$|p(t)| \leq \phi(t)\epsilon E_\alpha(t). \tag{14}$$

By using the same technique in as applied in the proof of Theorem 3.3 and using equation (14), we get

$$|u(t) - v(t)| \leq \phi(t)\epsilon E_\alpha(t)\left|\int_0^t e^{(t-x)}\ dx\right|$$

for all $t > 0$, where $K = \left|\int_0^t e^{(t-x)}\ dx\right|$ exists. Hence,

$$|u(t) - v(t)| \leq K\phi(t)\epsilon E_\alpha(t).$$

Thus the linear differential equation (1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. This finishes the proof. $\square$

4. Application

In this section, we will introduce some examples to make it easier to understand the main results of this paper.

First, we study the logistic growth model in a population

$$\frac{dw}{dt} = \xi w\left(1 - \frac{w}{\eta}\right), \tag{15}$$

where ξ and η are positive constants. Here $w = w(t)$ represents the population of the species at time t and $\xi w\left(1 - \frac{w}{\eta}\right)$ is the per capita growth rate, and η is the carrying capacity of the environment. Non-dimensionalization of equation (15) by setting

$$u(t) = \frac{w(t)}{\eta}, \quad t = \xi\tau,$$

results in

$$\frac{du}{dt} = u\,(1-u). \tag{16}$$

If $w(0) = w_0$, then $u(0) = \frac{w_0}{\eta}$, and the analytical solution of the equation (16) follows easily

$$u(\tau) = \frac{1}{1 + \left(\dfrac{\eta}{w_0 - 1}\right) e^{-t}}.$$

Now, we will apply Theorem 3.1 to the equation (16) to establish the Ulam stability.

Example 4.1. We consider the following logistic differential equation (16), it can be written as $u'(t) = u(t) - u^2$, where $g(u) = u^2$ is a nonlinear function.

If a continuously differentiable function $w : [0, \infty) \to \mathbb{R}$ of exponential order satisfies $|u'(t) - u(t) + u^2| \leq \epsilon$ for all $t \geq 0$ and for some $\epsilon > 0$, then Theorem 3.1 implies that there exists a solution $y : [0, \infty) \to \mathbb{R}$ of the differential equation (16) such that $y(t)$ is of exponential order and $|w(t) - y(t)| \leq K\epsilon$ for all $t \geq 0$.

Example 4.2. We consider the following non-homogeneous linear differential equation

$$u'(t) = u(t) - u^3, \tag{17}$$

where $g(u) = u^3$ is a nonlinear function. If a continuously differentiable function $w : [0, \infty) \to \mathbb{R}$ of exponential order satisfies

$$|u'(t) - u(t) + u^3| \leq \epsilon$$

for all $t \geq 0$ and for some $\epsilon > 0$, then Theorem 3.1 implies that there exists a solution $y : [0, \infty) \to \mathbb{R}$ of the differential equation (16) such that $y(t)$ is of exponential order and $|w(t) - y(t)| \leq K\epsilon$, for $t \geq 0$.

Remark 4.3. The above examples are also true when we replace ϵ and $K\epsilon$ with $\phi(t)\epsilon$ and $K\phi(t)\varepsilon$, respectively, where $\phi(t)$ is an increasing function. In this case, we see that the corresponding differential equations have the Hyers-Ulam-Rassias stability.

Remark 4.4. The differential equations (16) and (17) have the Mittag-Leffler-Hyers-Ulam stability. In particular, they also have the Mittag-Leffler-Hyers-Ulam-Rassias stability when $\phi(t)$ is an increasing function.

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