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A VARIANT OF WILSON'S FUNCTIONAL EQUATION ON SEMIGROUPS

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Abstract. Let S be a semigroup. We determine the complex-valued solutions of the following functional equation

$$f(xy) + \mu(y)f(\sigma(y)x) = 2f(x)g(y),\ x, y \in S,$$

where $\sigma : S \to S$ is an automorphism, and $\mu : S \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$.

1. Introduction

Stetkær [9] solved the variant of d'Alembert's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y),\ x, y \in S,$$

on a semigroup S, where $\sigma : S \to S$ is an involutive automorphism, i.e., $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. The solutions are abelian and are of the form $f = \frac{\chi+\chi\circ\sigma}{2}$, where $\chi : S \to \mathbb{C}$ is a multiplicative function. In [5], Elqorachi and Redouani determined the complex-valued solutions of the variant of Wilson's functional equation

$$f(xy) + \mu(y)f(\sigma(y)x) = 2f(x)g(y),\ x, y \in G, \tag{1.1}$$

where G is a group, and $\mu : S \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Fadli et al. [6] obtained the solutions of (1.1) with $\mu = 1$ on groups. Their results were extended to the case that σ is an automorphism not necessarily involutive by Sabour [7]. In a recent paper [2], Ajebbar and Elqorachi solved (1.1) on semigroups generated by their squares. Ebanks [4] solved the partially Pexiderized d'Alembert-type equation

$$f(x\sigma(y)) + h(\tau(y)x) = 2f(x)k(y),\ x, y \in M,$$

for four unknown functions $f, g, h, k : M \to \mathbb{C}$, where $\sigma, \tau : M \to M$ are involutive automorphisms on monoids that are neither regular nor generated by their squares. There are some results about solutions of (1.1) with $\mu = 1$

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on abelian groups in the literature. See [1] and [8] for further contextual and historical discussion.

Our attention was drawn to (1.1) because in its solutions on groups and semigroups, the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x), \; x, y \in S, \tag{1.2}$$

plays an important role, and in the recent papers [3,4] by Ebanks, the solutions of (1.2) are described in a general semigroup.

The contributions of the present paper to the knowledge about solutions of (1.1) are the following:

(1) The setting has S to be a semigroup not necessarily generated by its squares.

(2) The automorphism $\sigma : S \to S$ is not necessarily involutive.

(3) We relate the solutions of (1.1) to the sine addition law (1.2), and we find explicit formulas for the solutions, expressing them in terms of multiplicative, additive and sometimes arbitrary functions.

Our notation and notions are described in the following section.

2. Notations and terminology

Throughout this paper S denotes a semigroup. If S is a topological semigroup, $C(S)$ denotes the algebra of continuous functions from S into $\mathbb{C}$.

A function f on S is additive if $f(xy) = f(x) + f(y)$ for all $x, y \in S$.

A function f on S is multiplicative if $f(xy) = f(x)f(y)$ for all $x, y \in S$.

A function f on S is central if $f(xy) = f(yx)$ for all $x, y \in S$, and f is abelian if f is central and $f(xyz) = f(xzy)$ for all $x, y, z \in S$.

The map $\sigma : S \to S$ denotes an automorphism and $\mu : S \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. For any subset $T \subseteq S$ define $T^2 := \{xy \,|\, x, y \in T\}$. If $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function, define the sets

$$I_\chi := \{x \in S \,|\, \chi(x) = 0\},$$

$$P_\chi := \{p \in I_\chi \backslash I_\chi^2 \,|\, up, pv, upv \in I_\chi \backslash I_\chi^2 \text{ for all } u, v \in S \backslash I_\chi\}.$$

For any function $f : S \to \mathbb{C}$, define $f^* : S \to \mathbb{C}$ by $f^*(x) := \mu(x)f(\sigma(x))$ for all $x \in S$, and the functions $f^e := \frac{f+f^*}{2}$, $f^\circ := \frac{f-f^*}{2}$.

3. Main result

In the following lemma we give some key properties of solutions of (1.1).

Lemma 3.1. *Let $f, g : S \to \mathbb{C}$ be a solution of (1.1). For all $a \in S$, define the function $f_a : S \to \mathbb{C}$ by $f_a(x) = f(ax) - f(a)g(x)$ for all $x \in S$. The following statements hold*:

(1) *$f_a(xy) = f_a(x)g(y) + f_a(y)g(x)$ for all $a, x, y \in S$, and hence f_a and g are abelian, in particular central.*

(2) *$f^\circ(xy) = f(x)g(y) - f^*(y)g(x)$ for all $x, y \in S$.*

(3) *$f(xy) = 2f(x)g(y) + 2f(y)g(x) - 4f^e(y)g(x) + f^*(xy)$ for all $x, y \in S$.*

(4) *If f is central and $g \neq 0$, then f^e and g are linearly dependent.*

(5) *If f^e and g are linearly independent, then there exist two functions $h_1, h_2 : S \to \mathbb{C}$ such that*

$$g(xy) = f(x)h_1(y) + g(x)h_2(y) \quad \text{for all} \quad x, y \in S.$$

(6) *If g is a non-zero multiplicative function, then there exists a function $h : S \to \mathbb{C}$ such that*

$$f(xy) = f(x)g(y) + g(x)h(y) \quad \text{for all} \quad x, y \in S.$$

Proof. (1) Let $a, x, y \in S$ be arbitrary. We use similar computations to those of [9]. We apply (1.1) to the pair (ax, y), we obtain

$$f(axy) + \mu(y)f(\sigma(y)ax) = 2f(ax)g(y).$$

Now if we apply (1.1) to the pair $(\sigma(y)a, x)$ and multiply the identity obtained by $-\mu(y)$ we get

$$-\mu(y)f(\sigma(y)ax) - \mu(xy)f(\sigma(x)\sigma(y)a) = -2\mu(y)f(\sigma(y)a)g(x).$$

By applying (1.1) to the pair (a, xy), we get

$$f(axy) + \mu(xy)f(\sigma(x)\sigma(y)a) = 2f(a)g(xy).$$

By adding these three identities, we obtain

$$f(axy) = f(a)g(xy) + f(ax)g(y) + g(x)[f(ay) - 2f(a)g(y)].$$

Since a, x, y are arbitrary, we deduce that for all $a \in S$ the pair (f_a, g) satisfies the sine addition law

$$f_a(xy) = f_a(x)g(y) + f_a(y)g(x), \quad x, y \in S. \tag{3.1}$$

According to [4, Theorem 3.1] we deduce that f_a and g are abelian, in particular central. This is the case (1).

(2) By applying (1.1) to the pair $(\sigma(y), x)$ and multiplying the identity obtained by $-\mu(y)$ we get

$$-\mu(y)f(\sigma(y)x) - f^*(xy) = -2f^*(y)g(x). \tag{3.2}$$

By adding (3.2) to (1.1) we obtain

$$f(xy) - f^*(xy) = 2f(x)g(y) - 2f^*(y)g(x). \tag{3.3}$$

This implies that

$$f^\circ(xy) = f(x)g(y) - f^*(y)g(x). \tag{3.4}$$

This is the result (2) of Lemma 3.1.

(3) The identity (3.3) implies that

$$f(xy) = 2f(x)g(y) - 2f^*(y)g(x) + f^*(xy).$$

Since $f^* = 2f^e - f$, we get

$$f(xy) = 2f(x)g(y) - 2(2f^e(y) - f(y))g(x) + f^*(xy).$$

So

$$f(xy) = 2f(x)g(y) + 2f(y)g(x) - 4f^e(y)g(x) + f^*(xy). \tag{3.5}$$

This occurs in part (3).

(4) Suppose that f is central. Since σ is an automorphism, we can see that f^* is central. So, taking this into account in the identity (3.5) we deduce that

$$-4f^e(y)g(x) = -4f^e(x)g(y) \quad \text{for all} \quad x, y \in S.$$

Since $g \neq 0$, we get that $f^e = cg$ for some constant $c \in \mathbb{C}$. This is part (4).

(5) Suppose that f^e and g are linearly independent. Using the associativity of the semigroup operation, we can compute $f^\circ(xyz)$ first as $f^\circ(x(yz))$ and then as $f^\circ((xy)z)$ using the identity (3.4) and compare the results. We obtain

$$f(xy)g(z) - f^*(z)g(xy) = f(x)g(yz) - f^*(yz)g(x). \tag{3.6}$$

Since $f = f^e + f^\circ$ and $f^* = f^e - f^\circ$ then by using the identity (3.4) we get

$$f(xy) = f^e(xy) + f^\circ(xy) = f^e(xy) + f(x)g(y) - f^*(y)g(x)$$

and

$$f^*(yz) = f^e(yz) - f^\circ(yz) = f^e(yz) - f(y)g(z) + f^*(z)g(y).$$

Substituting the last two identities in (3.6) we get after some rearrangement

$$\begin{aligned} &g(z)\left[g(x)f^e(y) - f^e(xy)\right] + f^*(z)\left[g(xy) - g(x)g(y)\right] \\ &= f(x)\left[g(y)g(z) - g(yz)\right] + g(x)\left[f^e(yz) - f^e(y)g(z)\right]. \end{aligned}$$

Furthermore, if f^* and g are linearly dependent, then since g is central, we get that f^* is central. Therefore f is central, since σ is an automorphism. So, according to (4), we deduce that f^e and g are linearly dependent. This is a contradiction. So f^* and g are linearly independent. By fixing $z = z_1$ and $z = z_2$ such that $g(z_1)f^*(z_2) - g(z_2)f^*(z_1) \neq 0$ in the identity above, we obtain two equations from which we get

$$g(xy) = f(x)h_1(y) + g(x)h_2(y), \tag{3.7}$$

for some functions $h_1, h_2 : S \to \mathbb{C}$. This occurs in (5).

(6) Suppose that g is a non-zero multiplicative function. Eq. (3.6) becomes

$$f(xy)g(z) - f^*(z)g(x)g(y) = f(x)g(y)g(z) - f^*(yz)g(x).$$

This implies that

$$g(z)\left(f(xy) - f(x)g(y)\right) = g(x)\left(f^*(z)g(y) - f^*(yz)\right). \tag{3.8}$$

By fixing $z = z_0$ such that $g(z_0) \neq 0$ in (3.8) we deduce that

$$f(xy) = f(x)g(y) + g(x)h(y) \tag{3.9}$$

for some function $h : S \to \mathbb{C}$. This is part (6). This completes the proof of Lemma 3.1. □

In the following lemma we give some properties of the subsets P_χ and $I_\chi \backslash P_\chi$ when χ is σ-invariant (see [3, Lemma 4.1]).

Lemma 3.2. *Let $\chi : S \to \mathbb{C}$ be a non-zero multiplicative function such that $\chi \circ \sigma = \chi$, where $\sigma : S \to S$ is an automorphism. Then*

(1) $\sigma(P_\chi) \subset P_\chi$.
(2) $\sigma(I_\chi \backslash P_\chi) \subset I_\chi \backslash P_\chi$.

Proof. (1) Let $x \in P_\chi$. Suppose that $\sigma(x) \notin P_\chi$, there exists $y \in S\backslash I_\chi$ such that $\sigma(x)y \in I_\chi^2$. Since σ is an automorphism and $\chi = \chi \circ \sigma$, there exists $z \in S\backslash I_\chi$ such that $y = \sigma(z)$, so $\sigma(x)y = \sigma(x)\sigma(z) = \sigma(a)\sigma(b)$ for some $a, b \in I_\chi$. Then $xz = ab \in I_\chi^2$, which implies that $x \notin P_\chi$ but this is a contradiction, so $\sigma(x) \in P_\chi$.

(2) For all $x \in I_\chi \backslash P_\chi$, $\sigma(x) \in I_\chi$ since $\chi = \chi \circ \sigma$. Suppose that $\sigma(x) \in P_\chi$ for all $y \in S\backslash I_\chi$ we have $\sigma(x)y \in I_\chi \backslash I_\chi^2$. Since σ is an automorphism, there exists $z \in S\backslash I_\chi$ such that $y = \sigma(z)$, so $\sigma(x)y = \sigma(xz) \in I_\chi \backslash I_\chi^2$, then since $\chi = \chi \circ \sigma$ we get that $xz \in I_\chi \backslash I_\chi^2$. Since y is arbitrary, then z is also arbitrary. This implies that $x \in P_\chi$. This is a contradiction, so $\sigma(x) \in I_\chi \backslash P_\chi$. This completes the proof of Lemma 3.2. □

Now we are ready to solve the functional equation (1.1).

Theorem 3.1. *The solutions $f, g : S \to \mathbb{C}$ of the functional equation* (1.1) *with $g \neq 0$ are the following pairs*:

(1) *$f = 0$ and $g \neq 0$ arbitrary.*
(2) *$f = \alpha\chi + \beta\chi^*$ and $g = \frac{\chi+\chi^*}{2}$, where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function and $\alpha, \beta \in \mathbb{C}$ are constants such that $(\alpha, \beta) \neq (0, 0)$. In addition, if $\beta \neq 0$, then $\chi \circ \sigma^2 = \chi$.*
(3)
$$f = \begin{cases} \chi(c + A) & \text{on} \quad S\backslash I_\chi, \\ 0 & \text{on} \quad I_\chi \backslash P_\chi, \\ \rho & \text{on} \quad P_\chi, \end{cases} \quad \text{and} \quad g = \chi,$$
where $c \in \mathbb{C}$ is a constant, $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function and $A : S\backslash I_\chi \to \mathbb{C}$ is an additive function such that $\chi^* = \chi$ and $A \circ \sigma = -A$, $\rho : P_\chi \to \mathbb{C}$ is the restriction of f to P_χ such that $\mu\rho \circ \sigma = -\rho$. In addition, we have the following conditions:

(I) *If $x \in \{up, pv, upv\}$ for $p \in P_\chi$ and $u, v \in S\backslash I_\chi$, then $x \in P_\chi$ and we have, respectively, $\rho(x) = \rho(p)\chi(u)$, $\rho(x) = \rho(p)\chi(v)$, or $\rho(x) = \rho(p)\chi(uv)$.*

(II) *$f(xy) = f(yx) = 0$ for all $x \in S\backslash I_\chi$ and $y \in I_\chi \backslash P_\chi$.*

Note that, off the exceptional case (1)*, f and g are abelian.*

Furthermore, off the exceptional case (1)*, if S is a topological semigroup and $f \in C(S)$, then $g, \chi, \chi^* \in C(S)$, $A \in C(S\backslash I_\chi)$ and $\rho \in C(P_\chi)$.*

Proof. We check by elementary computations that if f, g are of the forms (1)–(3), then (f, g) is a solution of (1.1), so left is that any solution (f, g) of (1.1) fits into (1)–(3).

Let $f, g : S \to \mathbb{C}$ be a solution of (1.1). If $f = 0$, then g is arbitrary, so we have the solution family (1). From now on we assume that $f \neq 0$. According to Lemma 3.1(2) we have

$$f^\circ(xy) = f(x)g(y) - f^*(y)g(x) \quad \text{for all} \quad x, y \in S. \tag{3.10}$$

Since $f = f^e + f^\circ$ and $f^* = f^e - f^\circ$, Eq. (3.10) can be written as

$$f^\circ(xy) = f^\circ(x)g(y) + f^\circ(y)g(x) + f^e(x)g(y) - f^e(y)g(x). \tag{3.11}$$

First case : f^e and g are linearly dependent. There exists a constant $c \in \mathbb{C}$ such that $f^e = cg$. So, for all $x, y \in S$ we have

$$f^e(x)g(y) - f^e(y)g(x) = cg(x)g(y) - cg(y)g(x) = 0.$$

Now, Eq. (3.11) becomes

$$f^\circ(xy) = f^\circ(x)g(y) + f^\circ(y)g(x) \quad \text{for all} \quad x, y \in S. \tag{3.12}$$

According to [4, Theorem 3.1] and taking into account that $g \neq 0$ we have the following possibilities:

(1) $f^\circ = c_1(\chi_1 - \chi_2)$ and $g = \frac{\chi_1+\chi_2}{2}$ for some constant $c_1 \in \mathbb{C}$ and $\chi_1, \chi_2 : S \to \mathbb{C}$ are two multiplicative functions such that $\chi_1 \neq \chi_2$. So, since $f^e = cg$, we deduce that $f = \alpha\chi_1 + \beta\chi_2$ for some constants $\alpha, \beta \in \mathbb{C}$. Substituting f and g in the functional equation (1.1), we get after some simplification that

$$\alpha\chi_1(x)\left[\chi_1^*(y) - \chi_2(y)\right] + \beta\chi_2(x)\left[\chi_2^*(y) - \chi_1(y)\right] = 0.$$

Since $\chi_1 \neq \chi_2$, then according to [8, Theorem 3.18] we get that

$$\begin{cases} \alpha\chi_1(x)\left[\chi_1^*(y) - \chi_2(y)\right] = 0, \\ \beta\chi_2(x)\left[\chi_2^*(y) - \chi_1(y)\right] = 0 \end{cases}$$

for all $x, y \in S$. Since $f \neq 0$, then at least one of α and β is not zero.

(i) If $\alpha \neq 0$ and $\beta = 0$, we deduce that $\chi_1 \neq 0$, and $\chi_1^*(y) = \chi_2(y)$ for all $y \in S$. The result occurs in part (2) with $\beta = 0$ and $\chi_1 = \chi$.

(ii) If $\alpha \neq 0$ and $\beta \neq 0$. Suppose that $\chi_1 = 0$, then $\chi_2 \neq 0$, so $\chi_1 = \chi_2^* \neq 0$. This is a contradiction. Thus $\chi_1 \neq 0$, $\chi_1^* = \chi_2$, and $\chi_2^* = \chi_1$. Now if we put $\chi = \chi_1$ we have $\chi^* = \chi_2$ and $\chi \circ \sigma^2 = \chi$. This occurs in case (2).

(iii) If $\alpha = 0$ and $\beta \neq 0$, then $\chi_2 \neq 0$, and $\chi_2^*(y) = \chi_1(y)$ for all $y \in S$. This occurs in part (2) with $\alpha = 0$, $\chi_1 = \chi$ and $\chi_2 = \chi^*$. In addition, $\chi_2^* = \chi_1$ implies that $\chi \circ \sigma^2 = \chi$.

(2)

$$f^\circ = \begin{cases} \chi A & \text{on} \quad S \backslash I_\chi, \\ 0 & \text{on} \quad I_\chi \backslash P_\chi, \\ \rho & \text{on} \quad P_\chi, \end{cases} \quad \text{and} \quad g = \chi,$$

where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function and $A : S \backslash I_\chi \to \mathbb{C}$ is an additive function, $\rho : P_\chi \to \mathbb{C}$ is a function satisfying condition (I), and f° satisfies condition (II). Since $f^e = c\chi = 0$ on I_χ, then $f = f^\circ$ on I_χ, so f satisfies condition (II). Since $f = f^\circ + f^e = f^\circ + cg$, we obtain

$$f = \begin{cases} \chi(c + A) & \text{on} \quad S \backslash I_\chi, \\ 0 & \text{on} \quad I_\chi \backslash P_\chi, \\ \rho & \text{on} \quad P_\chi. \end{cases}$$

By applying the identity (3.10) to the pair $(\sigma(y), x)$ and multiplying the identity obtained by $\mu(y)$, we get

$$\mu(y)f^{\circ}(\sigma(y)x) = f^{*}(y)g(x) - f^{*}(x)g^{*}(y). \tag{3.13}$$

By adding (3.13) to (3.10), we get that

$$f^{\circ}(xy) + \mu(y)f^{\circ}(\sigma(y)x) = f(x)g(y) - f^{*}(x)g^{*}(y). \tag{3.14}$$

Now, by subtracting (3.14) from (1.1) we get

$$f^{e}(xy) + \mu(y)f^{e}(\sigma(y)x) = f(x)g(y) + f^{*}(x)g^{*}(y).$$

Since $f^e = c\chi$, we deduce that

$$c\chi(x)\chi(y) + c\chi(x)\chi^{*}(y) = f(x)\chi(y) + f^{*}(x)\chi^{*}(y).$$

This implies that

$$\chi(y)\left[c\chi(x) - f(x)\right] + \chi^{*}(y)\left[c\chi(x) - f^{*}(x)\right]. \tag{3.15}$$

If $\chi \neq \chi^*$, we get from (3.15) since χ and χ^* are non-zero that

$$f(x) = c\chi(x) \quad \text{and} \quad f^{*}(x) = c\chi(x)$$

for all $x \in S$. Since $f \neq 0$, we have $c \neq 0$ and $f^* = c\chi^* = c\chi$, so $\chi = \chi^*$. This is a contradiction. So $\chi = \chi^*$, and the functional equation (1.1) implies that

$$\chi(xy)\left(c + A(xy)\right) + \mu(y)\chi(\sigma(y)x)\left(c + A(\sigma(y)x)\right) = 2\chi(y)\chi(x)\left(c + A(x)\right)$$

for all $x, y \in S \backslash I_\chi$. Since A is additive and $\chi(xy) \neq 0$, the identity above reduces to $A \circ \sigma = -A$. For $x \in S\backslash I_\chi$ and $y \in P_\chi$ we have $xy \in P_\chi$ and by Lemma 3.2(1) we get $\sigma(y) \in P_\chi$, so $\sigma(y)x \in P_\chi$, then Eq. (1.1) can be written as

$$\rho(y)\chi(x) + \mu(y)\rho \circ \sigma(y)\chi(x) = 0,$$

which implies that $\mu\rho \circ \sigma = -\rho$ since $\chi \neq 0$. Now, if $y \in I_\chi \backslash P_\chi$, then by Lemma 3.2(2) we get $\sigma(y) \in I_\chi\backslash P_\chi$. It follows from condition (II) that $f(xy) = f(\sigma(y)x) = 0$, so

$$f(xy) + \mu(y)f(\sigma(y)x) = 0 = 2f(x)\chi(y),$$

since $\chi(y) = 0$. This is part (3) of Theorem 3.1.

<u>Second case :</u> f^e and g are linearly independent. According to Lemma 3.1(5) there exist two functions $h_1, h_2 : S \to \mathbb{C}$ such that

$$g(xy) = f(x)h_1(y) + g(x)h_2(y) \quad \text{for all} \quad x, y \in S. \tag{3.16}$$

According to Lemma 3.1(2), (f_a, g) satisfies the sine addition law (1.2). So we have the following possibilities:

<u>Subcase A :</u> $f_a = 0$ for all $a \in S$. That is $f(xy) = f(x)g(y)$ for all $x, y \in S$. According to the proof of [2, Theorem 4.2 (case 1)], this case leads to $f = \lambda\chi$ and $g = \chi$, where $\chi : S \to \mathbb{C}$ is non-zero multiplicative function and $\lambda \in \mathbb{C}$ is a constant. Thus, f is central. So, since $g \neq 0$, we get according to Lemma 3.1(4) that f^e and g are linearly dependent. This case does not occur.

<u>Subcase B :</u> $f_a \neq 0$ for some $a \in S$. There exist two multiplicative functions

$\chi_1, \chi_2 : S \to \mathbb{C}$ such that $g = \frac{\chi_1+\chi_2}{2}$.
Subcase B.1 : $\chi_1 \neq \chi_2$.
Subcase B.1.1 : $h_1 = 0$. The identity (3.16) becomes

$$g(xy) = g(x)h_2(y) \quad \text{for all} \quad x, y \in S. \tag{3.17}$$

Since g is central and $g \neq 0$, we deduce from (3.17) that $h_2 = bg$ for some constant $b \in \mathbb{C}$, so (3.17) can be written as

$$g(xy) = bg(x)g(y) \quad \text{for all} \quad x, y \in S. \tag{3.18}$$

Since $g = \frac{\chi_1+\chi_2}{2}$, we deduce from the identity (3.18) that

$$(2-b)\left(\chi_1(xy) + \chi_2(xy)\right) = b\left(\chi_1(x)\chi_2(y) + \chi_1(y)\chi_2(x)\right). \tag{3.19}$$

Since $g \neq 0$, we can assume without loss of generality that $\chi_2(y_0) \neq 0$ for some $y_0 \in S$. Thus, if we put $y = y_0$ in the identity (3.19) we obtain

$$\left[(2-b)\chi_1(y_0) - b\chi_2(y_0)\right]\chi_1(x) + \left[(2-b)\chi_2(y_0) - b\chi_1(y_0)\right]\chi_2(x) = 0 \tag{3.20}$$

for all $x \in S$. Since $\chi_1 \neq \chi_2$, then by using [8, Theorem 3.18] we deduce from (3.20) that

$$\left[(2-b)\chi_1(y_0) - b\chi_2(y_0)\right]\chi_1 = 0, \tag{3.21}$$

$$\left[(2-b)\chi_2(y_0) - b\chi_1(y_0)\right]\chi_2 = 0. \tag{3.22}$$

Suppose that $b = 0$, we get from (3.22) that $\chi_2 = 0$. This is a contradiction. So $b \neq 0$. Now, if $b = 2$, we get from (3.21) that $\chi_1 = 0$. That is $g = \frac{\chi_2}{2}$. So Eq. (3.6) implies that for all $x, y, z \in S$

$$\chi_2(z)\left[f(xy) - \chi_2(y)f(x)\right] = \chi_2(x)\left[\chi_2(y)f^*(z) - f^*(yz)\right]. \tag{3.23}$$

Since $\chi_2(y_0) \neq 0$, then by putting $z = y_0$ in (3.23) we get that

$$f(xy) = \beta\chi_2(xy) + \chi_2(x)k(y) + \chi_2(y)f(x) \quad \text{for all} \quad x, y \in S, \tag{3.24}$$

where $k(y) = \frac{-f^*(yy_0)}{\chi_2(y_0)}$ and $\beta = \frac{f^*(y_0)}{\chi_2(y_0)} \in \mathbb{C}$. By applying the identity (3.24) to the pair $(\sigma(y), x)$ and multiplying the identity obtained by $\mu(y)$, we get

$$\mu(y)f(\sigma(y)x) = \beta\chi_2^*(y)\chi_2(x) + \chi_2^*(y)k(x) + \chi_2(x)f^*(y), \quad x, y \in S. \tag{3.25}$$

By adding (3.24) to (3.25), and taking into account that the pair (f, g) satisfies (1.1), we get after some rearrangement

$$\chi_2(x)\left[\beta(\chi_2(y) + \chi_2^*(y)) + f^*(y) + k(y)\right] = -\chi_2^*(y)k(x). \tag{3.26}$$

Since $\chi_2(y_0) \neq 0$, we deduce from (3.26) that $f^* + k = a_1\chi_2 + a_2\chi_2^*$ for some constants $a_1, a_2 \in \mathbb{C}$. Taking this into account in (3.26), we get that

$$\chi_2(x)\left[b_1\chi_2(y) + b_2\chi_2^*(y)\right] = -\chi_2^*(y)\left[a_1\chi_2(x) + a_2\chi_2^*(x) - f^*(x)\right] \tag{3.27}$$

for some constants $b_1, b_2 \in \mathbb{C}$. Since $\chi_2(y_0) \neq 0$, we deduce from (3.27) that $f^* = c_1\chi_2 + c_2\chi_2^*$, where $c_1, c_2 \in \mathbb{C}$ are constants. This implies that f^* is central, and then f is central since σ is an automorphism. So, according to Lemma 3.1(4) f^e and g are linearly dependent. This case does not occur.

Therefore $b \in \mathbb{C}\backslash\{0,2\}$ and $\chi_1 \neq 0$, so we get from Eq. (3.22) since $\chi_2(y_0) \neq 0$ that $\chi_1(y_0) = \frac{2-b}{b}\chi_2(y_0) \neq 0$. Taking this into account in (3.21), we obtain

$$\left[\frac{(2-b)^2}{b} - b\right]\chi_2(y_0)\chi_1 = 0.$$

Since $\chi_2(y_0) \neq 0$ and $\chi_1 \neq 0$, we get $(2-b)^2 - b^2 = 0$. So $b = 1$, and then we deduce from (3.18) that g is a multiplicative function. That is $\chi_1 = \chi_2$ which contradicts the assumption $\chi_1 \neq \chi_2$. This case does not occur.

Subcase B.1.2 : $h_1 \neq 0$. There exists $y_0 \in S$ such that $h_1(y_0) \neq 0$, so we get from the identity (3.16) by putting $y = y_0$ that

$$f(x) = d_1 g(xy_0) + d_2 g(x) \quad \text{for all} \quad x \in S,$$

where $d_1, d_2 \in \mathbb{C}$ are constants. Since $g = \frac{\chi_1+\chi_2}{2}$, where χ_1, χ_2 are multiplicative functions, we deduce that $f = e_1\chi_1 + e_2\chi_2$ for some constants $e_1, e_2 \in \mathbb{C}$. This implies that f is central. Thus, according to Lemma 3.1(4) f^e and g are linearly dependent. This case does not occur.

Subcase B.2 : $\chi_1 = \chi_2$. In this case, g is a multiplicative function. So, according to Lemma 3.1(5) there exists a function $h : S \to \mathbb{C}$ such that

$$f(xy) = f(x)g(y) + g(x)h(y) \quad \text{for all} \quad x, y \in S. \tag{3.28}$$

Subcase B.2.1 : $h = 0$. From (3.28) we deduce that $f(xy) = f(x)g(y)$ for all $x, y \in S$. This implies that $f_a = 0$, which contradicts the assumption $f_a \neq 0$. This case does not occur.

Subcase B.2.2 : $h \neq 0$. If we put $x = a$ in (3.28), we get $f_a(y) = g(a)h(y)$ for all $y \in S$. In addition, $f_a \neq 0$ implies that $g(a) \neq 0$, so $h = \frac{1}{g(a)} f_a$. Then h is central, since f_a is central. On the other hand, if we apply the identity (3.28) to the pair $(\sigma(y), x)$ and multiply the identity obtained by $\mu(y)$, we get

$$\mu(y)f(\sigma(y)x) = f^*(y)g(x) + g^*(y)h(x) \quad \text{for all} \quad x, y \in S. \tag{3.29}$$

By adding (3.28) to (3.29), and taking into account that (f, g) satisfies the functional equation (1.1), we get

$$f(x)g(y) = g(x)\left[h(y) + f^*(y)\right] + g^*(y)h(x) \quad \text{for all} \quad x, y \in S. \tag{3.30}$$

Since $g \neq 0$, we deduce from (3.30) that

$$f(x) = f_1 g(x) + f_2 h(x) \quad \text{for all} \quad x \in S, \tag{3.31}$$

for some constants $f_1, f_2 \in \mathbb{C}$. Since g and h are central, we deduce from (3.31) that f is central. Then, according to Lemma 3.1(4) f^e and g are linearly dependent. This case does not occur.

For the topological statements, suppose that f is continuous and $f \neq 0$. The continuity of g follows easily from the continuity of f and the functional equation (1.1). Let $y_0 \in S$ such that $f(y_0) \neq 0$, we get from (1.1) that

$$g(x) = \frac{f(xy_0) + \mu(y_0)f(\sigma(y_0)x)}{2f(y_0)} \quad \text{for } x \in S.$$

The functions $x \mapsto f(xy_0)$ and $x \mapsto f(\sigma(y_0)x)$ are continuous, since S is a topological semigroup so that the right translation $x \mapsto xy_0$ and the left translation $x \mapsto \sigma(y_0)x$ are continuous. Then g is continuous.

In case (2) we get the continuity of χ and χ^* from the continuity of g by the help of [8, Theorem 3.18]. For case (3) the function ρ is continuous by restriction, since f is continuous. In addition, we have

$$\chi A = f - c\chi \quad \text{on} \quad S \backslash I_\chi.$$

So $g = \chi$ is continuous. Thus A is continuous, since $\chi \neq 0$. This completes the proof of Theorem 3.1. □

Remark 3.1. For a semigroup S such that $S^2 \neq S$, there exists a non-zero function f such that

$$f(xy) + \mu(y)f(\sigma(y)x) = 0 \quad \text{for all} \quad x, y \in S. \tag{3.32}$$

Let $S = \{0, 1\}$ and define the semigroup operation as $xy = 0$ for all $x, y \in S$. We let $\sigma(x) = x$ for all $x \in S$, and f be the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, \end{cases}$$

so we can see that f satisfies (3.32).

4. Examples

In this section we give some examples of non-zero continuous solutions of the functional equation (1.1) with $\mu = 1$.

Example 4.1. Let G be the $(ax + b)$–group defined by

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, \quad b \in \mathbb{R} \right\}.$$

We consider the following automorphism on G

$$\sigma \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 1 \end{pmatrix},$$

so σ is not involutive. According to [8, Example 3.13], the continuous non-zero multiplicative functions on G are of the form

$$\chi_\lambda : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto a^\lambda,$$

where $c, \lambda \in \mathbb{C}$. We can see that $\chi_\lambda \circ \sigma = \chi_\lambda$. The only additive function A on G such that $A \circ \sigma = -A$ is $A = 0$, so we deduce that the non-zero continuous solutions of (1.1) are

$$\begin{cases} f : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \alpha a^\lambda, \\ g : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto a^\lambda, \end{cases}$$

where $\alpha \in \mathbb{C}\setminus\{0\}$ and $\lambda \in \mathbb{C}$.

Example 4.2. Let $S = (\mathbb{C}, +)$ and $\sigma(z) = 2z$ for all $z \in \mathbb{C}$. The functional equation (1.1) is written as

$$f(z+z') + f(z+2z') = 2f(z)g(z'),\ z, z' \in S.$$

The continuous characters on S are the functions of the form $\chi(z) = e^{az}$, $z \in \mathbb{C}$, where $a \in \mathbb{C}$. If A is additive on S such that $A \circ \sigma = -A$, then $A = 0$. In addition, $\chi = \chi \circ \sigma^2$ implies that $\chi = 1$, so we deduce that the continuous non-zero solutions of (1.1) are

$$\begin{cases} f(z) = \alpha e^{az}, \\ g(z) = \frac{e^{az}+e^{2az}}{2}, \end{cases}$$

where $\alpha \in \mathbb{C}\setminus\{0\}$.

Example 4.3. Let $S = H_3$ be the Heisenberg group defined by

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

We consider the following automorphism

$$\sigma \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 2z \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{pmatrix}.$$

According to [8, Example 3.14], the continuous non-zero multiplicative functions on S have the form

$$\chi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{ax+by},$$

where $a, b \in \mathbb{C}$. The only additive function A on S such that $A \circ \sigma = -A$ is $A = 0$. On the other hand, $\chi \circ \sigma^2 = \chi$ implies that $\chi = e^{ax}$. So the continuous non-zero solutions of Eq. (1.1) are

$$\begin{cases} f : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \alpha e^{ax+by}, \\ g : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \frac{e^{ax+by}+e^{ax+2by}}{2}, \end{cases}$$

where $\alpha \in \mathbb{C}\setminus\{0\}$.

Example 4.4. Let $S = (]-1, 1[, \cdot)$ and $\sigma(x) = x$ for all $x \in S$. S is not generated by its squares and if χ is a continuous multiplicative function on S, then χ have one of the forms

$$\chi = 1,\ \chi(x) := \begin{cases} |x|^{\alpha} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases} \text{ or } \chi(x) := \begin{cases} |x|^{\alpha} sgn(x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases} \tag{4.1}$$

where $\alpha \in \mathbb{C}$ has a positive real part. The non-zero continuous solutions of (1.1) are

$$\begin{cases} f(x) = c\chi(x), \\ g(x) = \chi(x), \end{cases}$$

where $c \in \mathbb{C} \setminus \{0\}$, and χ have one of the three forms in (4.1).

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