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THE DIMENSION OF THE MAXIMAL SPECTRUM OF SOME RING EXTENSIONS

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Abstract. Let A be a ring and $\mathcal{J} = \{\text{ideals } I \text{ of } A \,|\, J(I) = I\}$. The Krull dimension of A, written $\dim A$, is the sup of the lengths of chains of prime ideals of A; whereas the dimension of the maximal spectrum, denoted by $\dim_{\mathcal{J}} A$, is the sup of the lengths of chains of prime ideals from $\mathcal{J}$. Then $\dim_{\mathcal{J}} A \leq \dim A$. In this paper, we will study the dimension of the maximal spectrum of some constructions of rings and we will be interested in the transfer of the property J-Noetherian to ring extensions.

1. Introduction

All rings considered in this paper are assumed to be commutative with identity. We denote by $Rad(A)$, $Nilp(A)$ and $Reg(A)$ the Jacobson radical, the nilradical and the set of regular elements of a ring A, respectively. If I is an ideal of A, $J(I)$ denotes the Jacobson radical of I, i.e., the intersection of all maximal ideals containing I.

Let $\mathcal{J} = \{\text{ideals } I \text{ of } A \,|\, J(I) = I\}$. By a chain of ideals of length n, we mean a sequence of ideals $I_0 \subset I_1 \subset \cdots \subset I_n$. The Krull dimension of A, written $\dim A$, is the sup of the lengths of chains of prime ideals of A; whereas the dimension of the maximal spectrum, denoted by $\dim_{\mathcal{J}} A$, is the sup of the lengths of chains of prime ideals from $\mathcal{J}$. Clearly, $\dim_{\mathcal{J}} A \leq \dim A$.

For any ideal I of A, a prime ideal P in $\mathcal{J}$ which contains I is called a component of I if P is minimal among the primes of $\mathcal{J}$ which contain I. Every $I \in \mathcal{J}$ is the intersection of its components. We say that A is J-Noetherian if the ideals of $\mathcal{J}$ satisfy the ascending chain condition. A is J-Noetherian implies that every ideal of A has only finitely many components (and the statements are equivalent when $\dim_{\mathcal{J}} A$ is finite). Moreover, any prime of $\mathcal{J}$ which contains I also contains a component of I. We refer the reader to Grothendieck ([12], p. 6, Paragr. 14) for a proper perspective of these concepts.

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There both dim and $\dim_{\mathcal{J}}$ are treated simultaneously by considering $spec(A) = \{$prime ideals of A, with Zariski topology$\}$ and the subspace $Max(A)$ consisting of the maximal ideals of A. Our set $\mathcal{J}$ corresponds to the collection of closed subsets of $Max(A)$, and $\dim_{\mathcal{J}} R$ is the combinatorial dimension of $Max(A)$ in Grothendieck's terminology.

For a proper ideal I of a ring A, a comaximal factorization is a product $I = I_1 I_2 \cdots I_n$ of proper ideals with $I_i + I_j = A$ for $i \neq j$. A proper ideal I is called pseudo-irreducible if it has no comaximal factorizations except for $I = I$. If the factors of a comaximal factorization $I = I_1 I_2 \cdots I_n$ are pseudo-irreducible, then the comaximal factorization $I = I_1 I_2 \cdots I_n$ is called complete. In [13], the authors showed that the complete comaximal factorization for every proper ideal of a ring A exists if and only if A is J-Noetherian.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [1, Theorem 3.2]. Suitable background on commutative trivial ring extensions can be found in [1, 2, 11, 14].

Let T be a ring and let M be an ideal of T. Denote by π the natural surjection $\pi : T \longrightarrow T/M$. Let D be a subring of T/M. Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T such that $D = R/M$. The ring R is known as the pullback associated to the following pullback diagram:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi|_R} & D = R/M \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\quad\pi\quad} & T/M \end{array}$$

where i and j are the natural injections.

A particular case of this pullback is the $D + M$-construction, when the ring T is of the form $K + M$, where K is a field and M is a maximal ideal of T, and R takes the form $D + M$. See for instance [10, 11].

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f B = \{(a, f(a) + j) : a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation (see [5, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (cf. [15, p. 2]), and the CPI extensions (in the sense of Boisen and Sheldon [3]) are strictly related to it (see [5, Example 2.7 and Remark 2.8]).

A particular case of this construction is the amalgamated duplication of a ring along an ideal I (introduced and studied by D'Anna and Fontana in [4, 8, 9]). Let A be a ring, and let I be an ideal of A. $A \bowtie I := \{(a, a+i) : a \in A,\ i \in i\}$ is called the amalgamated duplication of A along the ideal I. See for instance [4–9].

In this paper, we will study the dimension of the maximal spectrum of some constructions of rings and we will be interested in the transfer of the property J-Noetherian to ring extensions such as homomorphic image, localization, direct product, trivial ring extension and the amalgamation of rings.

Let A be a ring. We denote by $\mathcal{J}_A = \{\text{ideals } I \text{ of } A \mid J(I) = I\}$.

2. Main results

We start our study with the homomorphic image. We have the following result:

Proposition 2.1. *Let R be a ring and K an ideal of R.*

(1) *If $\frac{I}{K} \in \mathcal{J}_{\frac{R}{K}}$, then $I \in \mathcal{J}_R$.*

(2) *Let I be an ideal of R such that $K \subseteq I$. Then, $I \in \mathcal{J}_R$ if and only if $\frac{I}{K} \in \mathcal{J}_{\frac{R}{K}}$.*

Proof. (1) Let $\frac{I}{K} \in \mathcal{J}_{\frac{R}{K}}$. Then, $\frac{I}{K} = J(\frac{I}{K}) = \bigcap_{\substack{M \in Max(R) \\ I \subseteq M}} \frac{M}{K} = \frac{J(I)}{K}$.

Hence, $I = J(I)$ and so $I \in \mathcal{J}_R$.

(2) Let I be an ideal of R such that $K \subseteq I$. If $I \in \mathcal{J}_R$, then $I = J(I) = \bigcap_{\substack{M \in Max(R) \\ I \subseteq M}} M$. So,

$$\frac{I}{K} = \frac{J(I)}{K} = \bigcap_{\substack{M \in Max(R) \\ I \subseteq M}} \frac{M}{K} = J(\frac{I}{K}).$$

Therefore, $\frac{I}{K} \in \mathcal{J}_{\frac{R}{K}}$. The converse comes from (1). □

Now, we will see the dimension of the maximal spectrum of the homomorphic image and the transfer of the J-Noetherianity.

Theorem 2.2. *Let R be a ring and K an ideal of R.*

(1) (a) $\dim_{\mathcal{J}_{\frac{R}{K}}}(\frac{R}{K}) \leq \dim_{\mathcal{J}_R}(R)$.

(b) *If $K = \bigcap_{P \in (\mathcal{J}_R \bigcap Spect(R))} P$, then* $\dim_{\mathcal{J}_{\frac{R}{K}}}(\frac{R}{K}) = \dim_{\mathcal{J}_R}(R)$.

(2) *If R is J-Noetherian, then so is $\frac{R}{K}$.*

Proof. (1) (a) Let $\frac{P_0}{K} \subseteq \frac{P_1}{K} \subseteq \cdots \subseteq \frac{P_n}{K}$ be a maximal chain of prime ideals from $\mathcal{J}_{\frac{R}{K}}$. Then, P_i is a prime ideal from $\mathcal{J}_R$ for each $i = 0, 1, \ldots, n$ by Proposition 2.1. So, $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ is a chain of prime ideals from

$\mathcal{J}_R$ that has the same length of the chain $\frac{P_0}{K} \subseteq \frac{P_1}{K} \subseteq \cdots \subseteq \frac{P_n}{K}$. Hence, $\dim_{\mathcal{J}_{\frac{R}{K}}}(\frac{R}{K}) \leq \dim_{\mathcal{J}_R}(R)$.

(b) It follows from by (a) and Proposition 2.1.

(2) Suppose that R is J-Noetherian. Let $\frac{I_0}{K} \subseteq \frac{I_1}{K} \subseteq \frac{I_3}{K} \subseteq \cdots$ be a chain of ideals of $\mathcal{J}_{\frac{R}{K}}$. By Proposition 2.1(1), I_i is an ideal of $\mathcal{J}_R$ for $i = 1, 2, \ldots$. Then, we obtain the chain of ideals of $\mathcal{J}_R$, $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$. Since R is J-Noetherian, there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = \cdots$. Hence, $\frac{I_n}{K} = \frac{I_{n+1}}{K} = \frac{I_{n+2}}{K} = \cdots$ and so $\frac{R}{K}$ is a J-Noetherian ring. □

For localization, we have the following result:

Proposition 2.3. *Let A be a quasilocal ring and $S = Reg(A)$.*

(1) *If $S^{-1}I \in \mathcal{J}_{S^{-1}A}$, then $I \in \mathcal{J}_A$.*

(2) *Let I be an ideal of A such that there exists a maximal ideal M, $I \subseteq M$ and $M \cap S = \emptyset$. Then, $S^{-1}I \in \mathcal{J}_{S^{-1}A}$ if and only if $I \in \mathcal{J}_A$.*

Proof. (1) If $S^{-1}I \in \mathcal{J}_{S^{-1}A}$, then

$$\begin{aligned} S^{-1}I &= J(S^{-1}I) \\ &= \bigcap_{\substack{M \bigcap S = \emptyset \\ I \subseteq M \in Max(A)}} S^{-1}M \\ &= \bigcap_{I \subseteq M \in Max(A)} S^{-1}M \\ &= S^{-1}\Big(\bigcap_{I \subseteq M \in Max(A)} M\Big) \\ &\quad \text{(since } A \text{ is quasilocal and } S^{-1} \text{ commutes with finite intersection)} \\ &= S^{-1}(J(I)). \end{aligned}$$

Since $S = Reg(A)$ and $S^{-1}I = S^{-1}(J(I))$ then $I = J(I)$. Hence, $I \in \mathcal{J}_A$.

(2) Let I be an ideal of A such that there exists a maximal ideal M, $I \subseteq M$ and $M \cap S = \emptyset$. Suppose that $I \in \mathcal{J}_A$. Then, $I = J(I)$. So,

$$\begin{aligned} S^{-1}I &= S^{-1}(J(I)) \\ &= S^{-1}\Big(\bigcap_{I \subseteq M \in Max(A)} M\Big) \\ &= \bigcap_{I \subseteq M \in Max(A)} S^{-1}M \\ &= \bigcap_{\substack{M \bigcap S = \emptyset \\ I \subseteq M \in Max(A)}} S^{-1}M \\ &\quad \text{(Since there exists a maximal ideal } M,\ I \subseteq M \text{ and } M \cap S = \emptyset) \end{aligned}$$

$= J(S^{-1}I)$.

Hence, $S^{-1}I \in \mathcal{J}_{S^{-1}A}$. □

Our next theorem develops a result on the dimension of the maximal spectrum of localization.

Theorem 2.4. *Let A be a quasilocal ring and $S = Reg(A)$.*

(1) $\dim_{\mathcal{J}_{S^{-1}A}}(S^{-1}A) \leq \dim_{\mathcal{J}_A}(A)$.
(2) *If A is J-Noetherian, then so is $S^{-1}A$.*

Proof. (1) Let $S^{-1}P_0 \subseteq S^{-1}P_1 \subseteq \cdots \subseteq S^{-1}P_n$ be a maximal chain of prime ideals from $\mathcal{J}_{S^{-1}A}$. Then, P_i is a prime ideal from $\mathcal{J}_A$ for each $i = 0, 1, \ldots, n$ by Proposition 2.3. So, $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ is a chain of prime ideals from $\mathcal{J}_A$ that has the same length of the chain $S^{-1}P_0 \subseteq S^{-1}P_1 \subseteq \cdots \subseteq S^{-1}P_n$. Hence, $\dim_{\mathcal{J}_{S^{-1}A}}(S^{-1}A) \leq \dim_{\mathcal{J}_A}(A)$.

(2) Suppose that A is J-Noetherian. Let $S^{-1}K_1 \subseteq S^{-1}K_2 \subseteq S^{-1}K_3 \subseteq \cdots$ be a chain of ideals of $\mathcal{J}_{S^{-1}A}$. By Proposition 2.3(1), K_i is an ideal of $\mathcal{J}_A$ for $i = 1, 2, \ldots$. Then, we obtain the chain of ideals from $\mathcal{J}_A$, $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$. Since A is J-Noetherian, there exists a positive integer n such that $K_n = K_{n+1} = K_{n+2} = \cdots$. Hence, $S^{-1}K_n = S^{-1}K_{n+1} = S^{-1}K_{n+2} = \cdots$ and so $S^{-1}A$ is a J-Noetherian ring. □

In [16, Lemma 3.13], the author gives a necessary and sufficient condition for the ring $R \times S$, where R and S are two rings, to be J-Noetherian.

Proposition 2.5 (Lemma 3.13, [16])**.** *Let R and S be two rings. Then R and S are J-Noetherian if and only if $R \times S$ is J-Noetherian.*

For the dimension of the maximal spectrum of direct products, we give the following result.

Proposition 2.6. *Let R and S be two rings. Then*

$$\dim_{\mathcal{J}_{R\times S}}(R \times S) = \sup\{\dim_{\mathcal{J}_S} S, \dim_{\mathcal{J}_R} R\}.$$

For the proof of this proposition we will need the following lemma:

Lemma 2.7. *Let R and S be two rings. Then*

(1) *Let I be an ideal of R. Then $J(I \times S) = J(I) \times S$.*
(2) *Let K be an ideal of S. Then $J(R \times K) = R \times J(K)$.*

Proof. Obviously by the definition of the Jacobson radical of an ideal. □

Proof of Proposition 2.6. Prime ideals of $R \times S$ have the form $P \times S$ or $R \times Q$, where P (resp., Q) is a prime ideal of R (resp., S). The result follows from Lemma 2.7. □

Now, we study the trivial ring extension of the ring A by an A-module E, $R := A \propto E$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [1, Theorem 3.2].

The following proposition gives the relation between the sets $\mathcal{J}_R$ and $\mathcal{J}_A$.

Proposition 2.8. *Let A be a ring, E an A-module and $R = A \propto E$.*

(1) *For all $K \in \mathcal{J}_R$, K has the form $I \propto E$ for some ideal I of A.*
(2) *$I \in \mathcal{J}_A$ if and only if $I \propto E \in \mathcal{J}_R$.*

Proof. (1) All maximal ideals of R have the form $M \propto E$, where M is a maximal ideal of A. Then, $\mathcal{J}_R = \{\text{ideals } K \text{ of } R \mid J(K) = K\} = \{\text{ideals } K \text{ of } R \mid J(K) = \bigcap_{\substack{M \in Max(A) \\ K \subseteq M \propto E}} (M \propto E) = K\}$. So, $K \in \mathcal{J}_R$ if and only if $\bigcap_{\substack{M \in Max(A) \\ K \subseteq M \propto E}} (M \propto E) = K$ and since $0 \propto E \subseteq \bigcap_{\substack{M \in Max(A) \\ K \subseteq M \propto E}} (M \propto E) = K$ then K has the form $I \propto E$ for some ideal I of A.

(2) $I \in \mathcal{J}_A$ if and only if $I = J(I) = \bigcap_{\substack{M \in Max(A) \\ I \subseteq M}} M$ that is equivalent to $I \propto E = \bigcap_{\substack{M \in Max(A) \\ I \subseteq M}} (M \propto E) = J(I \propto E)$ and so $I \propto E \in \mathcal{J}_R$. Therefore, $I \in \mathcal{J}_A$ if and only if $I \propto E \in \mathcal{J}_R$. □

We give the following result for the dimension of the maximal spectrum of the trivial ring extension and the transfer of the J-Noetherianity to it.

Theorem 2.9. *Let A be a ring, E an A-module and $R = A \propto E$. Then:*

(1) $\dim_{\mathcal{J}_A} A = \dim_{\mathcal{J}_R} R$.
(2) *A is J-Noetherian if and only if R is J-Noetherian.*

Proof. (1) Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be a maximal chain of prime ideals from $\mathcal{J}_A$. Then, $P_i \propto E$ is a prime ideal from $\mathcal{J}_R$ for each $i = 0, 1, \ldots, n$ by Proposition 2.8. So, $P_0 \propto E \subsetneq P_1 \propto E \subsetneq \cdots \subsetneq P_n \propto E$ is a chain of prime ideals from $\mathcal{J}_R$ that has the same length of the chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$. Hence, $\dim_{\mathcal{J}_R}(A \propto E) \geq \dim_{\mathcal{J}_A} A$.

Let $H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_m$ be a maximal chain of prime ideals from $\mathcal{J}_R$. By Proposition 2.8, $H_i = P_i \propto E$, where P_i is a prime ideal from $\mathcal{J}_A$ for $i = 0, \ldots, m$. Then there exists a chain of prime ideals from $\mathcal{J}_A$, $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m$. Hence, $\dim_{\mathcal{J}_A} A \geq \dim_{\mathcal{J}_R}(A \propto E)$.

We conclude that $\dim_{\mathcal{J}_A} A = \dim_{\mathcal{J}_R} R$.

(2) Suppose that A is J-Noetherian. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be a chain of ideals of $\mathcal{J}_R$. By Proposition 2.8(1), $K_i = I_i \propto E$, where I_i is an ideal of $\mathcal{J}_A$ for $i = 1, 2, \ldots$. Then, we obtain the chain of ideals of $\mathcal{J}_A$, $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$.

Since A is J-Noetherian there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = \cdots$. Hence, $I_n \propto E = I_{n+1} \propto E = I_{n+2} \propto E = \cdots$ and so R is a J-Noetherian ring.

Conversely, suppose that R is J-Noetherian. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ a chain of ideals of $\mathcal{J}_A$. By Proposition 2.8(1), $I_i \propto E$ is an ideal of $\mathcal{J}_R$ for $i = 1, 2, \ldots$. Then, we obtain the chain of ideals of $\mathcal{J}_R$, $I_1 \propto E \subseteq I_2 \propto E \subseteq I_3 \propto E \subseteq \cdots$. Since R is J-Noetherian there exists a positive integer n such that $I_n \propto E = I_{n+1} \propto E = I_{n+2} \propto E = \cdots$. Hence, $I_n = I_{n+1} = I_{n+2} = \cdots$ and so A is a J-Noetherian ring. $\square$

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. The ring $R = A \bowtie^f J$ is the amalgamation of A and B along J with respect to f. For all $P \in Spec(A)$ and $Q \in Spec(B)$, set

$$P'^f = P \bowtie^f J = \{(p, f(p) + j) : p \in P,\ j \in J\},$$

$$\overline{Q}^f = \{(a, f(a) + j) : a \in A,\ j \in J,\ f(a) + j \in Q\}.$$

Recall that prime (resp., maximal) ideals of R are of the type P'^f or $\overline{Q}^f$ for $P \in Spec(A)$ (resp., $P \in Max(A)$) and $Q \in Spec(B) \setminus V(J)$ (resp., $Q \in Max(B) \setminus V(J)$), where $V(J) = \{P \in Spec(B) \mid J \subseteq P\}$ [6, Proposition 2.6].

As in the previous sections, the following proposition gives the relation between the sets $\mathcal{J}_{A\bowtie^f J}$ and $\mathcal{J}_A$.

Proposition 2.10. *Let A and B be rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. If $J \subset Rad(B)$, then*

(1) *$I \in \mathcal{J}_A$ if and only if $I \bowtie^f J \in \mathcal{J}_{A\bowtie^f J}$.*
(2) *All ideals of $\mathcal{J}_{A\bowtie^f J}$ have the form $I \bowtie^f J$, where I is an ideal in $\mathcal{J}_A$.*

Proof. By the condition $J \subseteq Rad(B)$, all maximal ideals of $A \bowtie^f J$ have the form $M \bowtie^f J$, where $M \in Max(A)$.

(1) Let $I \in \mathcal{J}_A$. Then $I = J(I) = \bigcap_{\substack{M\in Max(A)\\ I\subseteq M}} M$. Hence,

$$\begin{aligned} I \bowtie^f J = J(I) \bowtie^f J &= \left(\bigcap_{\substack{M\in Max(A)\\ I\subseteq M}} M \right) \bowtie^f J \\ &= \bigcap_{\substack{M\in Max(A)\\ I\subseteq M}} M \bowtie^f J = J(I \bowtie^f J). \end{aligned}$$

So, $I \bowtie^f J \in \mathcal{J}_{A\bowtie^f J}$.

Conversely, suppose that $I \bowtie^f J \in \mathcal{J}_{A\bowtie^f J}$. Then

$$I \bowtie^f J = J(I \bowtie^f J) = \bigcap_{\substack{M\in Max(A)\\ I\subseteq M}} M \bowtie^f J$$

$$= \left(\bigcap_{\substack{M \in Max(A) \\ I \subseteq M}} M \right) \bowtie^f J = J(I) \bowtie^f J.$$

Therefore, $I = J(I)$ and so $I \in \mathcal{J}_A$.

(2) Let $K \in \mathcal{J}_{A \bowtie^f J}$. $K = J(K) = \bigcap_{\substack{M \in Max(A) \\ K \subseteq M}} M \bowtie^f J$. Since $0 \times J \subseteq \bigcap_{\substack{M \in Max(A) \\ K \subseteq M}} M \bowtie^f J$ then $0 \times J \subseteq K$ and so $K = I \bowtie^f J$ for some ideal I in $\mathcal{J}_A$. □

Remark 2.11. If $I \in \mathcal{J}_A$, then $I \bowtie^f J \in \mathcal{J}_{A \bowtie^f J}$ without any conditions. Indeed,

$$J(I \bowtie^f J) = \bigcap_{\substack{\mathcal{M} \in Max(A \bowtie^f J) \\ (I \bowtie^f J) \subseteq \mathcal{M}}} \mathcal{M}.$$

All maximal ideals that contain $I \bowtie^f J$ have the form $M \bowtie^f J$, where M is a maximal ideal of A (since $I \bowtie^f J$ contains $0 \times J$). Hence,

$$J(I \bowtie^f J) = \bigcap_{\substack{M \in Max(A) \\ I \subseteq M}} M \bowtie^f J = \left(\bigcap_{\substack{M \in Max(A) \\ I \subseteq M}} M \right) \bowtie^f J$$
$$= J(I) \bowtie^f J = I \bowtie^f J.$$

The author of [16] gives a characterization for the amalgamated algebra along an ideal to be J-Noetherian, and for the convenience of the reader we include it here.

Proposition 2.12 (Proposition 3.14, [16])**.** *$A \bowtie^f J$ is J-Noetherian if and only if A and $f(A) + J$ are J-Noetherian.*

Now, we will see the dimension of the maximal spectrum of the amalgamation of rings and the transfer of the J-Noetherianty.

Proposition 2.13. *Let A and B be rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. If $J \subseteq Nilp(B)$, then*

$$\dim_{\mathcal{J}_{A \bowtie^f J}}(A \bowtie^f J) = \dim_{\mathcal{J}_A}(A).$$

Proof. If $J \subseteq Nilp(B)$, then all prime ideals of $A \bowtie^f J$ have the form $P \bowtie^f J$, where $P \in Spec(A)$.

Let $P_0 \bowtie^f J \subseteq P_1 \bowtie^f J \subseteq \cdots \subseteq P_n \bowtie^f J$ be a maximal chain of prime ideals from $\mathcal{J}_{A \bowtie^f J}$. Then, P_i is a prime ideal from $\mathcal{J}_A$ for each $i = 0, 1, \ldots, n$ by Proposition 2.10. So, $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ is a chain of prime ideals from $\mathcal{J}_A$. Hence, $\dim_{\mathcal{J}_{A \bowtie^f J}}(A \bowtie^f J) \leq \dim_{\mathcal{J}_A}(A)$. Conversely, let $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$

be a maximal chain of prime ideals from $\mathcal{J}_A$. Then, $P_0 \bowtie^f J \subseteq P_1 \bowtie^f J \subseteq \cdots \subseteq P_n \bowtie^f J$ is a chain of prime ideals from $\mathcal{J}_{A\bowtie^f J}$ by Proposition 2.10 and so $\dim_{\mathcal{J}_{A\bowtie^f J}}(A \bowtie^f J) \geq \dim_{\mathcal{J}_A}(A)$. We conclude that $\dim_{\mathcal{J}_{A\bowtie^f J}}(A \bowtie^f J) = \dim_{\mathcal{J}_A}(A)$. □

Remark 2.14. Without the condition $J \subseteq Nilp(B)$, the following results are still true: $\dim_{\mathcal{J}_{A\bowtie^f J}}(A \bowtie^f J) \geq \dim_{\mathcal{J}_A}(A)$. It suffices to see that the condition $(J \subseteq Nilp(B))$ is not used in the second part in the proof of Theorem 2.13.

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