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REMARKS ON THE GRADIENT FLOW OF α ENERGY POTENTIAL ON THE LINE

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Abstract. We are interested in the gradient flow of α energy potential. We provide basic estimates and study asymptotic behaviors for the case $N = 2, \ldots, 5$.

1. Introduction

In this work, we study the gradient flow of the following α energy potential

$$\mathcal{V}_\alpha(\mathbb{X}) := \begin{cases} \displaystyle\sum_{i=1}^N \frac{1}{2}x_i^2 + \frac{1}{2\alpha}\sum_{1\le i<k\le N}\frac{1}{|x_i-x_k|^{2\alpha}} & \text{for } \alpha = 1, 2, \ldots, \\ \displaystyle\sum_{i=1}^N \frac{1}{2}x_i^2 - \sum_{1\le i<k\le N}\frac{1}{2}\log|x_i-x_k|^2 & \text{for } \alpha = 0, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}$ and $\mathbb{X} = \{x_1, x_2, \ldots, x_N\}$.

The log-gas system [11] is given by the following Hamiltonian

$$H_L = \sum_{i=1}^N \frac{1}{2}p_i^2 + \sum_{i=1}^N \frac{1}{2}x_i^2 - \sum_{1\le i<k\le N}\frac{1}{2}\log|x_i-x_k|^2,$$

where $(x_1, x_2, \ldots, x_N)$ and $(p_1, p_2, \ldots, p_N) = (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_N}{dt})$ are positions and momenta of N particles, respectively. The particles in the log-gas system are described via a logarithmic interaction potential and confined in a harmonic trap. The Hamiltonian of the Calogero-Moser system [4–6, 10] is given by

$$H_{CM} = \sum_{i=1}^N \frac{1}{2}p_i^2 + \sum_{i=1}^N \frac{1}{2}x_i^2 + \sum_{1\le i<k\le N}\frac{1}{2}\frac{1}{|x_i-x_j|^2}.$$

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Here we consider the initial value problem of the gradient flow associated with the potential $\mathcal{V}_\alpha$:

$$\frac{d\mathbb{X}}{dt} = -\nabla_{\mathbb{X}}\mathcal{V}_\alpha(\mathbb{X}), \qquad \mathbb{X}(0) = \mathbb{X}_0, \qquad t > 0, \tag{2}$$

which can be rewritten as

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{k\neq i} \frac{1}{(x_i - x_k)^{2\alpha+1}} - x_i, \quad i = 1, 2, \ldots, N, \\ x_i(0) &= x_i^0, \end{aligned} \tag{3}$$

where we denote $\sum_{k\neq i} f_k = f_1 + \cdots + f_{i-1} + f_{i+1} + \cdots + f_N$. It is enough to consider the case of $x_1^0 < x_2^0 < \cdots < x_N^0$.

We investigate asymptotic behaviors of the solution to (3). Note that the study of H_L and H_{CM} leads us to the second order differential equations while (3) is the system of first order ODEs. Moreover, α energy potential (1) generalizes the potentials of H_L ($\alpha = 0$) and H_{CM} ($\alpha = 1$). Some special solutions of (3) with $\alpha = 0$ were studied in [2, 3, 12]. In particular, it was proved in [3] that the ODEs (3) with $\alpha = 0$ admit unique global solutions.

In Section 2, we prove Propositions 2.1 and 2.2. The estimate (5) and Proposition 2.1 exclude the possibility of $|x_{k+1}(t) - x_k(t)| \to 0$ as $t \to t_0$ for some k and $t_0 > 0$. Therefore the solution to (3) exists for all $t > 0$. Proposition 2.2 tells us that the center of mass $M(t) := \frac{1}{N}\sum_{i=1}^N x_i(t)$ converges to 0 exponentially and $M(t) = 0$ is an invariant space. In Sections 3 and 4, we study the case of $N = 2, \ldots, 5$ and investigate the asymptotic behaviors of solutions more precisely.

2. Basic properties and results

First of all, we will show that a collision of particles does never happen for the initial data $x_1^0 < x_2^0 < \cdots < x_N^0$. For the solution of the gradient flow (2), it is easy to check that

$$\mathcal{V}_\alpha(\mathbb{X})(t) \le \mathcal{V}_\alpha(\mathbb{X})(0). \tag{4}$$

Then we can derive from (4), for $\alpha \ge 1$,

$$|x_i - x_k|^{2\alpha}(t) \ge \frac{1}{2\alpha\mathcal{V}_\alpha(\mathbb{X})(0)}, \tag{5}$$

which excludes the possibility of $|x_{k+1}(t) - x_k(t)| \to 0$ as $t \to t_0$ for some k and $t_0 > 0$.

For $\alpha = 0$, we have

$$-\log\left(\Pi_{1\le i<k\le N}|x_i - x_k|\right) \le \mathcal{V}_\alpha(X)(0),$$

which implies that

$$\Pi_{1\le i<k\le N}|x_i - x_k| \ge e^{-\mathcal{V}_\alpha(X)(0)}.$$

In this case, we can not exclude the possibility of $|x_{k+1}(t) - x_k(t)| \to 0$ as $t \to t_0$ for some k and $t_0 > 0$. We define

$$\varepsilon(t) := \min_{k=1,\dots,N-1} (x_{k+1} - x_k)(t).$$

Proposition 2.1. *Let x_j be the solution of* (3) *with $\alpha = 0$. For the initial data satisfying $x_1^0 < x_2^0 < \cdots < x_N^0$, we have $x_1(t) < x_2(t) < \cdots < x_N(t)$ for all $t \geq 0$. More precisely, we obtain*

$$\frac{d\varepsilon(t)}{dt} \geq \frac{4}{N\varepsilon(t)} - \varepsilon(t).$$

Therefore, we have $\frac{d\varepsilon}{dt} > 0$ when $\varepsilon(t) < \frac{2}{\sqrt{N}}$.

Proof. Let $\varepsilon(t) = \min\limits_{k=1,\dots,N-1} (x_{k+1} - x_k)(t) = x_{i+1} - x_i$ for some i. Then we have

$$\begin{aligned}
&\frac{d}{dt}(x_{i+1} - x_i) \\
&= \sum_{k \neq i+1} \frac{1}{x_{i+1} - x_k} - \sum_{k \neq i} \frac{1}{x_i - x_k} - (x_{i+1} - x_i) \\
&= \sum_{\substack{k \neq i+1 \\ k \neq i}} \left(\frac{1}{x_{i+1} - x_k} - \frac{1}{x_i - x_k} \right) + \frac{1}{x_{i+1} - x_i} - \frac{1}{x_i - x_{i+1}} - \varepsilon \\
&= \sum_{k<i} \left\{ \frac{x_i - x_{i+1}}{(x_{i+1} - x_k)(x_i - x_k)} \right\} + \sum_{k>i+1} \left\{ \frac{x_i - x_{i+1}}{(x_{i+1} - x_k)(x_i - x_k)} \right\} + \frac{2}{\varepsilon} - \varepsilon.
\end{aligned}$$

Since $|x_i - x_k| \geq |i-k|\varepsilon$, we have

$$\begin{aligned}
\sum_{k<i} \left\{ \frac{-\varepsilon}{(x_{i+1} - x_k)(x_i - x_k)} \right\} &\geq \sum_{k<i} \frac{-\varepsilon}{(i+1-k)\varepsilon(i-k)\varepsilon} \\
&= -\frac{1}{\varepsilon} \sum_{k<i} \left\{ \frac{1}{i-k} - \frac{1}{i+1-k} \right\} \\
&= -\frac{1}{\varepsilon} \left(1 - \frac{1}{i} \right),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k>i+1} \left\{ \frac{-\varepsilon}{(x_{i+1} - x_k)(x_i - x_k)} \right\} &\geq \sum_{k>i+1} \frac{-\varepsilon}{(k-i-1)\varepsilon(k-i)\varepsilon} \\
&= -\frac{1}{\varepsilon} \sum_{k>i+1} \left\{ \frac{1}{k-i-1} - \frac{1}{k-i} \right\} \\
&= -\frac{1}{\varepsilon} \left(1 - \frac{1}{N-i} \right).
\end{aligned}$$

Then we arrive at

$$\begin{aligned}\frac{d\varepsilon}{dt}(t) &\geq -\frac{1}{\varepsilon(t)}\left(1-\frac{1}{i}\right)-\frac{1}{\varepsilon(t)}\left(1-\frac{1}{N-i}\right)+\frac{2}{\varepsilon(t)}-\varepsilon(t)\\ &=\frac{1}{\varepsilon(t)}\left(\frac{N}{i(N-i)}\right)-\varepsilon(t)\\ &\geq \frac{1}{\varepsilon(t)}\left(\frac{4}{N}\right)-\varepsilon(t).\end{aligned}$$

We have $\frac{d}{dt}(x_{i+1}-x_i)=\frac{d\varepsilon}{dt}>0$ when $\varepsilon(t)<\frac{2}{\sqrt{N}}$. Therefore it is impossible to have $x_{k+1}(t)-x_k(t)\to 0$ as $t\to t_0$ for some k and $t_0>0$. □

Let us consider the center of mass $M(t)=\frac{1}{N}\sum_{i=1}^N x_i(t)$.

Proposition 2.2. *Let x_j be the solution of* (3). *Then the center of mass M satisfies*

$$\frac{dM}{dt}+M=0. \tag{6}$$

Proof. Adding (3) for $i=1,2,\ldots,N$ and considering $\frac{1}{(x_i-x_k)^{2\alpha+1}}+\frac{1}{(x_k-x_i)^{2\alpha+1}}$ $=0$, we derive (6). □

Remark 2.3. The equation (6) can be solved by $M(t)=M(0)e^{-t}$. So the center of mass $M(t)$ converges to 0 exponentially. Also we have an invariant space $M(t)=0$ which implies $\sum_{i=1}^N x_i(t)=0$ for the initial data satisfying $\sum_{i=1}^N x_i^0=0$.

We introduce the diagonally dominant matrix and its properties. We refer to [8] for more information.

Definition. A square matrix is said to be diagonally dominant if, for every row of the matrix, the magnitude of the diagonal entry is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is diagonally dominant if

$$|a_{ii}|\geq\sum_{j\neq i}|a_{ij}| \quad \text{for all } i,$$

where a_{ij} denotes the entry in the i-th row and j-th column. If a strict inequality ($>$) is used, this is called strictly diagonally dominant matrix.

It is known that a symmetric strictly diagonally dominant matrix with positive diagonal entries is positive definite. We can check that

$$\partial_i\partial_j\mathcal{V}_\alpha(\mathbb{X})=\begin{cases}1+(2\alpha+1)\sum_{k\neq i}\frac{1}{(x_i-x_k)^{2\alpha+2}} & \text{for } i=j,\\ -(2\alpha+1)\frac{1}{(x_i-x_j)^{2\alpha+2}} & \text{for } i\neq j.\end{cases}$$

Therefore the Hessian of $\mathcal{V}_\alpha$ is a strictly diagonally dominant matrix with positive entries on the diagonal. This implies that it is positive definite and thus

$\mathcal{V}_\alpha$ is strictly convex. This implies that all the trajectories of its gradient flow converge exponentially fast to the equilibrium. More precisely, we have, for any solution $x_j(t)$, the following bound

$$|\mathbb{X}(t) - \mathbb{X}_e| \le c_1 e^{-c_2 t} |\mathbb{X}(0) - \mathbb{X}_e|,$$

where $\mathbb{X}_e$ is the equilibrium point and c_1, c_2 are positive constants. By the equilibrium point we mean the only one in each connected component of the domain.

We finish this section by noting equilibrium points of (3). It is well known that the equilibrium points of (3) with $\alpha = 0$ are given by zeros of the Hermite polynomials [7, 9]. This one is the unique global minimizer of $\mathcal{V}_0(\mathbb{X})$ up to symmetry and an asymptotically stable equilibrium point of the ODEs (3) with $\alpha = 0$. It is also known in [1] that zeros of the Hermite polynomials are included in the equilibrium points of (3) with $\alpha = 1$.

3. The cases of $N = 2$ and $N = 3$

We study the cases of $N = 2$, 3 and investigate the asymptotic behaviors of solutions more precisely.

3.1. $N = 2$

The equations (3) read as for $N = 2$

$$\frac{dx_1}{dt} = \frac{1}{(x_1 - x_2)^{2\alpha+1}} - x_1,$$
$$\frac{dx_2}{dt} = \frac{1}{(x_2 - x_1)^{2\alpha+1}} - x_2.$$

Then we have

$$\frac{d}{dt}(x_2 - x_1) = \frac{2}{(x_2 - x_1)^{2\alpha+1}} - (x_2 - x_1),$$

which is solved by

$$(x_2 - x_1)^{2\alpha+2}(t) = 2 + \Big((x_2^0 - x_1^0)^{2\alpha+2} - 2\Big)e^{-(2\alpha+2)t}.$$

Note that $x_2^0 - x_1^0 > 0$. On the other hand, we have from the equation (6)

$$(x_1 + x_2)(t) = (x_1^0 + x_2^0)e^{-t}.$$

Then we have

$$x_1(t) \to -2^{-\frac{2\alpha+1}{2\alpha+2}} \quad \text{and} \quad x_2(t) \to 2^{-\frac{2\alpha+1}{2\alpha+2}}$$

as $t \to \infty$.

3.2. $N = 3$

The equations (3) read as for $N = 3$

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{1}{(x_1 - x_2)^{2\alpha+1}} + \frac{1}{(x_1 - x_3)^{2\alpha+1}} - x_1,\\ \frac{dx_2}{dt} &= \frac{1}{(x_2 - x_1)^{2\alpha+1}} + \frac{1}{(x_2 - x_3)^{2\alpha+1}} - x_2,\\ \frac{dx_3}{dt} &= \frac{1}{(x_3 - x_1)^{2\alpha+1}} + \frac{1}{(x_3 - x_2)^{2\alpha+1}} - x_3.\end{aligned} \tag{7}$$

With the notation $x = x_2 - x_1$ and $y = x_3 - x_2$, we can derive from (7)

$$\begin{aligned}\frac{dx}{dt} + x &= \frac{2}{x^{2\alpha+1}} - \frac{1}{y^{2\alpha+1}} + \frac{1}{(x+y)^{2\alpha+1}},\\ \frac{dy}{dt} + y &= \frac{2}{y^{2\alpha+1}} - \frac{1}{x^{2\alpha+1}} + \frac{1}{(x+y)^{2\alpha+1}},\end{aligned} \tag{8}$$

where we use $x_3 - x_1 = y + x$. Then we can derive

$$\frac{d}{dt}(x-y) + x - y + \frac{3(x^{2\alpha+1} - y^{2\alpha+1})}{(xy)^{2\alpha+1}} = 0. \tag{9}$$

Note that $(x - y)(t) = 0$ is an invariant subspace of (9).

For the initial data satisfying $x(0) = y(0)$, the system (8) reduces to

$$\frac{dx}{dt} + x = \left(1 + \frac{1}{2^{2\alpha+1}}\right)\frac{1}{x^{2\alpha+1}}$$

from which we have $x \to \left(1 + \frac{1}{2^{2\alpha+1}}\right)^{\frac{1}{2\alpha+2}}$ as $t \to \infty$.

It is easy to show that $(x - y)(t) \neq 0$ if $(x - y)(0) \neq 0$. Without loss of generality, we assume that $(x - y)(t) > 0$. Since

$$\begin{aligned}\frac{x^{2\alpha+1} - y^{2\alpha-1}}{(xy)^{2\alpha+1}} &= \frac{(x-y)(x^{2\alpha} + x^{2\alpha-1}y + \cdots + xy^{2\alpha-1} + y^{2\alpha})}{x^{2\alpha+1}y^{2\alpha+1}}\\ &= (x-y)\left\{\frac{1}{xy^{2\alpha+1}} + \frac{1}{x^2y^{2\alpha}} + \cdots + \frac{1}{x^{2\alpha+1}y}\right\}\\ &:= (x-y)P(t),\end{aligned}$$

the equation (9) can be rewritten as

$$\frac{d}{dt}(x-y) + (1 + 3P(t))\,(x-y) = 0.$$

Then we have

$$0 < (x-y)(t) = (x_0 - y_0)e^{-\int_0^t (1+3P(s))ds}.$$

Noting that $P(s) \geq 0$, we have $x - y \to 0$ as $t \to \infty$ which implies that

$$x_1 - 2x_2 + x_3 \to 0.$$

We also know from Proposition 2.2 that

$$x_1 + x_2 + x_3 \to 0 \quad \text{as } t \to \infty.$$

Then we have, as $t \to \infty$,

$$x_1 + x_3 \to 0 \quad \text{and} \quad x_2 \to 0.$$

Let us consider the case of $x_1 = -x_3$ and $x_2 = 0$. Then the system (7) reduces to

$$\frac{dx_3}{dt} = \left(1 + \frac{1}{2^{2\alpha+1}}\right) \frac{1}{x_3^{2\alpha+1}} - x_3,$$

from which we have $x_3 \to \left(1 + \frac{1}{2^{2\alpha+1}}\right)^{\frac{1}{2\alpha+2}}$ as $t \to \infty$.

4. The cases of $N = 4$ and $N = 5$

For $N = 2m$, we consider the case of $x_k = -x_{2m-k+1}$ for $k = 1, 2, \ldots, m$. Then (3) reduces to the system of ODEs consisting of $x_{m+1}, \ldots, x_{2m}$. For $N = 2m+1$, we consider the case of $x_k = -x_{2m-k+2}$ for $k = 1, 2, \ldots, m$ and $x_{m+1} = 0$. Then (3) reduces to the system of ODEs consisting of $x_{m+2}, \ldots, x_{2m+1}$.

4.1. $N = 4$

We consider the cases of $x_1 = -x_4$, $x_2 = -x_3$. Then (3) reduces to the system of ODEs consisting of $x = x_3$, $y = x_4$.

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{(x-y)^{2\alpha+1}} + \frac{1}{(x+y)^{2\alpha+1}} + \frac{1}{(2x)^{2\alpha+1}} - x, \\ \frac{dy}{dt} &= \frac{1}{(y-x)^{2\alpha+1}} + \frac{1}{(x+y)^{2\alpha+1}} + \frac{1}{(2y)^{2\alpha+1}} - y. \end{aligned} \tag{10}$$

Note that $0 < x < y$. Then we have

$$\frac{d}{dt}(y - x) = \frac{2}{(y-x)^{2\alpha+1}} + \frac{1}{2^{2\alpha+1}} \left(\frac{1}{y^{2\alpha+1}} - \frac{1}{x^{2\alpha+1}} \right) - (y - x).$$

Since $y - x > 0$, we have

$$\frac{d}{dt}(y - x) < \frac{2}{(y-x)^{2\alpha+1}} - (y - x),$$

which implies

$$0 < e^{(2\alpha+2)t}(y - x)^{2\alpha+2}(t) < (y - x)^{2\alpha+2}(0) + 2e^{(2\alpha+2)t} - 2.$$

Then we have $0 < y - x \le 2^{\frac{1}{2\alpha+2}}$ as $t \to \infty$.

To study more precise behaviors of solutions to (10), we consider the case of $\alpha = 0$. Then the equations (10) become

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{x-y} + \frac{1}{x+y} + \frac{1}{2x} - x, \\ \frac{dy}{dt} &= \frac{1}{y-x} + \frac{1}{x+y} + \frac{1}{2y} - y. \end{aligned} \tag{11}$$

To find equilibrium points, we have from the right hand sides of (11)

$$5x^2 - y^2 = 2x^2(x^2 - y^2) \quad \text{and} \quad 5y^2 - x^2 = 2y^2(y^2 - x^2),$$

from which we derive

$$2(x^2 + y^2) = (y^2 - x^2)^2 \quad \text{and} \quad 3(y^2 - x^2) = (y^2 - x^2)(y^2 + x^2).$$

Then we have $y^2 + x^2 = 3$ and $y^2 - x^2 = \sqrt{6}$. Note that $0 < x < y$. To study the behaviors of $y^2 + x^2$ and $y^2 - x^2$, we can derive from (11)

$$\begin{aligned} \frac{dX}{dt} &= 6 - 2X, \\ \frac{dY}{dt} &= \frac{4X}{Y} - 2Y, \end{aligned} \tag{12}$$

where $X = x^2 + y^2$ and $Y = y^2 - x^2$. The solutions of (12) are given by

$$X(t) = e^{-2t}\Big(X(0) + 3e^{2t} - 3\Big),$$

$$e^{4t}Y^2(t) = Y^2(0) + \int_0^t 8e^{4s}X(s)ds.$$

Then we can conclude that

$$X(t) \to 3 \quad \text{and} \quad Y(t) \to \sqrt{6} \quad \text{as} \quad t \to \infty.$$

4.2. $N = 5$

We consider the cases of $x_1 = -x_5$, $x_2 = -x_4$ and $x_3 \equiv 0$. Then (3) reduces to the system of ODEs consisting of $x = x_4$, $y = x_5$.

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{(x-y)^{2\alpha+1}} + \frac{1}{(x+y)^{2\alpha+1}} + \left(1 + \frac{1}{2^{2\alpha+1}}\right)\frac{1}{x^{2\alpha+1}} - x, \\ \frac{dy}{dt} &= \frac{1}{(y-x)^{2\alpha+1}} + \frac{1}{(x+y)^{2\alpha+1}} + \left(1 + \frac{1}{2^{2\alpha+1}}\right)\frac{1}{y^{2\alpha+1}} - y. \end{aligned} \tag{13}$$

Note that $0 < x < y$. For $\alpha = 0$, the equations (13) become

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{x-y} + \frac{1}{x+y} + \frac{3}{2x} - x, \\ \frac{dy}{dt} &= \frac{1}{y-x} + \frac{1}{x+y} + \frac{3}{2y} - y. \end{aligned} \tag{14}$$

Then the equilibrium of (14) is a root of the following equations.

$$\begin{aligned} 2x^2 + \frac{3}{2}(x^2 - y^2) - x^2(x^2 - y^2) &= 0, \\ 2y^2 + \frac{3}{2}(y^2 - x^2) - y^2(y^2 - x^2) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} &2(x^2 + y^2) - (x^2 - y^2)^2 = 0, \\ &5(x^2 - y^2) - (x^2 - y^2)(x^2 + y^2) = 0. \end{aligned}$$

Therefore we have $x^2 + y^2 = 5$ and $y^2 - x^2 = \sqrt{10}$. We also derive from (14)

$$\frac{dX}{dt} = 10 - 2X,$$
$$\frac{dY}{dt} = \frac{4X}{Y} - 2Y,$$

where $X = x^2 + y^2$ and $Y = y^2 - x^2$. Then we can derive that

$$X(t) \to 5 \quad \text{and} \quad Y(t) \to \sqrt{10} \quad \text{as} \quad t \to \infty.$$

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