Commun. Korean Math. Soc. **38** (2023), No. 4, pp. 1019–1028
https://doi.org/10.4134/CKMS.c230018
pISSN: 1225-1763 / eISSN: 2234-3024

NONLINEAR MAPS PRESERVING THE MIXED PRODUCT ${}_*[X \diamond Y, Z]$ ON $*$-ALGEBRAS

Raof Ahmad Bhat, Abbas Hussain Shikeh,
and Mohammad Aslam Siddeeque

Abstract. Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital prime $*$-algebras such that $\mathfrak{A}$ contains a nontrivial projection. In the present paper, we show that if a bijective map $\Theta : \mathfrak{A} \to \mathfrak{B}$ satisfies $\Theta({}_*[X \diamond Y, Z]) = {}_*[\Theta(X) \diamond \Theta(Y), \Theta(Z)]$ for all $X, Y, Z \in \mathfrak{A}$, then Θ or $-\Theta$ is a $*$-ring isomorphism. As an application, we shall characterize such maps in factor von Neumann algebras.

1. Introduction

Throughout the text, by algebra we mean an associative algebra over the field of complex numbers $\mathbb{C}$. An algebra $\mathfrak{A}$ is called prime if for any $\mathfrak{p}, \mathfrak{q} \in \mathfrak{A}$, $\mathfrak{p}\mathfrak{A}\mathfrak{q} = \{0\}$ implies that either $\mathfrak{p} = 0$ or $\mathfrak{q} = 0$. The centre of an algebra $\mathfrak{A}$ is denoted by $\mathcal{Z}(\mathfrak{A})$. Let $\mathfrak{A}$ be a $*$-algebra. For $x, y \in \mathfrak{A}$, we denote $xy + yx$ by $x \circ y$, $xy - yx$ by $[x, y]$, $xy + yx^*$ by $x \triangleleft y$, $xy^* - yx$ by ${}_*[x, y]$, $xy - yx^*$ by $[x, y]_*$, $xy^* - yx^*$ by $[x, y]_\bullet$, $x^*y + y^*x$ by $x \bullet y$, $x^*y + yx^*$ by $x \Diamond y$ and $xy^* + yx^*$ by $x \diamond y$. If $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-algebras, then a map $\Theta : \mathfrak{A} \to \mathfrak{B}$ is said to preserve '$*$' if $\Theta(X^*) = \Theta(X)^*$ for all $X \in \mathfrak{A}$. Moreover, a map $\Theta : \mathfrak{A} \to \mathfrak{B}$ is called a $*$-isomorphism if it preserves '$*$' and is an isomorphism.

In recent years, numerous authors have studied the problems concerning the characterization of maps on operator algebras that leave certain relations invariant. The most remarkable result in this direction was obtained by Martindale [12], who proved that if $\mathcal{A}$ is a prime ring containing a nontrivial idempotent and $\mathcal{B}$ is any ring, then a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi(XY) = \Phi(X)\Phi(Y)$ for all $X, Y \in \mathcal{A}$ is necessarily additive. Inspired from this, the problem of characterizing the maps preserving the products $x \circ y$, $[x, y]$, $[x, y]_*$, $[x, y]_\bullet$ and $x \bullet y$ between rings or operator algebras have received a lot of attention from various algebraists. For example, Cui and Li [2] proved that a bijective map preserving the product $[x, y]_*$ between factor von Neumann algebras is either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. This result was extended to von Neumann algebras by Bai and Du (see [1]). Li et al. [9] showed

Received February 1, 2023; Revised June 21, 2023; Accepted July 13, 2023.
2020 *Mathematics Subject Classification.* Primary 16W20, 47B48.
Key words and phrases. $*$-algebra, isomorphism, von Neumann algebra.

that a bijective map preserving the product $x \bullet y$ between von Neumann algebras with no central abelian projections is sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. Taghavi et al. [14] proved that a bijective map preserving the triple product $x\diamondsuit y\diamondsuit z$ between prime $*$-algebras is a $*$-ring isomorphism. Quiet recently, Zang et al. [18] proved that a bijective map preserving the triple product $x \bullet y \bullet z$ between factor von Neumann algebras is a linear $*$-isomorphism or the negative of a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism. For other results see [1–9, 11] and their bibliographic content.

Recently, nonlinear maps preserving the mixed products have received a fair amount of attention. For instance, Yang and Zhang [17, 19] studied the nonlinear maps preserving the mixed products $[[x, y], z]_*$ and $[[x, y]_*, z]$ on factor von Neumann algebras. Zhao et al. [21] established that a bijective map preserving the mixed product $[x \triangleleft y, z]$ between factor von Neumann algebras is a linear $*$-isomorphism or the negative of a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism. These results demonstrate that some new products can completely determine the isomorphisms between operator algebras. For other results see [15–17, 20, 21] and references therein.

Motivated by the results mentioned above, in this paper, we will investigate the structure of the nonlinear maps preserving the mixed product ${}_*[X \diamond Y, Z]$ on prime $*$-algebras.

2. Results

Throughout the text $\mathbf{i}$ denotes the imaginary unit. We begin with the following lemma which plays a pivotal role in the proof of our main result.

Lemma 2.1. *Let $\mathfrak{A}$ be a unital $*$-algebra and $A, B \in \mathfrak{A}$ such that $AX^* = XB$ for all $X \in \mathfrak{A}$. Then $A = B = 0$.*

Proof. Suppose $AX^* = XB$ for all $X \in \mathfrak{A}$. Taking $X = I$, we get $A = B$. Hence $AX^* = XA$ for all $X \in \mathfrak{A}$. Replacing X by $\mathbf{i}X$ in the last relation, we find that $AX^* = -XA$ for all $X \in \mathfrak{A}$. Adding the previous two relations, we get $XA = 0$ for all $X \in \mathfrak{A}$. Hence $A = 0$. □

Now we are ready to state our main result.

Theorem 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital prime $*$-algebras such that $\mathfrak{A}$ contains a nontrivial projection. Suppose that a bijective map $\Theta : \mathfrak{A} \to \mathfrak{B}$ satisfies $\Theta({}_*[X \diamond Y, Z]) = {}_*[\Theta(X) \diamond \Theta(Y), \Theta(Z)]$ for all $X, Y, Z \in \mathfrak{A}$. Then Θ or $-\Theta$ is a $*$-ring isomorphism.*

Proof. Let E_1 be a nontrivial projection of $\mathfrak{A}$. By the Peirce decomposition of $\mathfrak{A}$ with respect to E_1, we have $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$. Note that any $X \in \mathfrak{A}$ can be written as $X = X_{11} + X_{12} + X_{21} + X_{22}$, where $X_{ij} \in \mathfrak{A}_{ij}$ for $i, j = 1, 2$ and $E_2 = 1 - E_1$. The proof is organized in the following claims.

Claim I. $\Theta(0)=0$.
Since Θ is surjective, there is $X\in\mathfrak{A}$ such that $\Theta(X)=0$. Hence

$$\Theta(0)=\Theta({}_*[0\diamond X,0])={}_*[\Theta(0)\diamond\Theta(X),\Theta(0)]=0.$$

Claim II. For any $X_{11}\in\mathfrak{A}_{11}, Y_{12}\in\mathfrak{A}_{12}$, $Z_{21}\in\mathfrak{A}_{21}$ and $X_{22}\in\mathfrak{A}_{22}$, we have

$$\Theta(X_{11}+Y_{12}+Z_{21})=\Theta(X_{11})+\Theta(Y_{12})+\Theta(Z_{21})$$

and

$$\Theta(Y_{12}+Z_{21}+X_{22})=\Theta(Y_{12})+\Theta(Z_{21})+\Theta(X_{22}).$$

Choose $T\in\mathfrak{A}$ such that $\Theta(T)=\Theta(X_{11})+\Theta(Y_{12})+\Theta(Z_{21})$. Since for any $A_{22}\in\mathfrak{A}_{22}$, $X_{11}\diamond A_{22}=Z_{21}\diamond A_{22}=0$, we have

$$\begin{aligned}\Theta({}_*[A_{22}\diamond T,Z])&={}_*[\Theta(A_{22})\diamond\Theta(T),\Theta(Z)]\\&={}_*[\Theta(A_{22})\diamond\Theta(X_{11}),\Theta(Z)]+{}_*[\Theta(A_{22})\diamond\Theta(Y_{12}),\Theta(Z)]\\&\quad+{}_*[\Theta(A_{22})\diamond\Theta(Z_{21}),\Theta(Z)]\\&=\Theta({}_*[A_{22}\diamond Y_{12},Z])\end{aligned}$$

for all $Z\in\mathfrak{A}$. By the injectivity of Θ, we have ${}_*[A_{22}\diamond T,Z]={}_*[A_{22}\diamond Y_{12},Z]$. Hence ${}_*[A_{22}\diamond(T-Y_{12}),Z]=0$ for all $Z\in\mathfrak{A}$. Applying Lemma 2.1, we get $A_{22}\diamond(T-Y_{12})=0$, that is, $A_{22}(T^*-Y_{12}^*)=(Y_{12}-T)A_{22}^*$ for every $A_{22}\in\mathfrak{A}_{22}$. Replacing A_{22} by $\mathbf{i}A_{22}$ in the previous relation, we have $-A_{22}(T^*-Y_{12}^*)=(Y_{12}-T)A_{22}^*$ for every $A_{22}\in\mathfrak{A}_{22}$. Adding the last two relations, we find that $(T-Y_{12})A_{22}=0$ for every $A_{22}\in\mathfrak{A}_{22}$. By the primeness of $\mathfrak{A}$, we deduce that $TE_2=Y_{12}$. Consequently, $T_{22}=0$ and $T_{12}=Y_{12}$.

Since ${}_*[E_1\diamond X_{11},E_1]={}_*[E_1\diamond Y_{12},E_1]=0$, we have

$$\begin{aligned}\Theta({}_*[E_1\diamond T,E_1])&={}_*[\Theta(E_1)\diamond\Theta(T),\Theta(E_1)]\\&={}_*[\Theta(E_1)\diamond\Theta(X_{11}),\Theta(E_1)]+{}_*[\Theta(A_{22})\diamond\Theta(Y_{12}),\Theta(E_1)]\\&\quad+{}_*[\Theta(E_1)\diamond\Theta(Z_{21}),\Theta(E_1)]\\&=\Theta({}_*[E_1\diamond Z_{21},E_1]).\end{aligned}$$

By the injectivity of Θ, we get ${}_*[E_1\diamond(T-Z_{21}),Z]=0$. Applying Lemma 2.1, we have $E_1\diamond(T-Z_{21})=0$, that is, $E_1T^*+TE_1=Z_{21}+Z_{21}^*$. Consequently, $T_{21}=Z_{21}$.

Now for any $A_{11}\in\mathfrak{A}$, ${}_*[Y_{12}\diamond A_{11},\mathbf{i}(E_1-E_2)]={}_*[Z_{21}\diamond A_{11},\mathbf{i}(E_1-E_2)]=0$. Therefore,

$$\begin{aligned}\Theta({}_*[T\diamond A_{11},\mathbf{i}(E_1-E_2)]&={}_*[\Theta(T)\diamond\Theta(A_{11}),\Theta(\mathbf{i}(E_1-E_2))]\\&={}_*[\Theta(X_{11})\diamond\Theta(A_{11}),\Theta(\mathbf{i}(E_1-E_2))]\\&\quad+{}_*[\Theta(Y_{12})\diamond\Theta(A_{11}),\Theta((\mathbf{i}(E_1-E_2))]\\&\quad+{}_*[\Theta(Z_{21})\diamond\Theta(A_{11}),\Theta(\mathbf{i}(E_1-E_2))]\\&=\Theta({}_*[X_{11}\diamond A_{11},\mathbf{i}(E_1-E_2)]).\end{aligned}$$

By the injectivity of Θ, we have ${}_*[T\diamond A_{11},\mathbf{i}(E_1-E_2)]={}_*[X_{11}\diamond A_{11},\mathbf{i}(E_1-E_2)]$, that is, $TA_{11}^*+A_{11}T_{11}^*-A_{11}T^*E_2+T_{11}A_{11}^*+A_{11}T^*-E_2TA_{11}^*=2X_{11}A_{11}^*+$

$2A_{11}X^*_{11}$ for every $A_{11} \in \mathfrak{A}_{11}$. Replacing A_{11} by $\mathbf{i}A_{11}$ in the last relation and adding the two relations, we see that $A_{11}(T^*_{11} - X^*_{11}) = 0$ for every $A_{11} \in \mathfrak{A}_{11}$. Consequently, $T_{11} = X_{11}$. Analogously one can prove that $\Theta(Y_{12}+Z_{21}+X_{22}) = \Theta(Y_{12}) + \Theta(Z_{21}) + \Theta(X_{22})$. This proves Claim II.

Claim III. For any $X_{ij} \in \mathrm{S}_{ij}, 1 \leq i, j \leq 2$, we have

$$\Theta(\sum_{i,j=1}^{2} X_{ij}) = \sum_{i,j=1}^{2} \Theta(X_{ij}).$$

Since Θ is surjective, there is $T \in \mathfrak{A}$ such that $\Theta(T) = \Theta(X_{11}) + \Theta(X_{12}) + \Theta(X_{21}) + \Theta(X_{22})$. Now for every $A_{11} \in \mathfrak{A}_{11}$, we have ${}_*[E_1 \diamond X_{12}, A_{11}] = {}_*[E_1 \diamond X_{22}, A_{11}] = 0$. Therefore using Claim II, we find that

$$\begin{aligned}
\Theta({}_*[E_1 \diamond T, A_{11}]) &= {}_*[\Theta(E_1) \diamond \Theta(T), \Theta(A_{11})] \\
&= {}_*[\Theta(E_1) \diamond \Theta(X_{11}), \Theta(A_{11})] + {}_*[\Theta(E_1) \diamond \Theta(X_{12}), \Theta(A_{11})] \\
&\quad + {}_*[\Theta(E_1) \diamond \Theta(X_{21}), \Theta(A_{11})] + \Theta({}_*[E_1 \diamond X_{22}, \Theta(A_{11})]) \\
&= \Theta({}_*[E_1 \diamond X_{11}, A_{11}]) + \Theta({}_*[E_1 \diamond X_{21}, A_{11}]) \\
&= \Theta((X_{11}+X^*_{11})A^*_{11} - A_{11}(X_{11} + X^*_{11}) + X_{21}A^*_{11} - A_{11}X^*_{21}).
\end{aligned}$$

By the injectivity of Θ, we find that

$$\begin{aligned}
&(T^*_{11} + TE_1)A^*_{11} - A_{11}(E_1T^* + T_{11}) \\
&= (X_{11} + X^*_{11} + X_{21})A^*_{11} - A_{11}(X_{11} + X^*_{11} + X^*_{21}).
\end{aligned} \tag{1}$$

Similarly using $\mathbf{i}E_1$ instead of E_1 in the above calculation, we see that

$$\begin{aligned}
&(T^*_{11} - TE_1)A^*_{11} - A_{11}(E_1T^* - T_{11}) \\
&= (-X_{11} + X^*_{11} - X_{21})A^*_{11} - A_{11}(-X_{11} + X^*_{11} + X^*_{21}).
\end{aligned} \tag{2}$$

Replacing A_{11} by $\mathbf{i}A_{11}$ in the previous two relations, we get

$$\begin{aligned}
&(T^*_{11} + TE_1)A^*_{11} + A_{11}(E_1T^* + T_{11}) \\
&= (X_{11} + X^*_{11} + X_{21})A^*_{11} + A_{11}(X_{11} + X^*_{11} + X_{21})
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&(T^*_{11} - TE_1)A^*_{11} + A_{11}(E_1T^* - T_{11}) \\
&= (-X_{11} + X^*_{11} - X_{21})A^*_{11} + A_{11}(-X_{11} + X^*_{11} + X^*_{21}),
\end{aligned} \tag{4}$$

respectively. Adding (1) and (3), we deduce that $T^*_{11}+TE_1 = X_{11}+X^*_{11}+X_{21}$. Consequently, $T_{21} = X_{21}$ and

$$T^*_{11} + T_{11} = X_{11} + X^*_{11}. \tag{5}$$

Also adding (2) and (4), we find that $T^*_{11} - TE_1 = -X_{11} + X^*_{11} - X_{21}$. Hence

$$T^*_{11} - T_{11} = X^*_{11} - X_{11}. \tag{6}$$

From (5) and (6), we conclude that $T_{11} = X_{11}$. Now, using the same procedure as above with E_2 in place of E_1 and A_{22} in place of A_{11}, we can see that

$T_{12} = X_{12}$ and $T_{22} = X_{22}$.

Claim IV. For every $X_{ij}, Y_{ij} \in \mathfrak{A}_{ij}$ with $i \neq j$, we have

$$\Theta(X_{ij} + Y_{ij}) = \Theta(X_{ij}) + \Theta(Y_{ij}).$$

Since Θ is surjective, given $\Theta(X_{ij}) + \Theta(Y_{ij}) \in \mathfrak{B}$ and $\Theta(X_{ij}^*) + \Theta(Y_{ij}^*) \in \mathfrak{B}$, there exist $M, N \in \mathfrak{A}$ such that

$$\Theta(M) = \Theta(X_{ij}) + \Theta(Y_{ij}) \text{ and } \Theta(N) = \Theta(X_{ij}^*) + \Theta(Y_{ij}^*).$$

Now $E_i \diamond X_{ij} = 0$ and $E_i \diamond Y_{ij} = 0$. Hence for any $A \in \mathfrak{A}$, we have

$$\begin{aligned}\Theta({}_*[M \diamond E_i, A]) &= {}_*[\Theta(M) \diamond \Theta(E_i), \Theta(A)]\\ &= {}_*[\Theta(X_{ij}) \diamond \Theta(E_i), \Theta(A)] + {}_*[\Theta(Y_{ij}) \diamond \Theta(E_i), \Theta(A)] = 0.\end{aligned}$$

By injectiveness of Θ, we obtain ${}_*[M \diamond E_i, A] = 0$. Invoking Lemma 2.1, we get $M \diamond E_i = 0$, that is, $ME_i + E_iM^* = 0$. Similarly using $\mathbf{i}E_i$ instead of E_i in the previous calculation, we find that $ME_i - E_iM^* = 0$. From the last two relations, we get $ME_i = 0$. Consequently, $M_{ji} = M_{ii} = 0$. Now, repeating the above procedure with N in place of M and E_j instead of E_i, one can see that $NE_j = 0$. Hence $N_{jj} = N_{ij} = 0$. Therefore, $N = N_{ji} + N_{ii}$ and $M = M_{jj} + M_{ij}$. Now by Claims III and IV, we have

$$\begin{aligned}&\Theta({}_*[(E_i + X_{ij}) \diamond (E_j + Y_{ij}^*), E_j])\\ &= {}_*[(\Theta(E_i) + \Theta(X_{ij})) \diamond (\Theta(E_j) + \Theta(Y_{ij}^*)), \Theta(E_j)]\\ &= \Theta({}_*[E_i \diamond Y_{ij}^*, E_j]) + \Theta({}_*[E_j \diamond X_{ij}, E_j])\\ &= \Theta(Y_{ij} - Y_{ij}^*) + \Theta(X_{ij} - X_{ij}^*)\\ &= \Theta(X_{ij}) + \Theta(Y_{ij}) - \Theta(Y_{ij}^*) - \Theta(X_{ij}^*)\\ &= \Theta(M_{jj} + M_{ij} - N_{ji} - N_{ii}).\end{aligned}$$

Since Θ is injective, we have $X_{ij} + Y_{ij} - X_{ij}^* - Y_{ij}^* = M_{jj} + M_{ij} - N_{ji} - N_{ii}$. Consequently, $X_{ij} + Y_{ij} = M_{ij}$ and $M_{jj} = 0$.

Claim V. For any $X_{ii}, Y_{ii} \in \mathfrak{A}_{ii}, 1 \leq i \leq 2$, we have

$$\Theta(X_{ii} + Y_{ii}) = \Theta(X_{ii}) + \Theta(Y_{ii}).$$

Since Θ is surjective, there is $T \in \mathfrak{A}$ such that $\Theta(T) = \Theta(X_{ii}) + \Theta(Y_{ii})$. Now since $X_{ii} \diamond E_j = Y_{ii} \diamond E_j = 0$ for $i \neq j$, we have $\Theta({}_*[T \diamond E_j, A]) = 0$ for any $A \in \mathfrak{A}$. In view of injectiveness of Θ and Lemma 2.1, we obtain $T \diamond E_j = 0$, that is, $TE_j + E_jT^* = 0$. Similarly using $\mathbf{i}E_j$ instead of E_j in the above calculation, we get $TE_j - E_jT^* = 0$. Adding the previous two relations, we have $TE_j = 0$. Consequently, $T_{ij} = T_{jj} = 0$.

Now ${}_*[X_{ii} \diamond E_i, E_j] = {}_*[Y_{jj} \diamond E_i, E_j] = 0$. Therefore $\Theta({}_*[T \diamond E_i, E_j]) = 0$ and hence ${}_*[T \diamond E_i, E_j] = 0$, that is, $T_{ji} + T_{ji}^* = 0$. Multiplying the previous relation by E_i from right, we get $T_{ji} = 0$. Hence $T = T_{ii}$.

Next, for any $A_{ji} \in \mathfrak{A}_{ji}$, we have

$$\begin{aligned}\Theta({}_*[A_{ji} \diamond T, E_j]) &= \Theta({}_*[A_{ji} \diamond X_{ii}, E_j]) + \Theta({}_*[A_{ji} \diamond Y_{ii}, E_j])\\ &= \Theta(X_{ii}A_{ji}^* - A_{ji}X_{ii}^*) + \Theta(Y_{ii}A_{ji}^* - A_{ji}Y_{ii}^*).\end{aligned}$$

In view of Claims II and IV, and injectivity of Θ it follows that $T_{ii}A_{ji}^*+A_{ij}T_{ii}^* = X_{ii}A_{ji}^*-A_{ji}X_{ii}^*+Y_{ii}A_{ji}^*-A_{ji}Y_{ii}^*$ for any $A_{ji} \in \mathfrak{A}_{ji}$. Replacing A_{ji} by $\mathbf{i}A_{ji}$ in the last relation and adding the two relations, we find that $(T_{ii}-X_{ii}-Y_{ii})AEj = 0$ for all $A \in \mathfrak{A}$. Consequently $T_{ii} = X_{ii} + Y_{ii}$. This proves the claim.

Claim VI. Θ is additive.

Let $X = \sum_{i,j=1}^{2} X_{ij}$ and $Y = \sum_{i,j=1}^{2} Y_{ij}$. By Claims IV, V and VI, we have

$$\begin{aligned}\Theta(X+Y) &= \Theta(\sum_{i,j=1}^{2}(X_{ij}+Y_{ij})) \\ &= \sum_{i,j=1}^{2}\Theta(X_{ij}+Y_{ij}) \\ &= \sum_{i,j=1}^{2}\Theta(X_{ij}) + \sum_{i,j=1}^{2}\Theta(Y_{ij}) \\ &= \Theta(\sum_{i,j=1}^{2}X_{ij}) + \Theta(\sum_{i,j=1}^{2}Y_{ij}) = \Theta(X)+\Theta(Y).\end{aligned}$$

Claim VII. $\Theta(I)^* = \Theta(I) \in \mathcal{Z}(\mathfrak{B})$.

Choose $X \in \mathfrak{A}$ such that $\Theta(X) = I_{\mathfrak{B}}$. Then

$$\begin{aligned}2(\Theta(I)^* - \Theta(I)) &= {}_*[\Theta(X)\diamond\Theta(X), \Theta(I)] \\ &= \Theta({}_*[X\diamond X, I]) = 0.\end{aligned}$$

Hence $\Theta(I) = \Theta(I)^*$. Moreover, for any $Y \in \mathfrak{A}$, we have

$${}_*[\Theta(X)\diamond\Theta(Y), \Theta(I)] = \Theta({}_*[X\diamond Y, I]) = 0.$$

Thus $(\Theta(Y)+\Theta(Y)^*)\Theta(I) = \Theta(I)(\Theta(Y)+\Theta(Y)^*)$. By surjectiveness of Θ, we get $(Z+Z^*)\Theta(I) = \Theta(I)(Z+Z^*)$ for all $Z \in \mathfrak{B}$. Replacing Z by $\mathbf{i}Z$ in the last relation and adding the two relations, we find that $\Theta(I) \in \mathcal{Z}(\mathfrak{B})$.

Claim VIII. $\Theta(\mathbf{i}I)^* = -\Theta(\mathbf{i}I) \in \mathcal{Z}(\mathfrak{B})$ and $\Theta(\mathbf{i}I)^2 = -I_{\mathfrak{B}}$.

For any $Y \in \mathfrak{A}$, we have ${}_*[\Theta(I)\diamond\Theta(\mathbf{i}I), \Theta(Y)] = \Theta({}_*[I\diamond \mathbf{i}I, Y]) = 0$. In view of surjectiveness of Θ and Lemma 2.1, it follows that $\Theta(I)\diamond\Theta(\mathbf{i}I) = 0$. Consequently, $\Theta(\mathbf{i}I)^* = -\Theta(\mathbf{i}I)$. Now for any $X \in \mathfrak{A}$, we have

$$\begin{aligned}2\Theta(X^*-X) = \Theta({}_*[I\diamond I, X]) &= {}_*[\Theta(I)\diamond\Theta(I), \Theta(X)] \\ &= 2\Theta(I)^2(\Theta(X)^*-\Theta(X)).\end{aligned}$$

Thus Θ preserves symmetric elements in both directions. Now for any $A \in \mathfrak{A}$ with $A^* = A$, we have ${}_*[\Theta(A)\diamond\Theta(\mathbf{i}I), \Theta(Y)] = \Theta({}_*[A\diamond \mathbf{i}I, Y]) = 0$ for all $Y \in \mathfrak{B}$. Hence $\Theta(A)\diamond\Theta(\mathbf{i}I) = 0$, that is, $\Theta(A)\Theta(\mathbf{i}I) = \Theta(\mathbf{i}I)\Theta(A)$. Clearly any $B \in \mathfrak{B}$ can be written as $B = B_1 + \mathbf{i}B_2$, where $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathbf{i}}$. Therefore, $\Theta(\mathbf{i}I) \in \mathcal{Z}(\mathfrak{B})$ and hence $-4\Theta(\mathbf{i}I) = \Theta({}_*[\mathbf{i}I\diamond \mathbf{i}I, \mathbf{i}I]) = 4\Theta(\mathbf{i}I)^3$. Consequently, $\Theta(\mathbf{i}I)^2 = -I_{\mathfrak{B}}$.

Claim IX. For every $A \in \mathfrak{A}$, $\Theta(\mathbf{i}A) = \eta\Theta(\mathbf{i}I)\Theta(A)$, where $\eta = 1$ or $\eta = -1$.
It is easy to see that

$$-4\Theta(\mathbf{i}I) = \Theta({}_*[I \diamond I, \mathbf{i}I]) = {}_*[\Theta(I) \diamond \Theta(I), \Theta(\mathbf{i}I)] = -4\Theta(I)^2\Theta(\mathbf{i}I).$$

Hence $\Theta(I)^2 = I_{\mathfrak{B}}$. Consequently $\Theta(I) = I_{\mathfrak{B}}$ or $\Theta(I) = -I_{\mathfrak{B}}$. Note that for any $B^* = B \in \mathfrak{A}$, we have $-4\Theta(\mathbf{i}B) = \Theta({}_*[B \diamond I, \mathbf{i}I]) = {}_*[\Theta(B) \diamond \Theta(I), \Theta(\mathbf{i}I)] = -4\Theta(I)\Theta(\mathbf{i}I)\Theta(B)$. Hence $\Theta(\mathbf{i}B) = \eta\Theta(\mathbf{i}I)\Theta(B)$, where $\eta = 1$ or $\eta = -1$. Now any $A \in \mathfrak{A}$ can be written as $A = A_1 + \mathbf{i}A_2$, where $A_k^* = A_k \in \mathfrak{A},\ k = 1, 2$, we have

$$\begin{aligned}
\Theta(\mathbf{i}A) &= \Theta(\mathbf{i}A_1 - A_2)\\
&= \eta\Theta(\mathbf{i}I)\Theta(A_1) - \Theta(A_2)\\
&= \eta\Theta(\mathbf{i}I)(\Theta(A_1) + \eta\Theta(\mathbf{i}I)\Theta(A_2))\\
&= \eta\Theta(\mathbf{i}I)\Theta(A).
\end{aligned}$$

Hence $\Theta(\mathbf{i}A) = \eta\Theta(\mathbf{i}I)\Theta(A)$ for all $A \in \mathfrak{A}$.
Claim X. Θ preserves '$*$'.
Any $A \in \mathfrak{A}$ can be written as $A = A_1 + \mathbf{i}A_2$, where $A_k^* = A_k \in \mathfrak{A},\ k = 1, 2$. Hence by Claim IX, we have

$$\begin{aligned}
\Theta(A)^* &= \Theta(A_1 + \mathbf{i}A_2)^*\\
&= (\Theta(A_1) + \eta\Theta(\mathbf{i}I)\Theta(A_2))^*\\
&= \Theta(A_1) - \eta\Theta(\mathbf{i}I)\Theta(A_2)\\
&= \Theta(A_1 - \mathbf{i}A_2) = \Theta(A^*).
\end{aligned}$$

Thus Θ preserves '$*$'.
Claim XI. Θ or $-\Theta$ is a $*$-ring isomorphism.
Define the map $\Delta : \mathfrak{A} \to \mathfrak{B}$ such that $\Delta(X) = \eta\Theta(X)$. Then it can be easily verified that Δ is an additive bijection with $\Delta(\mathbf{i}X) = \Delta(\mathbf{i}I)\Delta(X)$, $\Delta(\mathbf{i}I) \in \mathcal{Z}(\mathfrak{B})$ and satisfies $\Delta({}_*[X \diamond Y, Z]) = {}_*[\Delta(X) \diamond \Delta(Y), \Delta(Z)]$ for all $X, Y, Z \in \mathfrak{A}$. Moreover, Δ preserves '$*$'. Hence for any $X, Y \in \mathfrak{A}$, we have

$$\begin{aligned}
-2\Delta(\mathbf{i}I)\Delta(X \diamond Y) &= \Delta({}_*[X \diamond Y, \mathbf{i}I])\\
&= {}_*[\Delta(X) \diamond \Delta(Y), \Delta(\mathbf{i}I)]\\
&= -2\Delta(\mathbf{i}I)\Delta(X) \diamond \Delta(Y).
\end{aligned}$$

Therefore,

$$\Delta(XY^* + YX^*) = \Delta(X)\Delta(Y^*) + \Delta(Y)\Delta(X^*). \tag{7}$$

Also,

$$\begin{aligned}
\Delta(\mathbf{i}I)\Delta(XY^* - YX^*) &= \Delta((\mathbf{i}X)Y^* + Y(\mathbf{i}X)^*)\\
&= \Delta((\mathbf{i}X)\Delta(Y)^* + \Delta(Y)\Delta(\mathbf{i}X)^*)\\
&= \Delta(\mathbf{i}I)(\Delta(X)\Delta(Y)^* - \Delta(Y)\Delta(X)^*).
\end{aligned}$$

Thus,

$$\Delta(XY^* - YX^*) = \Delta(X)\Delta(Y^*) - \Delta(Y)\Delta(X^*) \tag{8}$$

for all $X, Y \in \mathfrak{A}$. From (7) and (8), we conclude that $\Delta(XY^*) = \Delta(X)\Delta(Y^*)$ for all $X, Y \in \mathfrak{A}$. Therefore $\Delta(XY) = \Delta(X)\Delta(Y)$ for all $X, Y \in \mathfrak{A}$ and hence Θ or $-\Theta$ is a $*$-ring isomorphism. This completes the proof. □

By the similar method, we can prove the following theorem.

Theorem 2.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital prime $*$-algebras such that $\mathfrak{A}$ contains a nontrivial projection. Suppose that a bijective map $\Theta : \mathfrak{A} \to \mathfrak{B}$ satisfies $\Theta({}_*[X \bullet Y, Z]) = {}_*[\Theta(X) \bullet \Theta(Y), \Theta(Z)]$ for all $X, Y, Z \in \mathfrak{A}$. Then Θ or $-\Theta$ is a $*$-ring isomorphism.*

A von Neumann algebra $\mathcal{N}$ is a weakly closed self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator. Note that $\mathcal{N}$ is a factor von Neumann algebra if its center contains only the scalar operators. It is well-known that a factor von Neumann algebra is a prime algebra.

Li et al. [10, Theorem 2.5] showed that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are factor von Neumann algebras with $\dim(\mathcal{N}_1) \geq 2$ and if $\Theta : \mathcal{N}_1 \to \mathcal{N}_1$ is a bijective map satisfying $\Theta([X \triangleleft Y, Z]_*) = [\Theta(X) \triangleleft \Theta(Y), \Theta(Z)]_*$ for all $X, Y, Z \in \mathcal{N}$. Then Θ is a linear $*$-isomorphism or the negative of a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism. Analogously, as a corollary of Theorem 2.2, we characterize nonlinear bijective maps preserving the mixed product ${}_*[X \diamond Y, Z]$ on factor von Neumann algebras given as below.

Corollary 2.4. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two factor von Neumann algebras with $dim(\mathcal{N}_1) \geq 2$. Suppose that a bijective map $\Theta : \mathcal{N}_1 \to \mathcal{N}_1$ satisfies $\Theta({}_*[X \diamond Y, Z]) = {}_*[\Theta(X) \diamond \Theta(Y), \Theta(Z)]$ for all $X, Y, Z \in \mathcal{N}$. Then Θ is a linear $*$-isomorphism or the negative of a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.*

Proof. By Theorem 2.2, Θ or $-\Theta$ is a $*$-ring isomorphism. It is easy to show that Θ or $-\Theta$ is a map preserving absolute value. Now, by Theorem 2.5 of [13], the desired conclusion holds.

□

References

[1] Z. Bai and S. Du, *Maps preserving products $XY - YX^*$ on von Neumann algebras*, J. Math. Anal. Appl. **386** (2012), no. 1, 103–109. https://doi.org/10.1016/j.jmaa.2011.07.052

[2] J. Cui and C.-K. Li, *Maps preserving product $XY - YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **431** (2009), no. 5-7, 833–842. https://doi.org/10.1016/j.laa.2009.03.036

[3] L. Dai and F. Lu, *Nonlinear maps preserving Jordan $*$-products*, J. Math. Anal. Appl. **409** (2014), no. 1, 180–188. https://doi.org/10.1016/j.jmaa.2013.07.019

[4] D. Huo, B. Zheng, and H. Liu, *Nonlinear maps preserving Jordan triple η-$*$-products*, J. Math. Anal. Appl. **430** (2015), no. 2, 830–844. https://doi.org/10.1016/j.jmaa.2015.05.021

[5] C. Li, Q. Chen, and T. Wang, *Nonlinear maps preserving the Jordan triple $*$-product on factor von Neumann algebras*, Chinese Ann. Math. Ser. B **39** (2018), no. 4, 633–642. https://doi.org/10.1007/s11401-018-0086-4

[6] C. Li and F. Lu, *Nonlinear maps preserving the Jordan triple 1-$*$-product on von Neumann algebras*, Complex Anal. Oper. Theory **11** (2017), no. 1, 109–117. https://doi.org/10.1007/s11785-016-0575-y

[7] C. Li, F. Lu, and X. Fang, *Nonlinear mappings preserving product $XY+YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **438** (2013), no. 5, 2339–2345. https://doi.org/10.1016/j.laa.2012.10.015

[8] C. Li, F. Lu, and T. Wang, *Nonlinear maps preserving the Jordan triple $*$-product on von Neumann algebras*, Ann. Funct. Anal. **7** (2016), no. 3, 496–507. https://doi.org/10.1215/20088752-3624940

[9] C. Li, F. Zhao, and Q. Chen, *Nonlinear maps preserving product X^*Y+Y^*X on von Neumann algebras*, Bull. Iranian Math. Soc. **44** (2018), no. 3, 729–738. https://doi.org/10.1007/s41980-018-0048-3

[10] C. Li, Y. Zhao, and F. Zhao, *Nonlinear maps preserving the mixed product $[A \bullet B, C]_*$ on von Neumann algebras*, Filomat **35** (2021), no. 8, 2775–2781. https://doi.org/10.2298/fil2108775l

[11] L. Liu and G. X. Ji, *Maps preserving product X^*Y+YX^* on factor von Neumann algebras*, Linear Multilinear Algebra **59** (2011), no. 9, 951–955. https://doi.org/10.1080/03081087.2010.495390

[12] W. S. Martindale, *When are multiplicative mappings additive?*, Proc. Amer. Math. Soc. **21** (1969), 695–698. https://doi.org/10.2307/2036449

[13] A. Taghavi, *Additive mappings on C^*-algebras preserving absolute values*, Linear Multilinear Algebra **60** (2012), no. 1, 33–38. https://doi.org/10.1080/03081087.2010.533271

[14] A. Taghavi, M. Razeghi, M. Nouri, and V. Darvish, *Maps preserving triple product A^*B+BA^* on $*$-algebras*, Asian-Eur. J. Math. **12** (2019), no. 3, 1950038, 13 pp. https://doi.org/10.1142/S1793557119500384

[15] A. Taghavi and E. Tavakoli, *Additivity of maps preserving Jordan triple products on prime C^*-algebras*, Ann. Funct. Anal. **11** (2020), no. 2, 391–405. https://doi.org/10.1007/s43034-019-00009-0

[16] Z. Yang and J. Zhang, *Nonlinear maps preserving mixed Lie triple products on factor von Neumann algebras*, Ann. Funct. Anal. **10** (2019), no. 3, 325–336. https://doi.org/10.1215/20088752-2018-0032

[17] Z. Yang and J. Zhang, *Nonlinear maps preserving the second mixed Lie triple products on factor von Neumann algebras*, Linear Multilinear Algebra **68** (2020), no. 2, 377–390. https://doi.org/10.1080/03081087.2018.1506732

[18] D. Zhang, C. Li, and Y. Zhao, *Nonlinear maps preserving bi-skew Jordan triple product on factor von Neumann algebras*, Period. Math. Hungar. **86** (2023), no. 2, 578–586. https://doi.org/10.1007/s10998-022-00492-4

[19] J. H. Zhang and F. J. Zhang, *Nonlinear maps preserving Lie products on factor von Neumann algebras*, Linear Algebra Appl. **429** (2008), no. 1, 18–30. https://doi.org/10.1016/j.laa.2008.01.031

[20] F. Zhao and C. Li, *Nonlinear maps preserving the Jordan triple $*$-product between factors*, Indag. Math. (N.S.) **29** (2018), no. 2, 619–627. https://doi.org/10.1016/j.indag.2017.10.010

[21] Y. Zhao, C. Li, and Q. Chen, *Nonlinear maps preserving mixed product on factors*, Bull. Iranian Math. Soc. **47** (2021), no. 5, 1325–1335. https://doi.org/10.1007/s41980-020-00444-z

Raof Ahmad Bhat
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
Email address: raofbhat1211@gmail.com

Abbas Hussain Shikeh
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
Email address: abbasnabi94@gmail.com

Mohammad Aslam Siddeeque
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
Email address: aslamsiddeeque@gmail.com