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JOINT ESSENTIAL NUMERICAL SPECTRUM AND JERIBI ESSENTIAL NUMERICAL SPECTRUM OF LINEAR OPERATORS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce the concept of joint essential numerical spectrum $\sigma_{en}(\cdot)$ of q-tuple of operators on a Banach space and to study its properties. This notion generalize the notion of the joint essential numerical range.

1. Introduction

The study of the numerical range of an operator and their generalization has along and distinguished history, see [4, 5, 9]. This subject is related and has applications to many different branches of pure and applied science such as operator theory, functional analysis, quantum physics. The numerical range is very useful for studying linear operators acting on Hilbert and Banach spaces.

The notion of the numerical range has been generalized in different directions. In one direction, the joint numerical range was introduced to study the behavior of set of operators. For a q-tuple of linear operators $A = (A_1, \dots, A_q)$ on a Banach space, the joint numerical spectrum is defined by:

$$W(A) = \{\langle Ax, x'\rangle = (\langle A_1x, x'\rangle, \dots, \langle A_qx, x'\rangle) : x \in D(A_k), k = 1, \dots, q, \\ \|x\| = 1, x' \in J(x)\},$$

where $J(x) = \{x' \in X' : \langle x, x'\rangle = \|x\|^2 = \|x'\|^2\}$ and X' is the dual of X. For more details, we refer to [7, 21, 22]. In recent years, there have been many interest and significant results concerning the essential joint numerical range

$$W_e(A) = \bigcap_{K \in K^q(X)} cl(W(A + K)),$$

where $K^q(X)$ is the set of q-tuple of compact operators, for $A = (A_1, \dots, A_q)$ and $K = (K_1, \dots, K_q)$, $(A + K) = (A_1 + K_1, \dots, A_q + K_q)$ and $cl(S)$ denotes the closure of a complex set S.

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In a Hilbert space, unlike the classical numerical range, the joint numerical range is not always convex. Then, the set $W_e(A)$ does not expect to be convex. However, there are some results concerning the convexity of $W(A)$ and $W_e(A)$, see [5,17,20] and their references. For example, in [18], C. K. Li and Y. T. Poon proved that $W_e(A)$ is closed and convex in the case of a q-tuple of self-adjoint operators on a Hilbert space.

Due to the lack of an inner product, the classical numerical range of operators on a Banach space is not closed and convex. M. Adler, W. Dada and A. Radl [2] introduced a new notion called the numerical spectrum $\sigma_n(\cdot)$. For an unbounded operator A on a Banach space, the authors proved that $\sigma_n(A)$ is always closed convex and contains the spectrum. Later, B. Abdelhedi, W. Boubaker and N. Moalla [1] extended this concept and introduced the essential numerical spectrum $\sigma_{en}(A)$ of an unbounded linear operator on a Banach space X. They proved that $\sigma_{en}(A)$ is always closed convex and contains the essential spectrum. This definition is closely related to the essential numerical range, since in the case of bounded operators on a Hilbert space, the essential numerical spectrum coincides with the essential numerical range $W_e(A)$.

Recently, W. Boubaker, N. Moalla and A. Radl in [6], introduced a new definition of the joint numerical spectrum, $\sigma_n(A)$, of q-tuple of operators on a Banach space. Motived by this new notion, we purpose in this paper to describe the joint essential numerical spectrum. Our goal in this work is to generalize the notion of the essential numerical spectrum of linear operator A to the more general case of q-tuple of linear operators $A = (A_1, \dots, A_q)$ on a Banach space. More precisely, we combine the definition of the joint essential spectrum and the joint numerical spectrum to obtain a new notion, called *the joint essential numerical spectrum* of a q-tuple of operators on a Banach space

$$\sigma_{en}(A) = \bigcap_{K \in K^q(X)} \sigma_n(A + K).$$

We prove that $\sigma_{en}(A)$ is a closed convex set in $\mathbb{C}^q$ and satisfying some properties of the joint essential numerical range on a Hilbert space. Our notion covers the earlier definition of joint essential numerical range in the two cases of Hilbert and Banach spaces, and generalises thein to the case of unbounded operators. One of the most exciting development of the essential spectrum is the study of the Jeribi essential spectrum, defined by

$$\sigma_j(A) = \bigcap_{F \in W_*(X)} \sigma(A + F),$$

where $W_*(X)$ stands for each one of the sets $W(X)$, the family of weakly compact operator on X, and $S(X)$, the family of strictly singular operator on X. This notion has been previously studied by A. Jeribi in [11–14]. By analogy of the notion of $\sigma_j(A)$, we define the joint Jeribi essential numerical spectrum as

$$\sigma_{jn}(A) = \bigcap_{F \in W_*^q(X)} \sigma_n(A+F),$$

where $W_*^q(X)$ is the family of q-tuple operator on $W_*(X)$. We verify that the joint Jeribi essential numerical spectrum satisfy some properties of the joint essential numerical spectrum.

Our paper consists of three section organized as follows: Section 2 is dedicated to introduce the new concept of joint essential numerical spectrum, joint Jeribi essential numerical spectrum and develop their properties. In particular, we are interested, which results from the case of a single operator carry over to the case of $q > 1$. In Section 3, we prove several equivalent formulation of joint essential numerical spectrum and joint Jeribi essential numerical spectrum. More precisely, we obtain a description of $\sigma_{en}(A)$ and $\sigma_{jn}(A)$ in terms of a perturbation of one of the components of A. Also, we obtain an analog result of the separation theorem for a convex set. In Section 4, we applies the obtained results to investigate the joint multiplication operators of a q-tuple of multiplication operators.

2. The joint essential numerical spectrum

At the beginning of this section, we introduce some notation and preliminary results which will be used thought this paper.

For $a = (a_1, \ldots, a_q) \in \mathbb{C}^q$ with $\|a\| := (\sum_{k=1}^q |a_k|^2)^{\frac{1}{2}} = 1$ and for $\omega \in \mathbb{R}$, we denote by $H_{a,\omega}^+$ the open hyperplane

$$H_{a,\omega}^+ := \big\{\lambda \in \mathbb{C}^q : \mathrm{Re}\,\big(a.\lambda\big) > \omega\big\} \subset \mathbb{C}^q,$$

where $a.\lambda = \sum_{i=1}^q \overline{a_i}\lambda_i$ and $\overline{a_i}$ is the complex conjugate of a_i. Its complement is $H_{a,\omega}^- := \mathbb{C}^q \backslash H_{a,\omega}^+$. Let $\mathbb{C}_\omega := \big\{z \in \mathbb{C} : \mathrm{Re}(z) > \omega\big\}$, the distance between $a.\lambda$, $\lambda \in H_{a,\omega}^+$ and $\mu \in \partial\mathbb{C}_\omega$ is defined as

$$d(a.\lambda, \partial\mathbb{C}_\omega) := \inf\big\{\| \sum_{i=1}^q \overline{a_i}\lambda_i - \mu\| : \mu \in \partial\mathbb{C}_\omega\big\}.$$

For a q-tuple of operators $A = (A_1, \ldots, A_q)$ and $a = (a_1, \ldots, a_q)$, we define the operator $a.A = \sum_{i=1}^q \overline{a_i}A_i$ and we denote $\rho(a.A)$ the resolvent set and $R(a.\lambda, a.A) = (a.\lambda I - a.A)^{-1}$ the resolvent operator if $a.\lambda \in \rho(a.A)$. In [6, Definition 2.1 page 348], W. Boubaker, N. Moalla and A. Radl defined the notion of the joint numerical spectrum $\sigma_n(\cdot)$ of q-tuple of operators on a Banach space as:

Definition 2.1. Let $(A = (A_1, \ldots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \ldots, A_q$ are bounded linear operators on a Banach space X. Then $z \in \mathbb{C}^q$ belongs to the *joint numerical resolvent set* $\rho_n(A)$ of A if there exist $a \in \mathbb{C}^q$ with $\|a\| = 1$ and $\omega \in \mathbb{R}$ such that

(i) $z \in H^{+}_{a,\omega}$,
(ii) $\mathbb{C}_\omega \subseteq \rho(a.A)$,
(iii) $\|R\big(a.\lambda, a.A\big)\| \leq \frac{1}{d(a.\lambda, \partial\mathbb{C}_\omega)}$ for all $\lambda \in H^{+}_{a,\omega}$.

The complementary set

$$\sigma_n(A) = \mathbb{C}^q \backslash \rho_n(A)$$

is called the joint numerical spectrum of A.

For the reader's convenience, we recall some properties of the joint numerical spectrum proved in [6, Corollary 2.4, page 349].

Proposition 2.2. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X. Then $\lambda \in \rho_n(A)$ if and only if there exists $a \in \Omega$ such that $a.\lambda \in \rho_n(a.A)$, where $\Omega = \{c \in \mathbb{C}^q : \|c\| = 1\}$.*

We consider, the following definition of the joint spectrum of a q-tuple of linear operators as (see [4, 10]).

Definition 2.3. Let $A = (A_1, \dots, A_q)$ be a q-tuple of linear operators on a Banach space X, the joint spectrum is defined as

$$\sigma(A) = \sigma_l(A) \cup \sigma_r(A),$$

where the left (resp. right) joint spectrum $\sigma_l(A)$ (resp. $\sigma_r(A)$) of A is defined as the set of all $\lambda \in \mathbb{C}^q$ such that for all q-tuples of bounded operators $B = (B_1, \dots, B_q)$ we have $\sum_{i=1}^{q} B_i(\lambda_i - A_i) \neq I$ on $D(A)$ (resp. $\sum_{i=1}^{q} (\lambda_i - A_i)B_i \neq I$ on X).

The following properties summarize some basic results of the joint numerical spectrum established in [6, Proposition 2.6 and Properties 2.8].

Properties 2.4. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X. Then*

(i) $\sigma(A) \subset \sigma_n(A)$.
(ii) $\sigma_n(I) = \{(1, \dots, 1)\}$.
(iii) $\sigma_n(A) = \sigma_n(U^{-1}AU)$ *for all isometric isomorphisms U on X, where $U^{-1}AU$ is defined as $U^{-1}AU = (U^{-1}A_1U, \dots, U^{-1}A_qU)$.*
(iv) $\sigma_n(A) = \sigma_n(A')$, *where $A' = (A'_1, \dots, A'_q)$ is the adjoint of A.*
(v) $\sigma_n(\alpha A + \beta I) = \alpha\sigma_n(A) + \beta$ *for all $\alpha \in \mathbb{C}$ and $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{C}^q$, where $\alpha A = (\alpha A_1, \dots, \alpha A_q)$ and $\beta I = (\beta_1 I, \dots, \beta_q I)$.*

There are several types of joint essential spectrum in the literature. In this paper, we will consider the joint essential spectrum of a q-tuple of operators $A = (A_1, \dots, A_q)$ defined in [15]

$$\sigma_{ess}(A) = \bigcap_{K \in K^q(X)} \sigma(A + K).$$

Analogously to the previous definition, we introduce the notion of the joint essential numerical spectrum and the joint Jeribi essential numerical spectrum.

Definition 2.5. Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X.

(i) The joint essential numerical spectrum $\sigma_{en}(A)$ of A is defined by

$$\sigma_{en}(A) = \bigcap_{K \in K^q(X)} \sigma_n(A + K).$$

(ii) The joint Jeribi essential numerical spectrum $\sigma_{jn}(A)$ of A is defined by

$$\sigma_{jn}(A) = \bigcap_{F \in W_*^q(X)} \sigma_n(A + F).$$

In the sequel, $\sigma_{\gamma n}(A)$, where $\gamma \in \{e, j\}$, stands for each one the set of $\sigma_{jn}(A)$ and $\sigma_{en}(A)$.

Remark 2.6. (i) As a first observation, we note that $\sigma_{\gamma n}(A)$ is an intersection of closed convex sets, hence it is closed and convex, which is not true for the essential spectrum $\sigma_{ess}(A)$.

(ii) We note that $\sigma_{ess}(A) \subset \sigma_{en}(A)$ and $\sigma_j(A) \subset \sigma_{jn}(A)$, since $\sigma(A) \subset \sigma_n(A)$.

(iii) If A is a q-tuple of bounded linear operators on a Banach space X, it follows from [6, Corollary 2.10, page 354] that $\sigma_n(A) = cl(co(W(A)))$, then $\sigma_{en}(A)$ coincides with the joint algebraic essential numerical range $V_e(A)$. In particular, in the case $X = l_p$, $1 \leq p < \infty$ and $Int(\sigma_{en}(A)) \neq \varnothing$, there exists $K \in K(X)^q$ such that $\sigma_{en}(A) = \sigma_n(A+K)$, (see [19, Corollary 2.14, page 40]).

(iv) We note that since $K(X) \subset W_*(X)$, we have $\sigma_{jn}(A) \subset \sigma_{en}(A)$.

In what follows, we give some properties of the joint essential numerical spectrum of a given q-tuple of linear operators.

Properties 2.7. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X, we have the following assertions.*

(i) $\sigma_{en}(A + K) = \sigma_{en}(A)$ *for all* $K \in K^q(X)$, *and* $\sigma_{jn}(A + F) = \sigma_{jn}(A)$ *for all* $F \in W_*^q(X)$.

(ii) $\sigma_{\gamma n}(A') \subset \sigma_{\gamma n}(A)$, *where* $A' = (A_1', \dots, A_q')$ *is the adjoint of* A. *If X is a reflexive Banach space, then* $\sigma_{\gamma n}(A) = \sigma_{\gamma n}(A')$.

(iii) $\sigma_{\gamma n}(A) = \sigma_{\gamma n}(U^{-1}AU)$ *for all isometric isomorphisms U on X, where* $U^{-1}AU = (U^{-1}A_1U, \dots, U^{-1}A_qU)$.

(iv) $\sigma_{\gamma n}(\alpha A + \beta I) = \alpha\sigma_{\gamma n}(A) + \beta$ *for all* $\alpha \in \mathbb{C}$ *and* $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{C}^q$, *where* $\alpha A = (\alpha A_1, \dots, \alpha A_q)$ *and* $\beta I = (\beta_1 I, \dots, \beta_q I)$.

Proof. We sketch only the proof of properties for $\sigma_{en}(A)$ since the proof is exactly the same in the case of $\sigma_{jn}(A)$.

(i) This property follows immediately from the definition of $\sigma_{\gamma n}(\cdot)$.

(ii) It follows from Properties 2.4(iii) of the joint numerical spectrum that, for all $K \in K^q(X)$, $\sigma_n(A+K) = \sigma_n(A'+K')$. Hence,

$$\sigma_{en}(A) = \bigcap_{K\in K^q(X)} \sigma_n(A'+K') \supset \bigcap_{H\in K^q(X')} \sigma_n(A'+H) = \sigma_{en}(A').$$

Therefore $\sigma_{en}(A') \subset \sigma_{en}(A)$. If X is a reflexive Banach space, the second inclusion derives immediately, since $A'' = A$.

(iii) Let U be an isometric isomorphisms on X. By using Properties 2.4, we have

$$\begin{aligned}\sigma_{en}(U^{-1}AU) &= \bigcap_{K\in K^q(X)} \sigma_n(U^{-1}(A+UKU^{-1})U)\\ &= \bigcap_{K\in K^q(X)} \sigma_n(A+UKU^{-1}) = \sigma_{en}(A).\end{aligned}$$

(iv) First, we discuss the case $\alpha = 0$. From Properties 2.4, we have $\sigma_n(\beta I + K) = \sigma_n(K) + \beta$. Then $\sigma_{en}(\beta I) = \bigcap_{K\in K^q(X)} \sigma_n(K) + \beta = \sigma_{en}(0) + \beta$. Hence $\sigma_{en}(\beta I) = \beta$. Second, if $\alpha \neq 0$, then $\sigma_{en}(\alpha A + \beta I) = \bigcap_{K\in K^q(X)} \sigma_n(\alpha(A + \alpha^{-1}K) + \beta I)$. Using Properties 2.4 we have $\sigma_n(\alpha(A+\alpha^{-1}K)+\beta I) = \alpha\sigma_n(A+\alpha^{-1}K) + \beta$. Thus

$$\begin{aligned}\sigma_{en}(\alpha A + \beta I) &= \alpha \bigcap_{K'\in K^q(X)} \sigma_n(A+K') + \beta\\ &= \alpha\sigma_{en}(A) + \beta.\end{aligned}$$

□

In the following, we prove a compactness result of the joint essential numerical spectrum for a q-tuple of bounded linear operators.

Proposition 2.8. *Let $A = (A_1, \ldots, A_q)$ be a q-tuple of bounded linear operators. Then we have*

(1)
$$\sigma_{en}(A) \subset \{\lambda \in \mathbb{C}^q : \|\lambda\| \leq \|A\|_e\},$$
where $\|A\|_e = \inf_{K\in K^q(X)} \|A+K\|$, *and* $\|A+K\| = (\sum_{i=1}^q \|A_i + K_i\|^2)^{\frac{1}{2}}$.

(2)
$$\sigma_{jn}(A) \subset \{\lambda \in \mathbb{C}^q : \|\lambda\| \leq \|A\|_w\},$$
where $\|A\|_w = \inf_{F\in W_*^q(X)} \|A+K\|$, *and* $\|A+F\| = (\sum_{i=1}^q \|A_i + F_i\|^2)^{\frac{1}{2}}$ *(for more details, we can refer to* [11–14]*).*

Proof. (1) Let $\lambda \in \sigma_{en}(A) = \bigcap_{K\in K^q(X)} \sigma_n(A+K)$. Then for all $K \in K^q(X)$, $\lambda \in \sigma_n(A+K)$. It follows from [6] that for all $K \in K^q(X)$, $\|\lambda\| \leq \|A+K\|$. Hence,

$$\|\lambda\| \leq \inf_{K\in K^q(X)} \|A+K\| = \|A\|_e.$$

(2) The proof follows exactly as above. □

The following result establish a relationships between the sets $\sigma_{\gamma n}(A)$ and $\sigma_n(A)$, which generalizes a well known result for the joint numerical range of q-tuple of self-adjoint bounded linear operators on Hilbert space, see [17, Theorem 5.2].

Proposition 2.9. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X. Then*

$$\sigma_{\gamma n}(A) = \sigma_n(A) \quad \textit{if and only if} \quad Ext(\sigma_n(A)) \subseteq \sigma_{\gamma n}(A),$$

where $Ext(\sigma_n(A)) = \{\lambda \in \sigma_n(A)$ such that $\forall \alpha, \beta \in \sigma_n(A), \lambda \notin]\alpha, \beta[\}$.

Proof. We give only the proof of the case of the joint essential numerical spectrum since the joint Jeribi essential numerical spectrum follows by the same argument. The first implication is immediate. Conversely, if $Ext(\sigma_n(A)) \subseteq \sigma_{en}(A)$. Using the convexity of the set $\sigma_{en}(A)$, we obtain $co(Ext(\sigma_n(A))) \subseteq \sigma_{en}(A) \subseteq \sigma_n(A)$. Then we have the equality, since $\sigma_n(A) \subseteq co(Ext(\sigma_n(A)))$. □

In the sequel, we extend the notion of joint spatial essential numerical range of a q-tuple of bounded linear operators on a Banach space (see L. T. Mang [19, Definition 1.4, page 24]) to a q-tuple of unbounded linear operators.

Definition 2.10. Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, we define the set $N_e(A)$ by the set of all complex numbers $\lambda \in \mathbb{C}^q$ with the property that there are nets $(x_\alpha) \subset D(A)$, $(x'_\alpha) \subset X'$ such that $\|x_\alpha\| = 1, x'_\alpha \in J(x_\alpha)$ for all α, $x_\alpha \to 0$ weakly and $\langle Ax_\alpha, x'_\alpha\rangle \to \lambda$.

Remark 2.11. If X is a Hilbert space, $N_e(A)$ can be defined by weakly null sequence $(x_n) \subset X$ of unit norm, see [18, Theorem 2.1]. If X is a Banach space, we use the notion of nets which guaranties that $N_e(A)$ is not empty, however this set can be empty if we use a weakly null sequence $(x_n) \subset X$ of unit norm.

The following theorem relates the joint essential numerical spectrum $\sigma_{en}(\cdot)$ to the set $N_e(\cdot)$.

Theorem 2.12. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X. Then we have*

$$cl(co(N_e(A))) \subset \sigma_{en}(A).$$

Proof. For $\lambda \in co(N_e(A))$, we write λ as

$$\lambda = \sum_{i=1}^{p} \beta_i \lambda_i, \text{ where } \forall i = 1, \dots, p\ ; 0 \le \beta_i \le 1;\ \sum_{i=1}^{p} \beta_i = 1 \text{ and } \lambda_i \in N_e(A).$$

For all $i = 1, \dots, p$, there exists nets $(x_{i_\alpha}) \subset D(A), (x'_{i_\alpha}) \subset X'$ such that $\|x_{i_\alpha}\| = \|x'_{i_\alpha}\| = \langle x_{i_\alpha}, x'_{i_\alpha}\rangle = 1$, $x_{i_\alpha} \to 0$ weakly and $\langle Ax_{i_\alpha}, x'_{i_\alpha}\rangle \to \lambda_i$. Let $K \in K^q(X)$, we consider an open hyperplane $H^+_{a,\omega}$ such that $\sigma_n(A+K) \subset H^-_{a,\omega}$ and $a.(A+K)$ is the generator of a ω-contractive C_0-semigroups. By the Hille-Yosida and Lummer-Phillips Theorems [8], we have

$$\mathrm{Re}(\langle a.(A+K)x, x'\rangle) \le \omega$$

for all $x \in D(A)$, $x' \in X'$ such that $\|x\| = \|x'\| = \langle x, x'\rangle = 1$. Hence

$$\begin{aligned}
\mathrm{Re}(a.\lambda) &= \mathrm{Re}\Big(\sum_{i=1}^{p} \beta_i a.\lambda_i\Big) \\
&= \mathrm{Re}\Big(\sum_{i=1}^{p} \beta_i \lim_{\alpha\to+\infty} \langle a.Ax_{i_\alpha}, x'_{i_\alpha}\rangle\Big) \\
&= \sum_{i=1}^{p} \beta_i \Big(\lim_{\alpha\to+\infty} \mathrm{Re}(\langle a.(A+K)x_{i_\alpha}, x'_{i_\alpha}\rangle) - \lim_{\alpha\to+\infty} \mathrm{Re}(\langle a.Kx_{i_\alpha}, x'_{i_\alpha}\rangle)\Big).
\end{aligned}$$

Since K_j is compact and $x_{i_\alpha} \to 0$ weakly, then $\lim_{\alpha\to+\infty}\langle K_j x_{i_\alpha}, x'_{i_\alpha}\rangle = 0$. Therefore

$$\mathrm{Re}(a.\lambda) \le \sum_{i=1}^{p} \beta_i \omega = \omega,$$

which implies that $\lambda \in \sigma_n(A+K)$ for all $K \in K^q(X)$. Hence $cl(co(N_e(A))) \subset \sigma_{en}(A)$, since $\sigma_{en}(A)$ is closed. □

Remark 2.13. (i) The result of the previous theorem generalizes the one established in [19] in the case of joint algebraic essential numerical range of q-tuple of bounded operators.

(ii) We emphasize that Theorem 2.12 is proved in [1] for the case $q = 1$.

3. Other description of $\sigma_{\gamma n}(\cdot)$

In this section, we establish several equivalent formulation of the joint essential numerical spectrum. Due to the obtained results, we give more characterization of $\sigma_{\gamma n}(\cdot)$.

In the following result, we obtain a description of $\sigma_{\gamma n}(A)$ in terms of the perturbation of one of the components of A.

Proposition 3.1. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X. Then*

(1)

$$\sigma_{en}(A) = \bigcap_{p=1}^{q} \bigcap_{K \in \widetilde{K}_p^q(X)} \sigma_n(A+K),$$

where $\widetilde{K}_p^q(X) = \{0\}^{p-1} \times K(X) \times \{0\}^{q-p}$.

(2)

$$\sigma_{jn}(A) = \bigcap_{p=1}^{q} \bigcap_{F \in \widetilde{W_*}_p^q(X)} \sigma_n(A+F),$$

where $\widetilde{W_*}_p^q(X) = \{0\}^{p-1} \times W_*(X) \times \{0\}^{q-p}$.

Proof. (1) First, we observe that $\widetilde{K}_p^q(X) \subseteq K^q(X)$ for all $p \in \{1, \ldots, q\}$. Consequently, we have

$$\sigma_{en}(A) \subseteq \bigcap_{p=1}^{q} \bigcap_{K \in \widetilde{K}_p^q(X)} \sigma_n(A+K).$$

Second, let $\lambda \notin \sigma_{en}(A)$, then there exists $K \in K^q(X)$ so that $\lambda \in \rho_n(A+K)$. It follows from Proposition 2.2 that there exists $a \in \Omega$ such that $a.\lambda \in \rho_n(a.(A+K))$. Since $\|a\| = 1$, then there exists $a_p \neq 0$, $p \in \{1, \ldots, q\}$, and if we let $\widetilde{K} = (0, \ldots, 0, \frac{a.K}{a_p}, 0, \ldots, 0) \in \widetilde{K}_p^q(X)$, we have $a.\lambda \in \rho_n(a.(A+\widetilde{K}))$. Applying Proposition 2.2, we obtain $\lambda \in \rho_n(A+\widetilde{K})$ and consequently,

$$\lambda \notin \bigcap_{p=1}^{q} \bigcap_{K \in \widetilde{K}_p^q(X)} \sigma_n(A+K).$$

The proof of (2) follows by the same argument as above. □

In [6], Proposition 2.7, W. Boubaker et al. proved that $\sigma_n(A) \subseteq \prod_{i=1}^{q} \sigma_n(A_i)$. Using the above proposition, we can show a similar result for the joint essential numerical spectrum.

Corollary 3.2. *Let $(A = (A_1, \ldots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \ldots, A_q$ are bounded linear operators on a Banach space X. Then*

$$\sigma_{en}(A) \subseteq \prod_{i=1}^{q} \sigma_{en}(A_i).$$

Proof. For all $j \in \{1, \ldots, q\}$ and $K \in \widetilde{K}_j^q(X)$, we have

$$\sigma_n(A+K) \subset \prod_{i=1}^{j-1} \sigma_n(A_i) \times \sigma_n(A_j + K_j) \times \prod_{i=j+1}^{q} \sigma_n(A_i).$$

Then

$$\bigcap_{K\in\widetilde{K}_j^q(X)} \sigma_n(A+K) \subseteq \prod_{i=1}^{j-1} \sigma_n(A_i) \times \sigma_{en}(A_j) \times \prod_{i=j+1}^{q} \sigma_n(A_i).$$

Using Proposition 3.1, we obtain

$$\sigma_{en}(A) \subseteq \bigcap_{j=1}^{q} \prod_{i=1}^{j-1} \sigma_n(A_i) \times \sigma_{en}(A_j) \times \prod_{i=j+1}^{q} \sigma_n(A_i).$$

Since $\sigma_{en}(A_i) \subset \sigma_n(A_i)$, we have the desired result. □

We are now able to prove how the joint essential numerical spectrum contains more information than the joint essential spectrum. It is well know that, if K is a q-tuple of compact operators on an infinite dimensional Banach space, then $\sigma_e(K) = \{0_{\mathbb{C}^q}\}$. But the converse is not always true. We purpose to prove the equivalence by using the notion of the joint essential numerical spectrum. For this, we introduce the measure of non-compactness (see [3, 16, 20]).

Definition 3.3. Let A be a bounded linear operator on a Banach space X. We define the semi-norm $\|\cdot\|_\mu$ in $B(X)$, called measure of non-compactness, as:

$$\|A\|_\mu = \inf\{\|A|_M\| : M \subset X \text{ a subspace of finite codimension}\}.$$

We list some useful properties of the measure of non-compactness.

Lemma 3.4. *Let A be a bounded linear operator on Banach space X. Then we have*

(i) $\|A\|_\mu = 0$ *if and only if A is compact.*
(ii) $\|A\|_\mu$ *is an algebra semi-norm in $B(X)$, i.e., for all $T, S \in B(X)$ and $\alpha \in \mathbb{C}$, we have*

$$\begin{aligned}\|T+S\|_\mu &\le \|T\|_\mu + \|S\|_\mu,\\ \|TS\|_\mu &\le \|T\|_\mu \cdot \|S\|_\mu,\\ \|\alpha T\|_\mu &= |\alpha|\|T\|_\mu.\end{aligned}$$

Also, we will use the following characterization of $co(N_e(A))$ proved in [19, Proposition 1.9].

Lemma 3.5. *Let A be a q-tuple of bounded linear operators on a Banach space X. Then*

$$co(N_e(A)) = \{(f(A_1), \dots, f(A_q)) : f \in \mathfrak{B}\},$$

where $\mathfrak{B} = \{f \in B(X)', |f(R)| \le \|R\|_\mu$ *for all* $R \in B(X)$, $f(I) = 1, f(K(X)) = \{0\}\}$.

In the next we generalizes the result of [1] to the case of a q-tuple of operators.

Theorem 3.6. *Let $A = (A_1, \dots, A_q)$ be a q-tuple of bounded linear operators. Then $\sigma_{en}(A) = \{0_{\mathbb{C}^q}\}$ if and only if A is a q-tuple of compact linear operators.*

Proof. If A is a q-tuple of compact linear operators, then

$$\sigma_{en}(A) \subset \sigma_n(0) = \{0_{\mathbb{C}^q}\}.$$

By Theorem 2.12, we have $co(N_e(A)) \subset \sigma_{en}(A)$. Since $co(N_e(A))$ is not empty, we obtain $\sigma_{en}(A) = \{0_{\mathbb{C}^q}\}$. For the converse, we consider A a q-tuple of bounded linear operators such that $\sigma_{en}(A) = \{0_{\mathbb{C}^q}\}$. In particular $co(N_e(A)) = \{0_{\mathbb{C}^q}\}$. Let $i \in \{1, \ldots, q\}$ and $\lambda \in co(N_e(A_i))$. Then there exists $f \in \mathfrak{B}$ such that $\lambda = f(A_i)$. Using Lemma 3.5, we have

$$(f(A_1), \ldots, f(A_{i-1}), \lambda, f(A_{i+1}), \ldots, f(A_q)) \in co(N_e(A)) = \{0_{\mathbb{C}^q}\}.$$

Then $\lambda = 0$ and $co(N_e(A_i)) = \{0\}$. On the other hand, we have

$$e^{-1}\|A_i\|_\mu \leq \max\{|\lambda| : \lambda \in co(N_e(A_i))\} \leq \|A_i\|_\mu,$$

(see [3, page 5]). This implies that $\|A_i\|_\mu = 0$ for all $i \in \{1, \ldots, q\}$. Due to Lemma 3.4, we deduce that $A \in K^q(X)$. □

An immediate consequence is:

Corollary 3.7. *Let $A = (A_1, \ldots, A_q)$ be a q-tuple of bounded linear operators and $\lambda \in \mathbb{C}^q$. Then $\sigma_{en}(A) = \{\lambda\}$ if and only if there exists $K \in K^q(X)$ such that $A = \lambda I + K$.*

We now characterize the joint essential numerical spectrum in terms of the intersection of essential numerical spectrum of linear combinations of components of A.

Proposition 3.8. *Let $(A = (A_1, \ldots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \ldots, A_q$ are bounded linear operators on a Banach space X. Then,*

$$\lambda \in \sigma_{\gamma n}(A) \quad \textit{if and only if for all} \quad a \in \mathbb{C}^q,\ a.\lambda \in \sigma_{\gamma n}(a.A).$$

Proof. Let $\lambda \in \sigma_{en}(A)$ and $a \in \mathbb{C}^q$, we want to show that $a.\lambda \in \sigma_{en}(a.A)$. We discuss only the case $a \neq 0$, since the result is hold for the case $a = 0$. Due to Proposition 3.1 and Proposition 2.2, we have $\frac{a}{\|a\|}.\lambda \in \sigma_n(\frac{a}{\|a\|}.(A + K))$ for all $p \in \{1, \ldots, q\}$, $K \in \widetilde{K}_p^q(X)$. By Properties 2.7(iv), we obtain $a.\lambda \in \sigma_n(a.(A + K))$ for all $p \in \{1, \ldots, q\}$, $K \in \widetilde{K}_p^q(X)$. Since $(a_1, \ldots, a_q) \neq 0_{\mathbb{C}^q}$, then there exists $i \in \{1, \ldots, q\}$ such that $a_i \neq 0$. So, for all $K_i \in K(X)$ we have $a.\lambda \in \sigma_n(a.A + \overline{a_i}K_i)$. Let $K \in K(X)$ and $K_i = \frac{K}{\overline{a_i}}$. Then $a.\lambda \in \sigma_n(a.A + K)$. We deduce that $a.\lambda \in \sigma_{en}(a.A)$. For the converse, let $\lambda \in \mathbb{C}^q$ such that for all $a \in \mathbb{C}^q$, $a.\lambda \in \sigma_{en}(a.A)$. We suppose that $\lambda \notin \sigma_{en}(A)$. Then from Proposition 3.1, there exist $p \in \{1, \ldots, q\}$ and $K \in \widetilde{K}_p^q(X)$ such that $\lambda \in \rho_n(A + K)$. By Proposition 2.2, there exists $b \in \Omega$ such that $b.\lambda \in \rho_n(b.(A + K))$. Hence $b.\lambda \in \rho_n(b.A + \overline{b_p}K_p)$ which is a contradiction. The result is checked for $\sigma_{jn}(A)$ and the proof is similar of the previous proof. □

We give now the following result analog of the separation theorem for a convex set.

Proposition 3.9. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are bounded linear operators on a Banach space X. Then*

(1) *$\lambda \notin \sigma_{en}(A)$ if and only if there exist $K \in K(X)$, $c \in \mathbb{C}^q$ and $r > 0$ such that for all $(x, x') \in \mathcal{A}$, $Re\langle (c.(A - \lambda) + K)x, x'\rangle > r$,*

(2) *$\lambda \notin \sigma_{jn}(A)$ if and only if there exist $F \in W_*(X)$, $c \in \mathbb{C}^q$ and $r > 0$ such that for all $(x, x') \in \mathcal{A}$, $Re\langle (c.(A - \lambda) + F)x, x'\rangle > r$,*

where $\mathcal{A} = \mathcal{A}_1 \bigcup \mathcal{A}_2$ and

- *$\mathcal{A}_1 = \{(x, x') \in X \times X' : x \in D(A)$ and $x' \in J(x)\}$,*
- *$\mathcal{A}_2 = \{(x, x') \in X \times X' : x \in D(A), x' \in D(A')$ such that there exists $\lambda \in \mathbb{C} : (c.A + K)'x' = \lambda x$ and $\langle x, x'\rangle = 1\}$.*

Proof. (1) Let $\lambda \notin \sigma_{en}(A)$. From Proposition 3.8, there exists $c \in \mathbb{C}^q$ such that $c.\lambda \notin \sigma_{en}(c.A)$. Hence, there exists $K \in K(X)$ such that $c.\lambda \notin \sigma_n(c.A + K)$. Since the numerical spectrum is a convex set, by the separation theorem, there exists $r > 0$ such that

$$\mathrm{Re}(c.\lambda) + r < \mathrm{Re}\, d \quad \text{for all } d \in \sigma_n(c.A + K). \tag{3.1}$$

Let $(x, x') \in \mathcal{A}$, it is easy to show that $\langle (c.A+K)x, x'\rangle \in W(c.A+K) \cup \sigma_p((c.A+K)')$. Using the fact that, $\sigma_n(c.A + K) = cl(coW(c.A + K)) \bigcup \sigma_p((c.A + K)')$ (see [2]) and inequality (3.1), we deduce that

$$\mathrm{Re}\langle (c.(A - \lambda) + K)x, x'\rangle > r, \quad \forall (x, x') \in \mathcal{A}.$$

For the converse we argue by contradiction, we assume that there exist $\lambda \in \sigma_{en}(A)$, $K \in K(X)$, $c \in \mathbb{C}^q$ and $r > 0$ such that $\mathrm{Re}\langle (c.(A - \lambda) + K)x, x'\rangle > r$, $\forall (x, x') \in \mathcal{A}$. Due to Proposition 3.8, $c.\lambda \in \sigma_{en}(c.A)$. So, for all $K \in K(X)$ $c.\lambda \in \sigma_n(c.A+K) = cl(co(W(c.A+K)) \bigcup \sigma_p((c.A+K)')$. On the one hand, if $c.\lambda \in cl(co(W(c.A+K))$, then $c.\lambda = \lim_{n\to+\infty} y_n$, where $y_n \in co(W(c.A+K))$. We write y_n as $y_n = \sum_{i=1}^{p} \alpha_{i_n} z_{i_n}$, where $\sum_{i=1}^{p} \alpha_{i_n} = 1$ and $z_{i_n} = \langle (c.A + K)x_{i_n}, x'_{i_n}\rangle$, $(x_{i_n}, x'_{i_n}) \in \mathcal{A}_1$. By assumption we have $\mathrm{Re}\, z_{i_n} > \mathrm{Re}(c.\lambda) + r$, which implies that $\mathrm{Re}(c.\lambda) = \lim_{n\to+\infty} \mathrm{Re}\, y_n \geq \mathrm{Re}(c.\lambda) + r$. Then $r \leq 0$ and this is contradiction. On the other hand, if $c.\lambda \in \sigma_p((c.A + K)')$, then there exists $(x, x') \in \mathcal{A}_2$ such that $c.\lambda = \langle (c.A + K)x, x'\rangle$. Therefore $\langle (c.(A - \lambda) + K)x, x'\rangle = 0 > r$, this contradict our supposition.

(2) The proof is similar of the previous proof. □

The next result shows that $\sigma_{en}(A)$ can be expressed as the intersection of half spaces. This result generalizes the one proved by C. K. Li and Y. T. Poon [18, Theorem 4.2] in the case of joint essential numerical range of a q-tuple of bounded self adjoint operators on a Hilbert space. It generalizes also the result of L. T. Mang [19, Corollary 3.3, page 45] proved in the case of joint algebraic essential numerical range of a q-tuple of bounded operators on a Banach space.

Theorem 3.10. *Let $(A = (A_1, \dots, A_q), D(A))$ be a q-tuple of linear operators, where A_1 is closed and densely defined linear operator and $A_2, \dots, A_q$ are* *bounded linear operators on a Banach space X. Then*

$$\sigma_{\gamma n}(A) = \bigcap_{c\in\Omega}\{\lambda \in \mathbb{C}^q \text{ such that } \operatorname{Re}(c.\lambda) \geq \min \operatorname{Re}\sigma_{\gamma n}(c.A)\}.$$

Proof. We give only the proof of the theorem in the case of the joint essential numerical spectrum since the joint Jeribi essential numerical spectrum follows by the same argument. Let $\lambda \in \sigma_{en}(A)$. Due to Proposition 3.8, we have $c.\lambda \in \sigma_{en}(c.A)$ for all $c \in \mathbb{C}^q$. Hence $\operatorname{Re}(c.\lambda) \geq \min \operatorname{Re}\sigma_{en}(c.A)$ for all $c \in \mathbb{C}^q$. Moreover,

$$\sigma_{en}(A) \subset \bigcap_{c\in\Omega}\{\lambda \in \mathbb{C}^q \text{ such that } \operatorname{Re}(c.\lambda) \geq \min \operatorname{Re}\sigma_{en}(c.A)\}.$$

On the other hand, let $\lambda \in \bigcap_{c\in\Omega}\{\lambda \in \mathbb{C}^q \text{ such that } \operatorname{Re}(c.\lambda) \geq \min \operatorname{Re}\sigma_{en}(c.A)\}$. We argue by contradiction, we assume that $\lambda \notin \sigma_{en}(A)$. By Proposition 3.9, there exist $K \in K(X)$, $c \in \mathbb{C}^q$ and $r > 0$ such that for all $(x, x') \in \mathcal{A}$

$$Re\langle (c.(A-\lambda) + K)x, x'\rangle > r. \tag{3.2}$$

We want to prove that $\min(Re(\sigma_n(c.A+K))) > r + Re(c.\lambda)$. Let $y \in \sigma_n(c.A+K) = cl(coW(c.A+K)) \bigcup \sigma_p((c.A+K)')$. If $y \in cl(coW(c.A+K))$, then $y = \lim_{n\to+\infty} y_n$, where $y_n \in co(W(c.A+K))$. We write y_n as $y_n = \sum_{i=1}^{p}\alpha_{i_n} z_{i_n}$ where $\sum_{i=1}^{p}\alpha_{i_n} = 1$ and $z_{i_n} = \langle (c.A+K)x_{i_n}, x'_{i_n}\rangle \in W(c.A+K)$, $(x_{i_n}, x'_{i_n}) \in \mathcal{A}_1$. Using the inequality (3.2), we have $\operatorname{Re} z_{i_n} > \operatorname{Re}(c.\lambda) + r$. Hence,

$$\operatorname{Re} y = \lim_{n\to+\infty} \operatorname{Re} y_n \geq r + \operatorname{Re}(c.\lambda). \tag{3.3}$$

If $y \in \sigma_p((c.A+K)')$, then there exists $(x, x') \in \mathcal{A}_2$ such that $\langle (c.A+K)x, x'\rangle = y$. Applying the inequality (3.2), we obtain

$$\operatorname{Re} y = \operatorname{Re}\langle (c.A+K)x, x'\rangle \geq r + \operatorname{Re}(c.\lambda). \tag{3.4}$$

We deduce from the inequalities (3.3) and (3.4) that for all $y \in \sigma_n(c.A+K)$, $\operatorname{Re} y \geq r + \operatorname{Re}(c.\lambda)$. Furthermore, $\min \operatorname{Re}\sigma_n(c.A+K) \geq r + \operatorname{Re}(c.\lambda)$. Since $\sigma_{en}(c.A) \subset \sigma_n(c.A+K)$, then $\min \operatorname{Re}\sigma_{en}(c.A) \geq r + \operatorname{Re}(c.\lambda)$. We assume $c \neq 0$, since in the case $c = 0$ we obtain immediately a contradiction. Using the fact that $\lambda \in \bigcap_{c\in\Omega}\{\lambda \in \mathbb{C}^q \text{ such that } \operatorname{Re}(c.\lambda) \geq \min \operatorname{Re}\sigma_{en}(c.A)\}$ together with the property (iv) of Properties 2.7, we deduce that $\operatorname{Re}(\frac{c}{\|c\|}.\lambda) \geq \min \operatorname{Re}\sigma_{en}(\frac{c}{\|c\|}.A) \geq \frac{r}{\|c\|} + \operatorname{Re}(\frac{c}{\|c\|}.\lambda)$. Thus, $r \leq 0$ and this is contradiction. $\square$

4. Example

Let Ω be a locally compact space, we consider the Banach space $X = C_0(\Omega)$, where

$$C_0(\Omega) := \{f \in C(\Omega) : \forall\varepsilon > 0 \text{ there exists compact } K_\varepsilon \subset \Omega \text{ such that } |f(s)| < \varepsilon, \forall s \in \Omega\backslash K_\varepsilon\}.$$

Let $\varphi : \Omega \to \mathbb{C}$ be a continuous and bounded function verifying the following hypothesis

$$(H): \ \varphi^{-1}(\{\lambda\}) \text{ is a countable set, } \forall \lambda \in \varphi(\Omega).$$

We consider $M_\varphi : C_0(\Omega) \to C_0(\Omega)$, $M_\varphi(f) = \varphi f$, the multiplication operator and $M_\phi = (M_\varphi, \dots, M_\varphi)$ the q-tuple of multiplication operators, where $\phi = (\varphi, \dots, \varphi)$. It is proved in [6] that $\sigma_n(M_\phi) = cl(co(\phi(\Omega)))$, where $\phi(\Omega) = \{(\varphi(x), \dots, \varphi(x)); x \in \Omega\}$. The case of essential numerical spectrum of multiplication operator has been previously considered by Abdelhedi et al. [1] in the particular case of $q = 1$. The following result generalize their result to q-tuple of multiplication operators.

Proposition 4.1. *Let φ satisfying the hypothesis (H) and $\phi = (\varphi, \dots, \varphi)$. Then*

$$\sigma_{en}(M_\phi) = cl(co(\phi(\Omega))).$$

Proof. On the one hand, we have $\sigma_{en}(M_\phi) \subset \sigma_n(M_\phi) = cl(co(\phi(\Omega)))$. On the other hand, let $\lambda \in co(\phi(\Omega))$, then $\lambda = \sum_{i=1}^p \beta_i \lambda_i$, where $\sum_{i=1}^p \beta_i = 1$ and $\forall i \in \{1, \dots, p\}$, $\lambda_i = (\varphi(x_i), \dots, \varphi(x_i))$ for some $x_i \in \Omega$. Let $a \in \mathbb{C}^q$, we have $a.\lambda = \sum_{j=1}^q \overline{a_j}(\sum_{i=1}^p \beta_i \varphi(x_i))$. Since $\sum_{i=1}^p \beta_i \varphi(x_i) \in co(\varphi(\Omega))$, then $a.\lambda \in (\sum_{j=1}^q \overline{a_j}) co(\varphi(\Omega))$. Using the result $\sigma_{en}(M_\varphi) = cl(co\varphi(\Omega))$ given in [1] together with property (iv) of Properties 2.7, we obtain

$$\begin{aligned} a.\lambda \in (\sum_{j=1}^q \overline{a_j}) \sigma_{en}(M_\varphi) &= \sigma_{en}(\sum_{j=1}^q \overline{a_j} M_\varphi) \\ &= \sigma_{en}(a.M_\phi), \forall a \in \mathbb{C}^q. \end{aligned}$$

From Proposition 3.8, we deduce that $\lambda \in \sigma_{en}(M_\phi)$. Thus, we have the following inclusion

$$co(\phi(\Omega)) \subset \sigma_{en}(M_\phi) \subset cl(co\phi(\Omega)).$$

Since $\sigma_{en}(M_\phi)$ is closed, we can derive the desired result. □

References

[1] B. Abdelhedi, W. Boubaker, and N. Moalla, *On the essential numerical spectrum of operators on Banach spaces*, Filomat **33** (2019), no. 7, 2191–2199. `https://doi.org/10.2298/fil1907191a`

[2] M. Adler, W. Dada, and A. Radl, *A semigroup approach to the numerical range of operators on Banach spaces*, Semigroup Forum **94** (2017), no. 1, 51–70. `https://doi.org/10.1007/s00233-015-9752-y`

[3] M. Barraa and V. Müller, *On the essential numerical range*, Acta Sci. Math. (Szeged) **71** (2005), no. 1-2, 285–298.

[4] F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Mathematical Society Lecture Note Series, 2, Cambridge Univ. Press, London, 1971.

[5] F. Bonsall and J. Duncan, *Numerical ranges. II*, London Mathematical Society Lecture Note Series, No. 10, Cambridge Univ. Press, New York, 1973.

[6] W. Boubaker, N. Moalla, and A. Radl, *On the joint numerical spectrum in Banach spaces*, Bull. Iranian Math. Soc. **45** (2019), no. 2, 345–358. https://doi.org/10.1007/s41980-018-0136-4
[7] A. Dash, *Joint numerical range*, Glasnik Mat. Ser. III **7(27)** (1972), 75–81.
[8] K.-J. Engel and R. J. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, 194, Springer, New York, 2000.
[9] K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Universitext, Springer, New York, 1997. https://doi.org/10.1007/978-1-4613-8498-4
[10] J. Janas, *Note on the joint spectrum of the Wiener-Hopf operators*, Proc. Amer. Math. Soc. **50** (1975), 303–308. https://doi.org/10.2307/2040557
[11] A. Jeribi, *Linear operators and their essential pseudospectra*, CRC Press, Boca Raton, 2018.
[12] A. Jeribi, *Denseness, bases and frames in Banach spaces and applications*, De Gruyter, Berlin, 2018. https://doi.org/10.1515/9783110493863
[13] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices*, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-17566-9
[14] A. Jeribi, *Perturbation theory for linear operators—denseness and bases with applications*, Springer, Singapore, 2021. https://doi.org/10.1007/978-981-16-2528-2
[15] J. C. Kim, *On joint Weyl and Browder spectra*, Bull. Korean Math. Soc. **37** (2000), no. 1, 53–62.
[16] A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Funct. Anal. **7** (1971), 1–26. https://doi.org/10.1016/0022-1236(71)90041-3
[17] C.-K. Li and Y.-T. Poon, *Convexity of the joint numerical range*, SIAM J. Matrix Anal. Appl. **21** (1999), no. 2, 668–678. https://doi.org/10.1137/S0895479898343516
[18] C.-K. Li and Y.-T. Poon, *The joint essential numerical range of operators: convexity and related results*, Studia Math. **194** (2009), no. 1, 91–104. https://doi.org/10.4064/sm194-1-6
[19] L. T. Mang, *The joint numerical range and the joint essential numerical range*, Mphil. THESIS, 2013.
[20] V. Müller, *The joint essential numerical range, compact perturbations, and the Olsen problem*, Studia Math. **197** (2010), no. 3, 275–290. https://doi.org/10.4064/sm197-3-5
[21] V. Wrobel, *Joint spectra and joint numerical ranges for pairwise commuting operators in Banach spaces*, Glasgow Math. J. **30** (1988), no. 2, 145–153. https://doi.org/10.1017/S0017089500007163
[22] Y. Yang, *Joint spatial numerical ranges of operators on Banach spaces*, Bull. Korean Math. Soc. **26** (1989), no. 2, 119–126.

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