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APPROXIMATELY LOCAL DERIVATIONS ON ℓ^1-MUNN ALGEBRAS WITH APPLICATIONS TO SEMIGROUP ALGEBRAS

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Abstract. At the present paper, we investigate bounded approximately local derivations of ℓ^1-Munn algebra $\mathbb{M}_I(\mathcal{A})$, where I is an arbitrary nonempty set and $\mathcal{A}$ is an approximately locally unital Banach algebra. Indeed, we show that if ${}_{\mathcal{A}}B(\mathcal{A},\mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A},\mathcal{A}^*)$ are reflexive, then every bounded approximately local derivation from $\mathbb{M}_I(\mathcal{A})$ into any Banach $\mathbb{M}_I(\mathcal{A})$-bimodule X is a derivation. Finally, we apply this result to study bounded approximately local derivations of the semigroup algebra $\ell^1(S)$, where S is a uniformly locally finite inverse semigroup.

1. Introduction and preliminaries

Suppose that $\mathcal{A}$ is a Banach algebra and X is a Banach $\mathcal{A}$-bimodule. We recall that a linear map $D : \mathcal{A} \to X$ is a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

We also say that D is a *local derivation* if for each $a \in \mathcal{A}$ there exists a derivation $D_a : \mathcal{A} \to X$ such that $D_a(a) = D(a)$. Kadison initiated the study of the notion of local derivation in 1990 by showing that continuous local derivations from a von Neumann algebra $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule X are derivations [10, Theorem A]. B. E. Johnson in [9, Theorem 5.3] gave a generalization of this result and proved that every local derivation from a C^*-algebra $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule X is a derivation. Moreover, E. Samei in [15] investigated local derivations over certain commutative semisimple Banach algebras and applied them for the Fourier algebra $A(G)$, where G is a locally compact group. For more details about local derivations and related topics, we refer the reader to [1], [6], [7] and [16].

On the other hand, E. Samei in [14] introduced and studied the more general concept of approximately local derivations. Indeed, an operator $D : \mathcal{A} \to X$

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is an *approximately local derivation* if for each $a \in \mathcal{A}$, there is a sequence of derivations (possibly unbounded) $\{D_{a,n}\}_{n=1}^{\infty}$ from $\mathcal{A}$ into X such that

$$D(a) = \lim_{n\to\infty} D_{a,n}(a).$$

In the case when D is bounded, we say that D is a bounded approximately local derivation. As a generalization of [9, Theorem 5.3], it is shown in [14] that every approximately local derivation from a C^*-algebra, a Banach algebra generated by idempotents, or a semisimple annihilator Banach algebra is a derivation.

It is worthwhile to mention that approximate local derivations problems have relationship with the study of reflexivity of linear space of bounded derivations introduced by Larson in [11]. Let X and Y be Banach spaces and $B(X, Y)$ be the space of bounded linear operators from X into Y. We say that $\mathcal{V} \subseteq B(X, Y)$ is *reflexive* if $\mathrm{ref}(\mathcal{V}) = \mathcal{V}$, where

$$\mathrm{ref}(\mathcal{V}) = \{T \in B(X, Y) : T(x) \in [\mathcal{V}x] \text{ for all } x \in X\},$$

and $[\mathcal{V}x]$ is the norm-closure of $\{V(x) : V \in \mathcal{V}\}$.

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. Throughout the paper, we denote the space of bounded derivations from $\mathcal{A}$ into X by $\mathcal{Z}^1(\mathcal{A}, X)$. Also, we assume that

$${}_{\mathcal{A}}B(\mathcal{A}, X) = \{T \in B(\mathcal{A}, X) : T(ab) = a \cdot T(b) \text{ for all } a, b \in \mathcal{A}\},$$

is the space of bounded right multipliers from $\mathcal{A}$ into X. Similarly, $B_{\mathcal{A}}(\mathcal{A}, X)$ denotes the space of bounded left multipliers from $\mathcal{A}$ into X. It is clear that $\mathcal{Z}^1(\mathcal{A}, X)$ is reflexive, if every bounded approximately local derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule X is a derivation. Moreover, it is proved that if $\mathcal{A}$ is an approximately locally unital Banach algebra such that ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^*)$ are reflexive, then every bounded approximately local derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule X is a derivation, see [14, Theorem 4.4]. Furthermore, for the group algebra $L^1(G)$, some results related to the reflexivity of $\mathcal{Z}^1(L^1(G), X)$ have been obtained in [15].

In this paper, our aim is to study reflexivity of the space of bounded derivations related to certain inverse semigroup algebras. For this purpose, we need to consider bounded approximately local derivations of the ℓ^1-Munn algebra $\mathbb{M}_I(\mathcal{A})$, where I is an arbitrary non-empty set and $\mathcal{A}$ is an approximately locally unital Banach algebra. In the particular case when $\mathfrak{A} = \mathbb{M}_I(\mathbb{C})$, we show that the spaces of ${}_{\mathfrak{A}}B(\mathfrak{A}, \mathfrak{A}^*)$ and $B_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A}^*)$ are reflexive. Using this, we prove that if ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^*)$ are reflexive, then every bounded approximately local derivation from $\mathbb{M}_I(\mathcal{A})$ into any Banach $\mathbb{M}_I(\mathcal{A})$-bimodule is a derivation. Moreover, as an application, we deduce that if S is a uniformly locally finite inverse semigroup, then every bounded approximately local derivation from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule is a derivation. By this fact, we conclude that the space $\mathcal{Z}^1(\ell^1(S), X)$ is reflexive, where S is a uniformly locally finite inverse semigroup.

2. Approximately local derivations of ℓ^1-Munn algebras

Let $\mathcal{A}$ be a Banach algebra and I be a non-empty set. We denote by $\mathbb{M}_I(\mathcal{A})$, the set of $I \times I$ matrices (a_{ij}) with entries in $\mathcal{A}$ such that

$$\|(a_{ij})\| := \sum_{i,j\in I} \|a_{ij}\| = \sup\{\sum_{i,j\in F} \|a_{ij}\| : F \subseteq I \text{ is a finite set}\} < \infty.$$

Then $\mathbb{M}_I(\mathcal{A})$ with the usual matrix multiplication is a Banach algebra that belongs to the class of ℓ^1-Munn algebras over $\mathcal{A}$ introduced in [5]. Some cohomological properties of ℓ^1-Munn algebras have been studied in many papers, for example see [5], [13] and [18]. It is worthwhile to mention that ℓ^1-Munn algebras are a generalization of Rees matrix semigroup algebras. We also refer the reader to [2] and [3], for some results related to derivations of Rees matrix semigroup algebras.

Moreover, local and approximate local derivations of matrix algebra $M_n(\mathcal{A})$ studied in [6] and [14]. Indeed, it is shown in [6, Corollary 2.8] that for a unital algebra $\mathcal{A}$ and any $n \geq 2$, every local derivation from $M_n(\mathcal{A})$ into any $M_n(\mathcal{A})$-bimodule is a derivation. Also, E. Samei in [14] extended this result for bounded approximately local derivations. Precisely, it is proved that if $\mathcal{A}$ is a Banach algebra generated by idempotents, then every bounded approximately local derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is a derivation.

In this section, our aim is to study bounded approximately local derivations of ℓ^1-Munn algebra $\mathbb{M}_I(\mathcal{A})$, where I is an arbitrary non-empty set and $\mathcal{A}$ is an approximately locally unital Banach algebra.

Definition 2.1. Suppose that $\mathcal{A}$ is a Banach algebra. We say that $\mathcal{A}$ is approximately locally unital if there are subsets $\mathcal{A}_l$ and $\mathcal{A}_r$ of $\mathcal{A}$ such that $\mathcal{A}$ is the closed linear span of both $\mathcal{A}_l$ and $\mathcal{A}_r$ and each element of $\mathcal{A}_l$ and $\mathcal{A}_r$ has a left identity and a right identity in $\mathcal{A}$, respectively.

We start with the following elementary lemma.

Lemma 2.2. *Let I be an arbitrary non-empty set. Then the ℓ^1-Munn algebra $\mathbb{M}_I(\mathbb{C})$ is an approximately locally unital Banach algebra.*

Proof. We put two subsets

$$(\mathbb{M}_I(\mathbb{C}))_l = (\mathbb{M}_I(\mathbb{C}))_r = \{E_{ij} : i, j \in I\},$$

where E_{ij} is the $I \times I$ matrix over $\mathbb{C}$ that has 1 as its (i, j)-th entry and 0 elsewhere. It is easy to see that for each $i, j \in I$, we have

$$E_{ii}E_{ij} = E_{ij} = E_{ij}E_{jj}.$$

This follows that each element of $(\mathbb{M}_I(\mathbb{C}))_l$ and $(\mathbb{M}_I(\mathbb{C}))_r$ has a left identity and a right identity, respectively. Moreover, for each $(\lambda_{ij}) \in \mathbb{M}_I(\mathbb{C})$, we have $(\lambda_{ij}) = \sum_{i,j\in I} \lambda_{ij}E_{ij}$, i.e., the closed linear span of $(\mathbb{M}_I(\mathbb{C}))_l = (\mathbb{M}_I(\mathbb{C}))_r$ is $\mathbb{M}_I(\mathbb{C})$, as required. □

In the following, we first obtain the reflexivity of the space of bounded left (right) multipliers on the ℓ^1-Munn algebra $\mathbb{M}_I(\mathbb{C})$.

Proposition 2.3. *Suppose that I is an arbitrary non-empty set and $\mathfrak{A} = \mathbb{M}_I(\mathbb{C})$. Then*

(i) ${}_{\mathfrak{A}}B(\mathfrak{A}, \mathfrak{A}^*)$ *and* $B_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A}^*)$ *are reflexive.*
(ii) *every bounded approximately local derivation from $\mathfrak{A}$ into any Banach $\mathfrak{A}$-bimodule X is a derivation.*
(iii) $\mathcal{Z}^1(\mathfrak{A}, X)$ *is reflexive, for each $\mathfrak{A}$-bimodule X.*

Proof. (i) We first suppose that $T \in \mathrm{ref}({}_{\mathfrak{A}}B(\mathfrak{A}, \mathfrak{A}^*))$ and choose $x, y \in \mathfrak{A}$ such that $xy = 0$. Therefore, there is a sequence of bounded right multipliers $\{T_{y,n}\}_{n=1}^{\infty}$ from $\mathfrak{A}$ into $\mathfrak{A}^*$ such that

$$T(y) = \lim_{n\to\infty} T_{y,n}(y).$$

Hence, we conclude that

$$\begin{aligned} x \cdot T(y) &= x \cdot (\lim_{n\to\infty} T_{y,n}(y)) \\ &= \lim_{n\to\infty} x \cdot T_{y,n}(y) \\ &= \lim_{n\to\infty} T_{y,n}(xy) \\ &= 0. \end{aligned}$$

Indeed, it follows that the operator T satisfies the condition $(f2)$ introduced in [4]. Thus, using [4, Theorem 4.7], we conclude that for each $x, z \in \mathfrak{A},\ r \in \mathfrak{B}$,

$$x \cdot T(rz) = xr \cdot T(z), \tag{1}$$

where $\mathfrak{B}$ is the closed linear span of the idempotent elements in $\mathfrak{A}$. Now, choose $\varepsilon > 0$ and $y = (y_{ij}) \in \mathfrak{A}$. Thus, there is a finite set $F_0 \subseteq I$ such that

$$\sum_{i,j\in I\setminus F_0} |y_{ij}| < \varepsilon.$$

This follows that for each $x, z \in \mathfrak{A}$ we have

$$\begin{aligned} &\|x \cdot T(yz) - xy \cdot T(z)\| \\ &= \|x \cdot T((y_{F_0} + y_{I\setminus F_0})z) - x(y_{F_0} + y_{I\setminus F_0}) \cdot T(z)\| \\ &\le \|x \cdot T(y_{F_0}z) - xy_{F_0} \cdot T(z)\| + \|x \cdot T(y_{I\setminus F_0}z) - xy_{I\setminus F_0} \cdot T(z)\|, \end{aligned}$$

where $y_{F_0} = \sum_{i,j\in F_0} y_{ij}E_{ij}$ and $y_{I\setminus F_0} = \sum_{i,j\in I\setminus F_0} y_{ij}E_{ij}$.

On the other hand, it is easy to see that y_{F_0} is generated by the idempotent elements of $M_{F_0}(\mathbb{C})$, see [4, Lemma 2.1]. This fact shows that $y_{F_0} \in \mathfrak{B}$. Now, by applying the equation (1), we have

$$\|x \cdot T(y_{F_0}z) - xy_{F_0} \cdot T(z)\| = 0.$$

Therefore,

$$\begin{aligned}\|x\cdot T(yz)-xy\cdot T(z)\| &\leq \|x\cdot T(y_{I\setminus F_0}z)-xy_{I\setminus F_0}\cdot T(z)\|\\ &\leq 2\|x\|\|T\|\|z\|\|y_{I\setminus F_0}\|\\ &= 2\|x\|\|T\|\|z\|(\sum_{i,j\in I\setminus F_0}|y_{ij}|)\\ &< 2\|x\|\|T\|\|z\|\varepsilon.\end{aligned}$$

This follows that the equality

$$x\cdot T(yz)=xy\cdot T(z),$$

holds for all $x,y,z\in\mathfrak{A}$. In the case where $x=x_1x_2\in\mathfrak{A}^2$, we conclude that

$$\begin{aligned}\langle T(yz),x\rangle &= \langle T(yz),x_1x_2\rangle\\ &= \langle x_2\cdot T(yz),x_1\rangle\\ &= \langle x_2y\cdot T(z),x_1\rangle\\ &= \langle x_2\cdot(y\cdot T(z)),x_1\rangle\\ &= \langle y\cdot T(z),x_1x_2\rangle\\ &= \langle y\cdot T(z),x\rangle.\end{aligned}$$

On the other hand, by Lemma 2.2, the Banach algebra $\mathfrak{A}$ is approximately locally unital and so $\overline{\mathfrak{A}^2}=\mathfrak{A}$. Hence,

$$T(yz)=y\cdot T(z)\qquad(y,z\in\mathfrak{A}).$$

Indeed, it follows that T is a left multiplier and so ${}_{\mathfrak{A}}B(\mathfrak{A},\mathfrak{A}^*)$ is reflexive. A similar result can be obtained for $B_{\mathfrak{A}}(\mathfrak{A},\mathfrak{A}^*)$.

(ii) This immediately follows from the clause (i), Lemma 2.2 and [14, Theorem 4.4].

(iii) It is clear from the clause (ii). □

Assume that A and B are Banach algebras. Throughout the paper, $A\widehat{\otimes}B$ denotes the *projective tensor product* equipped with the *projective norm* defined as

$$\|u\|_p=\inf\{\sum_{i=1}^{\infty}\|x_i\|\,\|y_i\|:u=\sum_{i=1}^{\infty}x_i\otimes y_i\}\quad(u\in A\widehat{\otimes}B).$$

Now, we are ready to present our main result.

Theorem 2.4. *Let I be an arbitrary non-empty set and $\mathcal{A}$ be an approximately locally unital Banach algebra. Then $\mathbb{M}_I(\mathcal{A})$ is an approximately locally unital Banach algebra. Moreover, in the case when ${}_{\mathcal{A}}B(\mathcal{A},\mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A},\mathcal{A}^*)$ are reflexive, we have every bounded approximately local derivation from $\mathbb{M}_I(\mathcal{A})$ into any Banach $\mathbb{M}_I(\mathcal{A})$-bimodule X is a derivation and so $\mathcal{Z}^1(\mathbb{M}_I(\mathcal{A}),X)$ is reflexive.*

Proof. It is an easy verification that the map $\theta : \mathbb{M}_I(\mathcal{A}) \to \mathbb{M}_I(\mathbb{C})\widehat{\otimes}\mathcal{A}$ defined by

$$\theta((a_{ij})) = \sum_{i,j\in I} E_{ij} \otimes a_{ij} \quad ((a_{ij}) \in \mathbb{M}_I(\mathcal{A})),$$

is an isometric isomorphism of Banach algebras, where $\widehat{\otimes}$ denotes the projective tensor product. Now, if $\mathcal{A}_l$ and $\mathcal{A}_r$ satisfy the assumption of Definition 2.1, then by considering

$$(\mathbb{M}_I(\mathcal{A}))_l = \{E_{ij} \otimes a : a \in \mathcal{A}_l\}, \quad (\mathbb{M}_I(\mathcal{A}))_r = \{E_{ij} \otimes a : a \in \mathcal{A}_r\},$$

we conclude that $\mathbb{M}_I(\mathcal{A})$ is an approximately locally unital Banach algebra.

Also, in the case when ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^*)$ are reflexive, the result immediately can be obtained from Lemma 2.2, Proposition 2.3 and [14, Theorem 4.6]. □

Example 2.5. It is easy to see that $C_0(\mathbb{R})$ is an approximately locally unital Banach algebra. Moreover, it follows from [9, Proposition 3.1] that the spaces

$${}_{C_0(\mathbb{R})}B(C_0(\mathbb{R}), C_0(\mathbb{R})^*) \quad \text{and} \quad B_{C_0(\mathbb{R})}(C_0(\mathbb{R}), C_0(\mathbb{R})^*),$$

are reflexive. Now, it follows from Theorem 2.4 that bounded approximately local derivations from $\mathbb{M}_I(C_0(\mathbb{R}))$ into any Banach $\mathbb{M}_I(C_0(\mathbb{R}))$-bimodule X are derivations.

In the sequel, as a consequence of Theorem 2.4, we can consider the ℓ^1-Munn algebra of certain classes of group algebras.

Corollary 2.6. *Suppose that G is a locally compact group. Then every bounded approximately local derivation from $\mathbb{M}_I(L^1(G))$ into any Banach $\mathbb{M}_I(L^1(G))$-bimodule X is a derivation, in either the following cases*:

(i) *G is an IN-group, i.e., it has a compact neighborhood of identity which is invariant under all inner automorphisms of G.*
(ii) *G is maximally almost periodic, i.e., finite-dimensional continuous unitary irreducible representations of G separate its point.*
(iii) *G is totally disconnected, i.e., $\{e_G\}$ is the largest connected subset of G containing e_G.*

Proof. By [17, Corollary 3.3], with any condition (i)-(iii) we know that $L^1(G)$ has an open subgroup of polynomial growth and so it is an approximately locally unital Banach algebra [17, Lemma 3.1]. On the other hand, by [14, Theorem 8.1], it is shown that the spaces

$${}_{L^1(G)}B(L^1(G), L^1(G)^*) \quad \text{and} \quad B_{L^1(G)}(L^1(G), L^1(G)^*),$$

are reflexive. Now, using Theorem 2.4, the proof is complete. □

3. Applications to semigroup algebras

Let S be a semigroup and $\ell^1(S) = \{f : S \to \mathbb{C} : \|f\|_1 = \sum_{s\in S} |f(s)| < \infty\}$. Define the convolution of two elements $f, g \in \ell^1(S)$ by

$$(f * g)(s) = \sum_{uv=s} f(u)g(v),$$

where $\sum_{uv=s} f(u)g(v) = 0$, when there are no elements $u, v \in S$ with $uv = s$. Then the Banach algebra $(\ell^1(S), *, \|\cdot\|_1)$ is called *the semigroup algebra* of S.

In this section, we will discuss some applications of Theorem 2.4 to study bounded approximately local derivations from the semigroup algebra $\ell^1(S)$ into a Banach $\ell^1(S)$-bimodule X, where S is a uniformly locally finite inverse semigroup. First, we give some definitions and basic properties of semigroups that we shall need.

A semigroup S is *inverse* if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. By [8, Proposition 5.2.1], for any inverse semigroup S, there is a partial order on S defined by

$$s \le t \iff s = ss^*t \qquad (s, t \in S). \tag{2}$$

For each $x \in S$, we set

$$(x] = \{y \in S : y \le x\}.$$

An inverse semigroup S is called *uniformly locally finite* if

$$\sup\{\mathrm{card}((x]) : x \in S\} < \infty.$$

Moreover, for each $q \in E(S) = \{p \in S : p^2 = p\}$, the maximal subgroup of S at q is denoted by G_q. It is easy to check that $G_q = \{s \in S : ss^* = s^*s = q\}$.

Definition 3.1 ([8, Proposition 5.1.2(4)]). Let S be an inverse semigroup. We define a relation $\mathcal{D}$ on S by $s\mathcal{D}t$ if and only if there exists $x \in S$ such that

$$s^*s = xx^*, \quad t^*t = x^*x.$$

We recall that if $\{\mathcal{A}_\alpha : \alpha \in I\}$ is a collection of Banach algebras, then we denote the ℓ^1-direct sum of $\{\mathcal{A}_\alpha\}$ by

$$\ell^1 - \bigoplus\{\mathcal{A}_\alpha : \alpha \in I\},$$

which is a Banach algebra with componentwise operations. The following theorem will be very useful which identifies the structure of the Banach algebra $\ell^1(S)$ for every uniformly locally finite inverse semigroup S.

Theorem 3.2 ([12, Theorem 2.18]). *Let S be a uniformly locally finite inverse semigroup and $\{D_\lambda : \lambda \in \Lambda\}$ be the collection of all $\mathcal{D}$-classes of S. Then*

$$\ell^1(S) \cong \ell^1 - \bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\},$$

as Banach algebras, where G_{p_λ} is the maximal subgroup of S at $p_\lambda \in E(D_\lambda)$.

We note that by [14, Theorem 6.1], it follows that in the case where S is a semilattice, then every bounded approximately local derivation from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule is a derivation. In the following theorem, we regard this problem for the inverse semigroup algebra $\ell^1(S)$, where S is uniformly locally finite.

Theorem 3.3. *Let S be a uniformly locally finite inverse semigroup. Then every bounded approximately local derivation from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule X is a derivation. In particular, $\mathcal{Z}^1(\ell^1(S), X)$ is reflexive.*

Proof. By the structure theorem [12, Theorem 2.18], we have

$$\ell^1(S) \cong \ell^1 - \bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\},$$

as Banach algebras. Now, for each $\lambda \in \Lambda$, we put

$$\mathfrak{A}_\lambda = \mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})),$$

and choose $T \in \mathrm{ref}[{}_{\mathfrak{A}_\lambda}B(\mathfrak{A}_\lambda, {\mathfrak{A}_\lambda}^*)]$. By regarding $\mathfrak{A}_\lambda^*$ as a right $\mathfrak{A}_\lambda$-module with the zero module action, we conclude that T is a bounded approximately local derivation from $\mathfrak{A}_\lambda$ into Banach $\mathfrak{A}_\lambda$-bimodule $\mathfrak{A}_\lambda^*$. Therefore, by Theorem 2.4, it follows that T is a derivation and so $T \in {}_{\mathfrak{A}_\lambda}B(\mathfrak{A}_\lambda, {\mathfrak{A}_\lambda}^*)$. This fact shows that ${}_{\mathfrak{A}_\lambda}B(\mathfrak{A}_\lambda, {\mathfrak{A}_\lambda}^*)$ is reflexive. With the similar argument, one can see that $B_{\mathfrak{A}_\lambda}(\mathfrak{A}_\lambda, {\mathfrak{A}_\lambda}^*)$ is also reflexive. On the other hand, since $\ell^1(G_{p_\lambda})$ is an approximately locally unital Banach algebra, it follows from Theorem 2.4 that $\mathfrak{A}_\lambda$ is also an approximately locally unital Banach algebra. Now, by [14, Theorem 4.5], we conclude that every bounded approximately local derivation from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule X is a derivation. □

Let G be a group and I be a non-empty set. Set

$$\mathcal{M}^0(G, I) = \{(g)_{ij} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{ij}$ denotes the $I \times I$-matrix with entry $g \in G$ in the (i, j) position and zero elsewhere. Then $\mathcal{M}^0(G, I)$ with the multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases} \qquad (g, h \in G,\ i, j, k, l \in I),$$

is an inverse semigroup with $(g)_{ij}^* = (g^{-1})_{ji}$, that is called the *Brandt semigroup* over G with index set I.

Corollary 3.4. *Let G be a group, I be a non-empty set and let $S = \mathcal{M}^0(G, I)$ be the Brandt semigroup over G with index set I. Then every bounded approximately local derivation from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule X is a derivation.*

Proof. We note that

$$E(S) = \{(e_G)_{ii} : i \in I\} \cup \{0\}.$$

Since $E(S)$ is a semilattice, it is easy to see that for each $x \in E(S)$,

$$(x] = \{y \in E(S) : y = yx\}$$
$$= \begin{cases} \{0\} & \text{if} \quad x = 0, \\ \{0, x\} & \text{if} \quad x \neq 0. \end{cases}$$

This follows that $(E(S), \leq)$ is uniformly locally finite. By [12, Proposition 2.14], we conclude that $(S, \leq)$ is a uniformly locally finite inverse semigroup. Now, by Theorem 3.3, the proof is complete. □

Corollary 3.5. *If S is an inverse semigroup such that $E(S)$ is finite, then bounded approximately local derivations from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule X are derivations.*

We recall that $A\widehat{\otimes}A$ is a Banach A-bimodule with the natural module actions, defined as

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

We also say that a Banach algebra $\mathcal{A}$ is *biflat* if there is a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ such that $\pi'' \circ \rho = \kappa_{\mathcal{A}}$, where $\kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}''$ is the natural embedding of $\mathcal{A}$ into its second dual and $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the multiplication map defined by $\pi(a \otimes b) = ab$.

Corollary 3.6. *If S is an inverse semigroup such that the semigroup algebra $\ell^1(S)$ is biflat, then bounded approximately local derivations from $\ell^1(S)$ into any Banach $\ell^1(S)$-bimodule X are derivations.*

Proof. Since $\ell^1(S)$ is biflat, we conclude that $(S, \leq)$ is uniformly locally finite, see [12, Theorem 3.7]. Now, the result easily obtain using Theorem 3.3. □

Finally, we conclude the paper with the following open question.

Question. For an arbitrary inverse semigroup S and any Banach $\ell^1(S)$-bimodule X, is $\mathcal{Z}^1(\ell^1(S), X)$ reflexive?

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