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GENERALIZED FOURIER–FEYNMAN TRANSFORMS AND CONVOLUTIONS FOR EXPONENTIAL TYPE FUNCTIONS OF GENERALIZED BROWNIAN MOTION PATHS

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Abstract. Let $C_{a,b}[0,T]$ denote the space of continuous sample paths of a generalized Brownian motion process (GBMP). In this paper, we study the structures which exist between the analytic generalized Fourier–Feynman transform (GFFT) and the generalized convolution product (GCP) for functions on the function space $C_{a,b}[0,T]$. For our purpose, we use the exponential type functions on the general Wiener space $C_{a,b}[0,T]$. The class of all exponential type functions is a fundamental set in $L_2(C_{a,b}[0,T])$.

1. Introduction

The concept of the Fourier–Wiener transform (FWT) introduced by Cameron and Martin [1, 2, 20] is now playing significant role in infinite dimensional functional analysis. The FWT and several analogies have developed in various research fields on infinite dimensional Banach spaces. For instance, the analytic Fourier–Feynman transform [10–13, 19] and the integral transform [6, 17, 18] are developed by many authors. In particular, Lee [15, 16] provided applications of the FWT to the study of differential equations on infinite dimensional Banach spaces. He defined the FWT on a class of exponential type analytic functions on abstract Wiener space and established theorems guaranteeing existence and regularity of solutions of the Cauchy problem. Also, Kuo [14] obtained several results involving the FWT of Brownian functionals, and used these results to solve a differential equation.

On the other hand, the concept of the "convolution product" corresponding to each transform and the structure between the transforms and the corresponding convolutions also have been established in the literature. Let B be an abstract Wiener space (or its complexification), let $\mathcal{F}$ denote the one of

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the transforms commented above, and let $*$ be a corresponding convolution product. Then the results in [6, 10–13, 17–20] say that

$$\mathcal{F}(F * G) = \mathcal{F}(F)\mathcal{F}(G) \tag{1.1}$$

under appropriate condition and with appropriate functions F and G on B.

In [5, 7], the authors used a GBMP to define an analytic GFFT for functions on a general Wiener space $C_{a,b}[0,T]$. By a study of Yeh [21, 22], the GBMP associated with continuous functions $a(\cdot)$ and $b(\cdot)$ induces the space $C_{a,b}[0,T]$. We refer the references [3–5, 7, 21, 22] for more detailed information about the definition of the GBMP and the construction of the function space $C_{a,b}[0,T]$. A standard Brownian motion is stationary in time, while in general, the GBMP is not stationary in time and is subject to the time dependent drift $a(t)$. From this reason, the GFFT and the GCP do not satisfy the homomorphism structure such as (1.1). For more details, see [8, Section 2].

Based on these background, we in this paper study the relationships between the GFFTs and the GCPs defined for functions on the very general function space $C_{a,b}[0,T]$. By an unusual behavior of the drift $a(t)$ of the GBMP, the relationships (see Theorems 3.3 and 3.5 below) between the transforms and the convolutions are more complicated than the relationships studied on the abstract Wiener space B.

2. Transforms and convolutions

In this section, we investigate interesting relationships between the GFFTs and the GCP for exponential type functions on $C_{a,b}[0,T]$. We adopt the notation and terminologies of those papers on the assumption that readers are familiar with the references [5, 7]. The basic concepts and definitions of the function space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ which forms a complete probability space, the scale-invariant measurability on $C_{a,b}[0,T]$, the Cameron–Martin space $(C'_{a,b}[0,T], (\cdot,\cdot)_{C'_{a,b}}, \|\cdot\|_{C'_{a,b}})$, and the analytic generalized Feynman integral $E_x^{\mathrm{anf}_q}[F(x)]$ may also be found in [3, 4]. In particular, we follow the definition in [3, 4] for the Paley–Wiener–Zygmund stochastic integral $(w,x)^\sim$. However, in order to propose our assertions in this paper, we shall restate the following definitions of the GFFT and the GCP.

Definition 2.1. Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let F be a complex-valued scale-invariant measurable function such that the function space integral $J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)d\mu(x)$ exists and is finite for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ. For $\lambda \in \mathbb{C}_+$, we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda).$$

For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y+x)].$$

For $p \in (1,2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula,

$$T_q^{(p)}(F)(y) = \underset{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}}{\mathrm{l.\,i.\,m.}}\, T_\lambda(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'} d\mu(y) = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula,

$$T_q^{(1)}(F)(y) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda(F)(y) \tag{2.1}$$

for s-a.e. $y \in C_{a,b}[0,T]$ if the limit exists.

We note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

Definition 2.2. Let F and G be scale-invariant measurable functions on $C_{a,b}[0,T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$, we define their GCP $(F * G)_\lambda$ (if it exists) by

$$(F * G)_\lambda(y) = \begin{cases} E_x^{\mathrm{an}_\lambda}\big[F\big(\frac{y+x}{\sqrt{2}}\big)G\big(\frac{y-x}{\sqrt{2}}\big)\big], & \lambda \in \mathbb{C}_+ \\ E_x^{\mathrm{anf}_q}\big[F\big(\frac{y+x}{\sqrt{2}}\big)G\big(\frac{y-x}{\sqrt{2}}\big)\big], & \lambda = -iq,\ q \in \mathbb{R} \setminus \{0\}. \end{cases}$$

When $\lambda = -iq$, we denote $(F * G)_\lambda$ by $(F * G)_q$.

Let $\mathcal{E}$ be the class of functions having the form

$$\Psi_w(x) = \exp\{(w,x)^\sim\} \tag{2.2}$$

for some $w \in C'_{a,b}[0,T]$ and for s-a.e. $x \in C_{a,b}[0,T]$, where $(w,x)^\sim$ denotes the Paley–Wiener–Zygmund stochastic integral [3, 4]. The functions given by equation (2.2) and linear combinations (with complex coefficients) of the Ψ_w's are called the (partially) exponential type functions on $C_{a,b}[0,T]$.

It was shown in [9] that the class $\mathcal{E} = \{\Psi_w : w \in C'_{a,b}[0,T]\}$ is a fundamental set in $L_2(C_{a,b}[0,T])$. Let

$$\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}.$$

Every exponential type function is scale-invariant measurable. Since we shall identify functions which coincide s-a.e. on $C_{a,b}[0,T]$, $\mathcal{E}(C_{a,b}[0,T])$ can be considered as the space of all s-equivalence classes of partially exponential type functions. For a more detailed illustration of the class $\mathcal{E}(C_{a,b}[0,T])$, see [3].

The following theorems are due to Chang and Choi [3].

Theorem 2.3. *Let $\Psi_w \in \mathcal{E}$ be given by equation (2.2). Then for all $p \in [1,2]$ and all $q \in \mathbb{R}\setminus\{0\}$, the L_p analytic GFFT of Ψ_w, $T_q^{(p)}(\Psi_w)$ exists and is given by the formula*

$$T_q^{(p)}(\Psi_w)(y) = \exp\left\{\frac{i}{2q}\|w\|^2_{C'_{a,b}} + (-iq)^{-1/2}(w,a)_{C'_{a,b}}\right\}\Psi_w(y) \tag{2.3}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Thus, $T_q^{(p)}(\Psi_w)$ is an element of $\mathcal{E}(C_{a,b}[0,T])$.

Theorem 2.4. *Given any $p \in [1,2]$ and $q \in \mathbb{R}\setminus\{0\}$, the L_p analytic GFFT, $T_q^{(p)} : \mathcal{E}(C_{a,b}[0,T]) \to \mathcal{E}(C_{a,b}[0,T])$ is an onto transform.*

Theorem 2.5. *Let Ψ_{w_1} and Ψ_{w_2} be exponential type functions in $\mathcal{E}$. Then the GCP of Ψ_{w_1} and Ψ_{w_2}, $(\Psi_{w_1} * \Psi_{w_2})_q$ exists for all $q \in \mathbb{R}\setminus\{0\}$ and is given by the formula*

$$\begin{aligned}(\Psi_{w_1} * \Psi_{w_2})_q(y)& \\ = \exp\left\{\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}} + (-2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\right\}&\Psi_{\frac{w_1+w_2}{\sqrt{2}}}(y)\end{aligned} \tag{2.4}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, $(\Psi_{w_1} * \Psi_{w_2})_q$ is an element of $\mathcal{E}(C_{a,b}[0,T])$.

Given any exponential type functions F and G in $\mathcal{E}(C_{a,b}[0,T])$, F and G can be written as

$$F \approx \sum_{j=1}^n \alpha_j \Psi_{w_j} \text{ and } G \approx \sum_{k=1}^m \beta_l \Psi_{\tau_k},$$

respectively, for finite sequences $\{w_1,\ldots,w_n\}$ and $\{\tau_1,\ldots,\tau_m\}$ in $C'_{a,b}[0,T]$, and finite sequences $\{\alpha_1,\ldots,\alpha_n\}$ and $\{\beta_1,\ldots,\beta_m\}$ in $\mathbb{C}\setminus\{0\}$, since $\mathcal{E}(C_{a,b}[0,T])$ $=$ Span$\mathcal{E}$. Thus, using the linearity of the L_p analytic GFFT $T_q^{(p)}$, the bilinearity of the GCP $(\,\cdot * \cdot\,)_q$, (2.3), and (2.4), it follows that for each $p \in [1,2]$,

$$\begin{aligned}T_q^{(p)}(F) &\approx \sum_{j=1}^n \alpha_j T_q^{(p)}(\Psi_{w_j}) \\ &\approx \sum_{j=1}^n \alpha_j \exp\left\{\frac{i}{2q}\|w_j\|^2_{C'_{a,b}} + (-iq)^{-1/2}(w_j,a)_{C'_{a,b}}\right\}\Psi_{w_j}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&(F*G)_q \\ \approx\ & \sum_{j=1}^n\sum_{k=1}^m \alpha_j\beta_k(\Psi_{w_j} * \Psi_{\tau_k})_q \\ \approx\ & \sum_{j=1}^n\sum_{k=1}^m \alpha_j\beta_k \exp\left\{\frac{i}{4q}\|w_j-\tau_k\|^2_{C'_{a,b}} + (-2iq)^{-1/2}(w_j-\tau_k,a)_{C'_{a,b}}\right\}\Psi_{\frac{w_j+\tau_k}{\sqrt{2}}},\end{aligned}$$

respectively.

In view of equations (2.5) and (2.3), we see that for every function in $\mathcal{E}(C_{a,b}[0,T])$, $T_q^{(p)}(F) \approx T_q^{(1)}(F)$ for all $p \in (1,2]$. Thus throughout the remainder of this paper, we work with the L_1 analytic GFFT for our assertions.

3. Relationship between GFFTs and GCPs

In view of equation (2.1), it follows that for s-a.e. $y \in C_{a,b}[0,T]$,

$$T_q^{(1)}(F)(y) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda(F)(y) = E_x^{\mathrm{anf}_q}[F(y+x)].$$

However, by the effect of the drift function $a(t)$ of the GBMP, we see that

$$E_x^{\mathrm{an}_\lambda}[F(x)] \neq E_x^{\mathrm{an}_\lambda}[F(-x)]$$

for almost every function F on $C_{a,b}[0,T]$. This yields the facts that

$$T_q^{(1)}(F)(y) \neq E^{\mathrm{anf}_q}[F(y-x)]$$

and the GCP of functions on $C_{a,b}[0,T]$ is not commutative.

The above discussion leads us to the following definition in order to specify another function space transform on $C_{a,b}[0,T]$. Given a scale-invariant measurable function F on $C_{a,b}[0,T]$, let

$$T_{q,+}^{(1)}(F)(y) = T_q^{(1)}(F)(y) \tag{3.1}$$

and let

$$T_{q,-}^{(1)}(F)(y) = E^{\mathrm{anf}_q}[F(y-x)]. \tag{3.2}$$

The generalized transforms $T_{q,+}^{(1)}(F)$ and $T_{q,-}^{(1)}(F)$ are called the p-GFFT and the n-GFFT, respectively, of scale-invariant measurable functions F on $C_{a,b}[0,T]$.

Lemma 3.1. *Let $\Psi_w \in \mathcal{E}$ be given by equation* (2.2)*. Then for all nonzero real numbers q and ρ, it follows that*

$$E_x^{\mathrm{anf}_q}[\Psi_w(\rho x)] = \exp\left\{ \frac{i\rho^2}{2q}\|w\|^2_{C'_{a,b}} + \rho(-iq)^{-1/2}(w,a)_{C'_{a,b}} \right\}. \tag{3.3}$$

Using (3.3) with $\rho = -1$, we have the following lemma.

Lemma 3.2. *Let $\Psi_w \in \mathcal{E}$ be given by equation* (2.2)*. Then for all $q \in \mathbb{R} \setminus \{0\}$, the analytic n-GFFT $T_{q,-}^{(1)}(\Psi_w)$ of Ψ_w exists and is given by the formula*

$$T_{q,-}^{(1)}(\Psi_w)(y) = \exp\left\{ \frac{i}{2q}\|w\|^2_{C'_{a,b}} - (-iq)^{-1/2}(w,a)_{C'_{a,b}} \right\}\Psi_w(y) \tag{3.4}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Thus, $T_{q,-}^{(1)}(\Psi_w)$ is an element of $\mathcal{E}(C_{a,b}[0,T])$.

We are now ready to provide our main assertions. In our next theorem, we establish that the p-GFFT of the GCP of functions F and G in $\mathcal{E}(C_{a,b}[0,T])$ is the product of their iterated transforms.

Theorem 3.3. *For any exponential type functions F and G in $\mathcal{E}(C_{a,b}[0,T])$, it follows that*

$$T^{(1)}_{q,+}\big((F*G)_q\big)(y)=T^{(1)}_{2q,+}\big(T^{(1)}_{2q,+}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T^{(1)}_{2q,-}\big(T^{(1)}_{2q,+}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big) \tag{3.5}$$

for all $q\in\mathbb{R}\setminus\{0\}$ and s-a.e. $y\in C_{a,b}[0,T]$.

Proof. In order to establish equation (3.5), it will suffice to show that for any functions Ψ_{w_1} and Ψ_{w_2} in $\mathcal{E}$,

$$\begin{aligned} &T^{(1)}_{q,+}\big((\Psi_{w_1}*\Psi_{w_2})_q\big)(y)\\ &=T^{(1)}_{2q,+}\big(T^{(1)}_{2q,+}(\Psi_{w_1})\big)\Big(\frac{y}{\sqrt{2}}\Big)T^{(1)}_{2q,-}\big(T^{(1)}_{2q,+}(\Psi_{w_2})\big)\Big(\frac{y}{\sqrt{2}}\Big)\end{aligned} \tag{3.6}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Using (2.3) with Ψ_w replaced with $(\Psi_{w_1}*\Psi_{w_2})_q$, and (2.4), it follows that

$$\begin{aligned} &T^{(1)}_{q,+}\big((\Psi_{w_1}*\Psi_{w_2})_q\big)(y)\\ &=\exp\bigg\{\frac{i}{2q}\Big\|\frac{w_1+w_2}{\sqrt{2}}\Big\|^2_{C'_{a,b}}+(-iq)^{-1/2}\Big(\frac{w_1+w_2}{\sqrt{2}},a\Big)_{C'_{a,b}}\bigg\}(\Psi_{w_1}*\Psi_{w_2})_q(y)\\ &=\exp\bigg\{\frac{i}{4q}\|w_1+w_2\|^2_{C'_{a,b}}+(-2iq)^{-1/2}(w_1+w_2,a)_{C'_{a,b}}\bigg\}\\ &\quad\times\exp\bigg\{\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}}+(-2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\bigg\}\Psi_{\frac{w_1+w_2}{\sqrt{2}}}(y)\\ &=\exp\bigg\{\frac{i}{2q}\|w_1\|^2_{C'_{a,b}}+2(-2iq)^{-1/2}(w_1,a)_{C'_{a,b}}\bigg\}\Psi_{w_1}\Big(\frac{y}{\sqrt{2}}\Big)\\ &\quad\times\exp\bigg\{\frac{i}{2q}\|w_2\|^2_{C'_{a,b}}\bigg\}\Psi_{w_2}\Big(\frac{y}{\sqrt{2}}\Big)\end{aligned} \tag{3.7}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Next, using equation (3.1) with q replaced with $2q$ and applying (2.3) two times, it follows that

$$\begin{aligned} &T^{(1)}_{2q,+}\big(T^{(1)}_{2q,+}(\Psi_{w_1})\big)\Big(\frac{y}{\sqrt{2}}\Big)\\ &=\exp\bigg\{\frac{i}{2q}\|w_1\|^2_{C'_{a,b}}+2(-i2q)^{-1/2}(w_1,a)_{C'_{a,b}}\bigg\}\Psi_{w_1}\Big(\frac{y}{\sqrt{2}}\Big)\end{aligned} \tag{3.8}$$

for s-a.e. $y\in C_{a,b}[0,T]$, and using equation (3.4) with q and Ψ_w replaced with $2q$ and $T^{(1)}_{2q,+}(\Psi_{w_2})$, respectively, it also follows that

$$T^{(1)}_{2q,-}\big(T^{(1)}_{2q,+}(\Psi_{w_2})\big)\Big(\frac{y}{\sqrt{2}}\Big)=\exp\bigg\{\frac{i}{2q}\|w_2\|^2_{C'_{a,b}}\bigg\}\Psi_{w_2}\Big(\frac{y}{\sqrt{2}}\Big) \tag{3.9}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Equation (3.6) follows now from (3.7), (3.8) and (3.9). ∎

Remark 3.4. Using a same method used in the proof of Theorem 3.3, one can show that for every exponential type function F in $\mathcal{E}(C_{a,b}[0,T])$,

$$T_{q,+}^{(1)}\big(T_{q,-}^{(1)}(F)\big) \approx T_{q,-}^{(1)}(T_{q,+}^{(1)}(F))$$

for all $q \in \mathbb{R} \setminus \{0\}$. Thus equation (3.5) can be rewritten by the formula

$$T_{q,+}^{(1)}\big((F * G)_q\big)(y) = T_{2q,+}^{(1)}\big(T_{2q,+}^{(1)}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q,+}^{(1)}\big(T_{2q,-}^{(1)}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in C_{a,b}[0,T]$. Also, the following relationship between the n-GFFT and the GCP follows readily from the techniques developed in the proof of Theorem 3.3:

$$\begin{aligned} T_{q,-}^{(1)}\big((F * G)_q\big)(y) &= T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q,-}^{(1)}\big(T_{2q,-}^{(1)}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big) \\ &= T_{2q,+}^{(1)}\big(T_{2q,-}^{(1)}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q,-}^{(1)}\big(T_{2q,-}^{(1)}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big) \end{aligned} \tag{3.10}$$

for all $q \in \mathbb{R} \setminus \{0\}$ and s-a.e. $y \in C_{a,b}[0,T]$.

Theorem 3.5. *For any exponential type functions F and G in $\mathcal{E}(C_{a,b}[0,T])$, it follows that*

$$\begin{aligned} &\Big(T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(F)\big) * T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(G)\big)\Big)_{-q}(y) \\ &= T_{q,-}^{(1)}\bigg(T_{-2q,+}^{(1)}\big(T_{2q,+}^{(1)}(F)\big)\Big(\frac{\cdot}{\sqrt{2}}\Big)T_{-2q,-}^{(1)}\big(T_{2q,+}^{(1)}(G)\big)\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y) \end{aligned} \tag{3.11}$$

for all $q \in \mathbb{R} \setminus \{0\}$ and s-a.e. $y \in C_{a,b}[0,T]$.

Proof. It also suffices to show that equation (3.11) holds with F and G replaced with exponential functions Ψ_{w_1} and Ψ_{w_2} in $\mathcal{E}$.

Using (2.3) and (3.4), it first follows that

$$T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(\Psi_{w_1})\big)(y) = \exp\bigg\{\frac{i}{2q}\|w_1\|_{C'_{a,b}}^2\bigg\}\Psi_{w_1}(y)$$

and

$$T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(\Psi_{w_2})\big)(y) = \exp\bigg\{\frac{i}{2q}\|w_2\|_{C'_{a,b}}^2\bigg\}\Psi_{w_2}(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$. Next, applying (2.4) with q replaced with $-q$, it follows that

$$\begin{aligned} &\Big(T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(\Psi_{w_1})\big) * T_{2q,-}^{(1)}\big(T_{2q,+}^{(1)}(\Psi_{w_2})\big)\Big)_{-q}(y) \\ &= \exp\bigg\{\frac{i}{2q}\Big(\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2\Big)\bigg\}(\Psi_{w_1} * \Psi_{w_2})_{-q}(y) \\ &= \exp\bigg\{\frac{i}{2q}\Big(\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2\Big)\bigg\} \end{aligned} \tag{3.12}$$

$$\times \exp\bigg\{-\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}}+(2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\bigg\}\Psi_{\frac{w_1+w_2}{\sqrt{2}}}(y)$$

$$= \exp\bigg\{\frac{i}{4q}\|w_1+w_2\|^2_{C'_{a,b}}+(2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\bigg\}\Psi_{\frac{w_1+w_2}{\sqrt{2}}}(y)$$

for s-a.e. $y\in C_{a,b}[0,T]$.

On the other hand, applying (2.3) two times, it follows that

$$\begin{aligned}&T^{(1)}_{-2q,+}\big(T^{(1)}_{2q,+}(\Psi_{w_1})\big)(y)\\&= \exp\bigg\{(-2iq)^{-1/2}(w_1,a)_{C'_{a,b}}+(2iq)^{-1/2}(w_1,a)_{C'_{a,b}}\bigg\}\Psi_{w_1}(y)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$, and applying (3.4), it also follows that

$$\begin{aligned}&T^{(1)}_{-2q,-}\big(T^{(1)}_{2q,+}(\Psi_{w_2})\big)(y)\\&= \exp\bigg\{(-2iq)^{-1/2}(w_2,a)_{C'_{a,b}}-(2iq)^{-1/2}(w_2,a)_{C'_{a,b}}\bigg\}\Psi_{w_2}(y)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$. Next, using (3.2) with F replaced with

$$T^{(1)}_{-2q,+}(T^{(1)}_{2q,+}(\Psi_{w_1}))(\cdot/\sqrt{2})T^{(1)}_{-2q,-}(T^{(1)}_{2q,+}(\Psi_{w_2}))(\cdot/\sqrt{2}),$$

and (3.3) with $w=w_1+w_2$ and $\rho=-1/\sqrt{2}$, it follows that

$$\begin{aligned}(3.13)\quad &T^{(1)}_{q,-}\bigg(T^{(1)}_{-2q,+}\big(T^{(1)}_{2q,+}(\Psi_{w_1})\big)\Big(\frac{\cdot}{\sqrt{2}}\Big)T^{(1)}_{-2q,-}\big(T^{(1)}_{2q,+}(\Psi_{w_2})\big)\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y)\\
&= E^{\mathrm{anf}_q}_x\bigg[T^{(1)}_{-2q,+}\big(T^{(1)}_{2q,+}(\Psi_{w_1})\big)\Big(\frac{y-x}{\sqrt{2}}\Big)T^{(1)}_{-2q,-}\big(T^{(1)}_{2q,+}(\Psi_{w_2})\big)\Big(\frac{y-x}{\sqrt{2}}\Big)\bigg]\\
&= E^{\mathrm{anf}_q}_x\bigg[\Psi_{w_1}\Big(\frac{y-x}{\sqrt{2}}\Big)\Psi_{w_2}\Big(\frac{y-x}{\sqrt{2}}\Big)\bigg]\\
&\quad\times\exp\bigg\{(-2iq)^{-1/2}(w_1+w_2,a)_{C'_{a,b}}+(2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\bigg\}\\
&= E^{\mathrm{anf}_q}_x\bigg[\exp\bigg\{-\frac{1}{\sqrt{2}}(w_1+w_2,x)^{\sim}\bigg\}\bigg]\\
&\quad\times\exp\bigg\{(-2iq)^{-1/2}(w_1+w_2,a)_{C'_{a,b}}+(2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\bigg\}\Psi_{\frac{w_1+w_2}{\sqrt{2}}}(y)\\
&= \exp\bigg\{\frac{i}{4q}\|w_1+w_2\|^2_{C'_{a,b}}+(2iq)^{-1/2}(w_1-w_2,a)_{C'_{a,b}}\bigg\}\Psi_{\frac{w_1+w_2}{\sqrt{2}}}(y)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Now (3.12) and (3.13) yield equation (3.11) with F and G replaced with Ψ_{w_1} and Ψ_{w_2} as desired. □

We finish this section with generalized Feynman integration formulas involving the GFFTs and the GCP.

Corollary 3.6. *For any exponential type functions F and G in $\mathcal{E}(C_{a,b}[0,T])$, it follows that*

$$\begin{aligned}(3.14)\qquad &\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{-q}} T_{q,+}^{(1)}\big((F*G)_q\big)(y)d\mu(y)\\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q}} T_{-2q,+}^{(1)}\big(T_{2q,+}^{(1)}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{-2q,+}^{(1)}\big(T_{2q,+}^{(1)}(G)\big)\Big(-\frac{y}{\sqrt{2}}\Big)d\mu(y)\end{aligned}$$

and

$$\begin{aligned}(3.15)\qquad &\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{-q}} T_{q,-}^{(1)}\big((F*G)_q\big)(y)d\mu(y)\\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q}} T_{-2q,+}^{(1)}\big(T_{2q,-}^{(1)}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{-2q,+}^{(1)}\big(T_{2q,-}^{(1)}(G)\big)\Big(-\frac{y}{\sqrt{2}}\Big)d\mu(y)\end{aligned}$$

for all $q \in \mathbb{R}\setminus\{0\}$, where $\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)d\mu(x)$ denotes the analytic generalized Feynman integral of F with parameter q [3, 5, 7].

4. Final remark with the previous works

Choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then the GBMP associated with the functions $a(\cdot)$ and $b(\cdot)$ reduces a standard Brownian motion (Wiener process), and so the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. In this case, it follows that

$$T_q^{(1)}(F) \approx T_{q,+}^{(1)}(F) \approx T_{q,-}^{(1)}(F)$$

for all scale-invariant measurable functions F on $C_0[0,T]$. Thus equations (3.5) and (3.10) are rewritten as

$$(4.1)\qquad T_q^{(1)}\big((F*G)_q\big)(y) = T_{2q}^{(1)}\big(T_{2q}^{(1)}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1)}\big(T_{2q}^{(1)}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in C_0[0,T]$. Furthermore, a close examination of the right-hand side of (4.1) shows that for any functions F and G in $\mathcal{E}(C_{a,b}[0,T])$,

$$(4.2)\qquad T_q^{(1)}\big((F*G)_q\big)(y) = T_q^{(1)}(F)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(1)}(G)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in C_0[0,T]$. This result subsumes similar known results obtained by Huffman, Park and Skoug [10–13].

Also, in the case that $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, it follows that

$$T_{-q,+}^{(1)}(T_q^{(1)}(\Psi_w)) \approx T_{-q,-}^{(1)}(T_q^{(1)}(\Psi_w)) \approx \Psi_w$$

for all exponential functions $\Psi_w \in \mathcal{E}$. From this and in view of Theorem 2.4, one can conclude that the inverse transform $\{T_q^{(1)}\}^{-1}$ of $T_q^{(1)}$ is equal to $T_{-q}^{(1)}$

on $\mathcal{E}(C_0[0,T])$. Under these arguments, equation (3.11) is rewritten by

$$\big(T_q^{(1)}(F) * T_q^{(1)}(G)\big)_{-q}(y) = T_q^{(1)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y) \tag{4.3}$$

for s-a.e. $y \in C_{a,b}[0,T]$. This result subsumes similar a known result obtained by Park, Skoug and Storvick [19, Theorem 3.1].

Under the setting $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, equations (3.14) and (3.15) yield the same equation, respectively, as follows.

$$\int_{C_0[0,T]}^{\mathrm{anf}_{-q}} T_q^{(1)}\big((F*G)_q\big)(y)d\mu(y) = \int_{C_0[0,T]}^{\mathrm{anf}_q} F\Big(\frac{y}{\sqrt{2}}\Big)G\Big(-\frac{y}{\sqrt{2}}\Big)d\mu(y).$$

This result subsumes similar a known result obtained by Huffman, Park and Skoug [11, Theorem 3.4].

However, the GBMP has a time dependent drift, i.e., the GBMPs are not centered Gaussian processes. Thus the GFFTs $T_{q,+}^{(1)}$ and $T_{q,-}^{(1)}$ for functions on $C_{a,b}[0,T]$ do not have complete homomorphism structures, such as (4.2) and (4.3), with their GCP.

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