J. Korean Math. Soc. **60** (2023), No. 6, pp. 1233–1254
https://doi.org/10.4134/JKMS.j220625
pISSN: 0304-9914 / eISSN: 2234-3008

SOME ABELIAN MCCOY RINGS

Rasul Mohammadi, Ahmad Moussavi, and Masoome Zahiri

Abstract. We introduce two subclasses of abelian McCoy rings, so-called π-CN-rings and π-duo rings, and systematically study their fundamental characteristic properties accomplished with relationships among certain classical sorts of rings such as 2-primal rings, bounded rings etc. It is shown that a ring R is π-CN whenever every nilpotent element of index 2 in R is central. These rings naturally generalize the long-known class of CN-rings, introduced by Drazin [9]. It is proved that π-CN-rings are abelian, McCoy and 2-primal. We also show that, π-duo rings are strongly McCoy and abelian and also they are strongly right AB. If R is π-duo, then $R[x]$ has property (A). If R is π-duo and it is either right weakly continuous or every prime ideal of R is maximal, then R has property (A). A π-duo ring R is left perfect if and only if R contains no infinite set of orthogonal idempotents and every left R-module has a maximal submodule. Our achieved results substantially improve many existing results.

1. Introduction and motivation

Throughout the current article, our rings will be associative, not necessarily commutative unless specified this, and containing identity element 1. Almost all notions and notations are standard being in agreement with the well-known book [23]. The more special terminology will be explained in the sequel. Recall for completeness of the exposition that a ring R is said to be 2-*primal* if its Baer-McCoy radical (in other terms, the prime radical or the lower nil-radical) $Nil_*(R)$ coincides with the set $Nil(R)$ consisting of all nilpotents in R, that is, $Nil_*(R) = Nil(R)$.

A ring R is said to be Armendariz if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. Following [28], a ring R is called *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$, where $f(x), g(x)$ are non-zero polynomials in $R[x]$. *Left McCoy rings* are defined dually and they satisfy dual properties. A ring

Received December 20, 2022; Revised May 18, 2023; Accepted September 1, 2023.
2020 *Mathematics Subject Classification.* 16S34, 16U99, 16E50, 16W10, 13B99.
Key words and phrases. CN-rings, π-CN-rings, 2-primal rings, McCoy rings, duo-rings, π-duo rings, strongly AB-rings, property (A), nil radical.

R is called *McCoy* if it is both left and right McCoy. This name was chosen by Nielsen in [28] in recognition of McCoy's proof in [27, Theorem 2].

Each of the four ring theoretic properties, Armendarizness, reversibility, polynomial semicommutativity and right duoness, implies the right McCoy condition. A natural question is whether there is a subclass of right McCoy rings which encompasses all these ring classes. We prove that there is a subclass of abelian right McCoy rings, namely π-CN-rings. It has been shown by Chen et al. [24] that the class of semicommutative right McCoy rings properly contains both the classes of reversible rings and polynomial semicommutative rings. However, this class does not contain all Armendariz rings because there are Armendariz rings which are not semicommutative (see [20, Example 14]).

Recall the classical result of Drazin from [9, Theorem 1] stating that all idempotents of a given ring are always central when all its nilpotents are always central. A ring R is called a *CN*-ring if any nilpotent element of it is central. We shall call a ring R a π-*CN*-ring whenever every nilpotent element of R of index 2 is central, and an abelian ring whenever every idempotent element of R is central.

We have the following irreversible implications:

$$\begin{tikzcd}
 & \text{right duo} \arrow[r] & \pi\text{-duo} \arrow[d] \\
\textit{commutative} \arrow[ur] \arrow[r] \arrow[dr] & \textit{reversible} \arrow[r] & \text{ablian McCoy} \\
 & CN \arrow[r] & \pi\text{-}CN \arrow[u] \arrow[d] \\
 & & \text{2-primal}
\end{tikzcd}$$

A right ideal of R is called *bounded* if it contains a non-zero ideal of R, and Faith said a ring is *strongly right* (resp., *left*) *bounded* if every non-zero right (resp., left) ideal were bounded. Hwang et al. in [21], initiated and studied the notion of strongly right AB rings as a generalization of strongly right bounded ring. A ring R is *strongly right* AB if every non-zero right annihilator of R is bounded. Strongly left AB rings are defined dually and R is called *strongly* AB if it is both strongly right AB and strongly left AB. Note that strongly right AB property is not preserved under the construction of polynomial rings.

For 2-primal rings, there is a question of Birkenmeier, Heatherly and Lee [3, Problem 3, p. 373] which asks if the prime radical of a ring R contains all nilpotent elements of index two, is R a 2-primal ring? In [17, Example 1] Hirano, Huynh and Park give the answer to this question in the negative. By definition, π-CN-rings satisfy the property that $N_2(R) \subseteq Nil_*(R)$, and hence they are 2-primal. In fact for a π-CN-ring R, $N_2(R) \subseteq Nil_*(R)$ and $Nil_*(R)$ has IFP. By Proposition 2.9, a ring R is 2-primal if and only if $N_2(R) \subseteq Nil_*(R)$ and $Nil_*(R)$ has IFP.

In Lemma 2.7, we see that the lower radical coincides with the nilpotent elements in a π-CN-ring, so that by Corollary 2.8, every π-CN-ring R is 2-primal, and so $Nil(R[x]) = Nil(R)[x]$. By Proposition 2.11, π-CN-rings are abelian (the idempotent elements are central). Our main result Theorem 2.15 shows that π-CN-rings are McCoy. As a consequense, we show that when R is a right zip π-CN-ring, then $R[x]$ is right strongly AB. A ring R is called a *power serieswise McCoy* ring if any power series $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a zero-divisor in $R[[x]]$ if and only if there is a non-zero element $r \in R$ which satisfies $rf(x) = 0$, in this case. In Corollary 2.21, we show that, when R is a π-CN-ring and $Nil(R)$ is Noetherian as a right R-module, then R is a power series-wise McCoy ring. Consequently, right Noetherian π-CN-rings are power series-wise McCoy. This result generalizes Fields [12, Theorem 5] for noncommutative rings. We also prove that π-CN-rings are 2-primal. It is also shown that R is a right zip right McCoy ring which is strongly right AB if and only if $R[x]$ is a right zip and strongly right AB ring.

In Subsection 2.2, we introduce another subclass of abelian McCoy rings, so-called π-duo rings, and systematically study their fundamental characteristic properties. We say a ring R is left π-duo if for every $0 \neq a \in R$, there exists a positive integer k for which $0 \neq a^k R \subseteq Ra^k$. In [26], G. Marks shows that a noncommutative skew polynomial ring is never one sided duo. We provide a noncommutative ring R, for which, the polynomial ring $R[x]$ is a π-duo ring. We present and analyze the notion of π-duo rings as a generalization of duo rings in the current research, which is driven by the foregoing discussion, and prove that π-duo rings are 2-primal. Our primary finding is that π-duo rings are McCoy and abelian at the same time. When R is a one-sided π-duo ring, we show that R is reduced only if R is nonsingular. Furthermore, we elaborate on the fact that a right π-duo ring R has the right property (A) provided that either of the following requirements are met:

- every prime ideal of R is maximal.
- R is right weakly continuous.
- R is right FI-extending with ACC on direct summands.

In Subsection 2.3, we study the property (A) of π-duo rings. Our achievements here are Theorems 2.41, 2.44, 2.45. We use $R[x]$ and $R[[x]]$ to denote the ring of all polynomials and the ring of all formal power series over a ring R, respectively. The set of central elements of R is denoted by $\mathrm{Cent}(R)$ and, we use $r_R(X)$ and $\ell_R(X)$, respectively, to denote the right and left annihilator of X in R.

2. Main results and applications

Our chief results are distributed into four subsections as follows: The first one is devoted to properties of the so-defined by us π-CN-rings and in the second one we initiate and study the π-duo rings, whereas the third one is concerned with the direct relations of the π-CN-rings and π-duo rings to the theory of bounded rings, as well as the property (A).

2.1. π-CN-rings

According to M. P. Drazin [9], a ring R is a *CN-ring* whenever every nilpotent element is central. By [9, Theorem 1], the centrality of all the nilpotent elements of a given associative ring implies the centrality of every idempotent element; and by [9, Theorem 7] these two properties are in fact equivalent in any regular ring.

Now we give a natural generalization of the class of CN-rings. We then provide a necessary and sufficient condition, for the centrality of nilpotent elements in the wider class of rings, so-called π-CN-rings:

Definition 2.1. A ring R is a π-CN-ring if for every non-zero nilpotent element $a \in R$, $0 \neq a^k$ is central, for some positive integer k.

Let $N_2(R) = \{a \in Nil(R) \,|\, a \text{ is nilpotent of index } 2\}$. Now we show that a ring R is a π-CN-ring whenever every nilpotent element of R of index 2 is central.

Lemma 2.2. *A ring R is a π-CN-ring if and only if $N_2(R) \subseteq \mathrm{Cent}(R)$.*

Proof. Assume that R is a π-CN-ring and $0 \neq a \in N_2(R)$. So we get $a \in \mathrm{Cent}(R)$.

Conversely, assume that $N_2(R) \subseteq \mathrm{Cent}(R)$, and $0 \neq a \in Nil(R)$. Then there exists a minimal integer k such that $a^k = 0$. As $0 \neq a^{k-1} \in N_2(R)$ we get a^{k-1} is a non-zero central element in R. □

Lemma 2.3. *Let R be a π-CN-ring with $\mathrm{Char}(R) = 2$. Then $N_2(R)$ is an ideal of R.*

In the following, we provide an example of a π-CN-ring which is not a CN-ring.

Example 2.4. Let $R = \mathbb{Z}_2\langle a, b, c\rangle$, where a, b, c are noncommuting indeterminates. Let I be the ideal of R generated by the set $\{a^3, b^2, c^3, ab - ba, bc - cb, ca^2, c^2a, ac, cab\}$. Let $S = \frac{R}{I}$. Then $a, c \in Nil(S)$ and $ac = 0 \neq ca$ which implies that S is not a CN-ring. Note that any element of S is of the form $\gamma = s_1 + s_2a + s_3a^2 + s_4b + s_5c + s_6c^2 + s_7ca + s_8ab + s_9a^2b + s_{10}cb + s_{11}c^2b$, where $s_i \in \mathbb{Z}_2$, $1 \leq i \leq 11$. Clearly each nilpotent element of S is of the form

$$\beta = t_1a + t_2a^2 + t_3b + t_4c + t_5c^2 + t_6ca + t_7ab + t_8a^2b + t_9cb + t_{10}c^2b,$$

where $t_1, \ldots, t_{10} \in \mathbb{Z}_2$. As $\beta^2 = t_1^2a^2 + t_4^2c^2 + t_4t_1ca \in \mathrm{Cent}(S)$, we deduce that S is a π-CN-ring.

Let R be a ring and M an (R, R)-bimodule. Recall that the trivial extension of R and M is

$$R \propto M = \{(r, m) : r \in R, m \in M\},$$

with addition defined componentwise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

For any subset I of R, we denoted $I \propto M = \{(r, m) : r \in I, m \in M\}$.

Lemma 2.5. *Let $R \propto M$ be a trivial extension of R and M. Then $R \propto M$ is π-CN if and only if R is π-CN and $rm = mr$ for every $r \in R$ and $m \in M$.*

Proof. By Lemma 2.2, $R \propto M$ is a π-CN-ring if and only if $N_2(R \propto M) \subseteq \mathrm{Cent}(R \propto M)$. As $N_2(R \propto M) = N_2(R) \propto M$, we obtain that $R \propto M$ is a π-CN-ring if and only if $N_2(R) \propto M \subseteq \mathrm{Cent}(R \propto M)$. Thus $R \propto M$ is π-CN if and only if R is a π-CN-ring and $rm = mr$ for every $r \in R$ and $m \in M$. □

Corollary 2.6. *Let R be a π-CN-ring with* $\mathrm{Char}(R) = 2$. *Then $R \propto N_2(R)$ is π-CN.*

Proof. Assume that $a, b \in N_2(R)$. Then $(a + b)^2 = ab + ba = 2ab = 0$. Now let $c \in N_2(R)$ and $r \in R$. Then $(cr)^2 = crcr = c^2r^2 = 0$. Thus $N_2(R)$ is an ideal of R. □

The sum of the nilpotent ideals of R, called the Wedderburn radical, is denoted by $W(R)$, the lower nil radical $Nil_*(R)$ is the intersection of all prime ideals of R, the Levitzky radical $L\text{-}rad(R)$ is the largest locally nilpotent ideal of R, the upper nil radical $Nil^*(R)$ is the sum of all nil ideals, and $Nil(R)$ is the set of all nilpotent elements of R. The readers can be referred to [4], for more details.

Lemma 2.7. *Let R be a π-CN-ring. Then $W(R) = Nil_*(R) = L\text{-}rad(R) = Nil^*(R) = Nil(R)$.*

Proof. Assume that $a \in Nil(R)$. If $a^2 = 0$, then we get $a \in \mathrm{Cent}(R)$ and so $(aR)^2 = 0$, which implies that $a \in W(R)$, as desired.

If $k \geq 3$ is a minimal integer such that $a^k = 0$, then $2k - 2 \geq k$ and so $a^{k-1}a^{k-1} = 0$. As R is a π-CN-ring, we have $a^{k-1} \in \mathrm{Cent}(R)$. It follows that $a^{k-1}RaR = 0$, and that $(a^{k-2}RaRa)^2 = 0$ and so $a^{k-2}RaRa \subseteq \mathrm{Cent}(R)$. This means that

$$(a^{k-2}RaRa)Ra = Ra(a^{k-2}RaRa) = 0.$$

So $(a^{k-3}RaRaRaRa)^2 = 0$, which implies that $a^{k-3}RaRaRaRa \subseteq \mathrm{Cent}(R)$. So

$$(a^{k-3}RaRaRaRa)Ra = Ra(a^{k-3}RaRaRaRa) = 0.$$

Continuing in this way we get $(aR)^{2k} = 0$, as desired. □

Corollary 2.8. *Every π-CN-ring R is* 2*-primal, and so $Nil(R[x]) = Nil(R)[x]$.*

An ideal I of a ring R is said to have IFP if R/I is a semicommutative ring. Note that, for 2-primal rings, there is a question of Birkenmeier, Heatherly and Lee [3, Problem 3, p. 373] which asks if the prime radical of a ring R contains all nilpotent elements of index two, is R a 2-primal ring? In [17, Example 1] Hirano, Huynh and Park give the answer to this question in the negative.

By definition, π-CN-rings satisfy the property that $N_2(R) \subseteq Nil_*(R)$, and hence they are 2-primal. In fact for a π-CN-ring R, $N_2(R) \subseteq Nil_*(R)$ and $Nil_*(R)$ has IFP.

Proposition 2.9. *A ring R is 2-primal if and only if $N_2(R) \subseteq Nil_*(R)$ and $Nil_*(R)$ has IFP.*

Proof. First assume that R is a 2-primal ring. Then clearly $N_2(R) \subseteq Nil_*(R)$. For proving $Nil_*(R)$ has IFP, assume on the contrary that there exist $a, b, r \in R$ such that $ab \in Nil_*(R)$, but $arb \notin Nil_*(R)$. Then there exists a minimal prime ideal P of R such that $arb \notin P$. As every minimal prime ideal of 2-primal ring is completely prime and $ab \in Nil_*(R) \subseteq P$, we must have $a \in P$ or $b \in P$. So we have $arb \in P$, which is a contradiction. Thus $aRb \subseteq Nil_*(R)$ which implies that $Nil_*(R)$ has IFP.

Conversely, assume that $N_2(R) \subseteq Nil_*(R)$ and $Nil_*(R)$ has IFP. For proving R is a 2-primal ring, assume that $a \in Nil(R)$. Then there exists an integer k such that $a^k \neq 0$ but $a^{2k} = 0$. It follows that $a^k \in N_2(R)$. As $N_2(R) \subseteq Nil_*(R)$ we get $a^k \in Nil_*(R)$. As $Nil_*(R)$ has IFP we get $(aR)^k \subseteq Nil_*(R)$. This means that $a \in Nil_*(R)$, implying R is 2-primal. □

Corollary 2.10. *Let R be a π-CN-ring. Then $N_2(R) \subseteq Nil_*(R)$ and $Nil_*(R)$ has IFP.*

An idempotent $e \in R$ is called *left* (*resp. right*) *semicentral* if $xe = exe$ (resp. $ex = exe$) for all $x \in R$. An idempotent which is both right and semicentral is central.

Proposition 2.11. *Every π-CN-ring is abelian.*

Proof. Assume that e is an idempotent of R. Then $(eR(1-e))^2 = 0$ implies that $eR(1-e) \subseteq \mathrm{Cent}(R)$. So $eR(1-e) = e^2R(1-e) = e(eR(1-e)) = (eR(1-e))e = 0$. This means that $eR = eRe$, i.e., e is a right semicentral idempotent. Similarly, from $((1-e)Re)^2 = 0$ we get $(1-e)Re \subseteq \mathrm{Cent}(R)$ which implies that $(1-e)Re = (1-e)Re^2 = ((1-e)Re)e = e((1-e)Re) = 0$. Hence e is a left semicentral idempotent. As e is both left and right semicentral, it follows that e is a central idempotent, as desired. □

Proposition 2.12. *A π-CN-ring R is nonsingular if and only if it is reduced.*

Proof. Clearly any reduced ring is nonsingular. For the converse, assume that R is nonsingular and $a \in R$ such that $a^2 = 0$. Then $a \in \mathrm{Cent}(R)$ and so $(Ra)^2 = 0$. Next we show that $r_R(aR)$ is an essential right ideal of R. For, take $0 \neq b \in R$. If $aRb \neq 0$, then $0 \neq Rba = aRb \subseteq aR \subseteq r_R(aR)$. So $r_R(aR)$ is an essential right ideal of R. As R is nonsingular, we must have $aR = 0$, so $a = 0$. Therefore R is a reduced ring. □

Lemma 2.13. *Every minimal left ideal of a π-CN-ring is an ideal.*

Proof. Assume that L is a minimal left ideal of R. Then by Brauer's Lemma, [23, (10.22)], we get either $L^2 = 0$ or $L = Re$. If $L^2 = 0$, then since R is π-CN, we get $L \subseteq \mathrm{Cent}(R)$ and so $L = RL = LR$, which ends the proof. If $L = Re$, then $LR = ReR = eR$, because R is abelian. So the result follows. $\square$

Lemma 2.14. *Let R be a π-CN-ring. Then for each $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ with $f(x)g(x) = 0$, there exists an integer l such that $a_0^{l+1} g(x) = 0$.*

Proof. We will show that $a_0^{2j+1} b_j = 0$ for every $0 \le j \le n$. From the equation $f(x)g(x) = 0$, we get

$$a_0 b_0 = 0, \tag{1}$$

$$a_0 b_1 + a_1 b_0 = 0, \tag{2}$$

$$\vdots$$

$$a_0 b_{t+1} + a_1 b_t + \cdots + a_{t+1} b_0 = 0. \tag{3}$$

Multiplying equation (2) on the left and the right side by a_0, it gives $a_0^2 b_1 a_0 + a_0 a_1 b_0 a_0 = 0$. As $(b_0 a_0)^2 = 0$, we get $b_0 a_0 \in \mathrm{Cent}(R)$ and so

$$0 = a_0^2 b_1 a_0 + a_0 a_1 b_0 a_0 = a_0^2 b_1 a_0 + a_0 b_0 a_0 a_1 = a_0^2 b_1 a_0.$$

It follows that $(a_0^2 b_1)^2 = 0$ and so $a_0^2 b_1 \in \mathrm{Cent}(R)$. From this and $a_0^2 b_1 a_0 = 0$, we get $a_0^3 b_1 = 0$. Inductively assume that $a_0^{2t+1} b_j = 0$ for every $0 \le j \le t$. Hence $(b_j a_0^{2t+1})^2 = 0$ and so $b_j a_0^{2t+1} \in \mathrm{Cent}(R)$. Multiplying the equation (3) on the left side by a_0 and right side by a_0^{2t+1} we get

$$a_0^2 b_{t+1} a_0^{2t+1} + a_0 a_1 b_t a_0^{2t+1} + \cdots + a_0 a_{t+1} b_0 a_0^{2t+1} = 0.$$

As $(b_j a_0^{2t+1})^2 = 0$ for every $0 \le j \le t$, we get $b_j a_0^{2t+1} \in \mathrm{Cent}(R)$ for every $0 \le j \le t$. Thus

$$a_0^2 b_{t+1} a_0^{2t+1} + a_0 b_t a_0^{2t+1} a_1 + \cdots + a_0 b_0 a_0^{2t+1} a_{t+1} = 0. \tag{$*$}$$

Also for every $s \le 2t+1$, $(a_0^s b_j a_0^{2t-s+1})^2 = 0$ and so $a_0^s b_j a_0^{2t-s+1} \in \mathrm{Cent}(R)$ for every $0 \le j \le t$. So for every $0 \le j \le t$, we get

$$\begin{aligned} a_0 b_j a_0^{2t+1} &= (a_0 b_j a_0^{2t}) a_0 = a_0 (a_0 b_j a_0^{2t}) = (a_0^2 b_j a_0^{2t-1}) a_0 \\ &= a_0 (a_0^2 b_j a_0^{2t-1}) = \cdots = a_0^{2t} b_j a_0^2 = (a_0^{2t} b_j a_0) a_0 \\ &= a_0 (a_0^{2t} b_j a_0) = a_0^{2t+1} b_j a_0 = 0. \end{aligned}$$

From this and $(*)$ we get $a_0^2 b_{t+1} a_0^{2t+1} = 0$. As $a_0^{2+v} b_{t+1} a_0^{2t-v} \in \mathrm{Cent}(R)$ for every $v \le 2t$, we get

$$0 = (a_0^2 b_{t+1} a_0^{2t}) a_0 = a_0 (a_0^2 b_{t+1} a_0^{2t}) = a_0^3 b_{t+1} a_0^{2t} = \cdots = a_0^{2t+2} b_{t+1} a_0 = a_0^{2t+3} b_{t+1},$$

as needed. $\square$

Theorem 2.15. *Every π-CN-ring is a McCoy ring.*

Proof. We prove the right McCoy property, the left case is similar. Suppose that $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ and $g(x) \neq 0$. We will show, by induction on the degree of $f(x)$, that there is some non-zero element in $I_{g(x)}$ which annihilates $f(x)$ on the right, where $I_{g(x)}$ denotes the ideal generated by the coefficients of $g(x)$. Assume that $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ with $b_0 \neq 0$.

If $\deg(f(x)) = 0$, then the claim is trivial since $f(x)b_0 = 0$. So the base case of our induction is established. Now, suppose $\deg(f(x)) > 0$.

Case 1: If $a_0 g(x) = 0$, then we set $f_1(x) = (f(x) - a_0)/x$ and get $f_1(x)g(x) = 0$. But $\deg(f_1(x)) < \deg(f(x))$, and hence by induction there exists a non-zero element $c \in I_{g(x)}$ satisfying $f_1(x)c = 0$, whence $f(x)c = 0$.

Case 2: If $a_0 g(x) \neq 0$, then we get three subcases:

(i) $g(x) \in Nil(R[x])$. As R is a 2-primal ring there exists an integer t minimal with respect that $b_{i_1} b_{i_2} \cdots b_{i_t} = 0$, where $b_{i_1}, b_{i_2}, \ldots, b_{i_t}$ are arbitrary elements of C_g. So there exist $b_{j_1}, b_{j_2}, \ldots, b_{j_{t-2}} \in C_g$, such that $g_1(x) = g(x) b_{j_1} \cdots b_{j_{t-2}} \neq 0$. Clearly any coefficient of $g_1(x)$ is a nilpotent element with nilpotancy index 2 and hence $g_1(x) \in \mathrm{Cent}(R[x])$. As $f(x)g(x) = 0$, by Lemma 2.35, there exists an integer l such that $a_0^l g_1(x) = 0 \neq a_0^{l-1} g_1(x)$. As $g_1(x) \in \mathrm{Cent}(R[x])$ we get $g_1(x) a_0^{l-1} = a_0^{l-1} g_1(x) \neq 0$. Put $g_2(x) = g_1(x) a_0^{l-1}$, then from $a_0 g_2(x)$ and $f(x)g_2(x) = 0$ we get $x(a_1 + a_2 x + \cdots + a_m x^{m-1}) g_2(x) = 0$ and so $(a_1 + a_2 x + \cdots + a_m x^{m-1}) g_2(x) = 0$. But $\deg(a_1 + a_2 x + \cdots + a_m x^{m-1}) < \deg(f(x))$, and hence by induction there exists a non-zero element $c \in I_{g_2(x)} \leq I_{g(x)}$ satisfying $(a_1 + a_2 x + \cdots + a_m x^{m-1})c = 0$, whence $f(x)c = 0$.

(ii) $g(x) \notin Nil(R[x])$ and $g(x) C_{f(x)} \neq 0$. Then there exists $a_r \in C_{f(x)}$ such that $g(x) a_r \neq 0$. Put $g_3(x) = g(x) a_r$, then $g_3(x) \in Nil(R[x])$ and so we reduce to the case (i).

(iii) $g(x) \notin Nil(R[x])$ and $b_j f(x) = 0$ for every $0 \leq j \leq n$. As $a_0 g(x) \neq 0$, there exists an integer $0 \leq v \leq n$ such that $a_0 b_v \neq 0$. From $g(x) a_i = 0$ for every $0 \leq i \leq m$, we get $(a_0 b_v)^2 = 0$ and since R is a π-CN-ring, we get $a_0 b_v \in \mathrm{Cent}(R) \subseteq \mathrm{Cent}(R[x])$. Since $a_0 b_v f(x) = 0$ we get $f(x) a_0 b_v = 0$ and $a_0 b_v \in I_{g(x)}$, as desired. □

A ring R is called *right zip* if for every $X \subseteq R$, $r_R(X) = 0$ implies that $r_R(Y) = 0$ for a finite subset $Y \subseteq X$. Zelmanowitz [33] showed that any ring satisfying the descending chain condition on right annihilators is right zip, and he also showed that there exist commutative zip rings which do not satisfy the descending chain condition on (right) annihilators. Faith [10], proved that if a ring R is commutative zip and G is a finite abelian group, then the group ring $R[G]$ is zip. Cedó [6] shows that the right zip property is not preserved under the construction of polynomial rings, matrix rings and also the group rings of a finite group. Hashemi [14] showed that when R is a right McCoy ring, then $R[x]$ is right zip if and only if R is right zip. Also in [18], several extensions of rings with zip property were studied.

Corollary 2.16. *Let R be a π-CN-ring. Then R is a zip ring if and only if $R[x]$ is a zip ring.*

Theorem 2.17. *Let R be a right zip ring. Then R is right McCoy and strongly right AB if and only if $R[x]$ is strongly right AB.*

Proof. ($\Rightarrow$) Assume on the contrary that $X \subseteq R[x]$ with $r_{R[x]}(X) \neq 0$ but $r_{R[x]}(XR[x]) = 0$. As R is right zip right McCoy, $R[x]$ is a right zip ring. So there exists a finite subset $\{f_1, \ldots, f_n\}$ of $XR[x]$ such that $r_{R[x]}(\{f_1, \ldots, f_n\}) = 0$. Note that $f_i = \sum_{j=1}^{n_i} g_j^i h_j^i$, $1 \leq i \leq n$, where $g_j^i \in X$ and $h_j^i \in R[x]$. Put $Y = \{g_j^i \mid 1 \leq i \leq n,\ 1 \leq j \leq n_i\}$. Then $r_{R[x]}(Y) \neq 0$ and the McCoy condition guarantee that $r_R(C_Y) \neq 0$. As R is strongly right AB we get $r_R(C_Y R) \neq 0$. Thus $r_R(YR) \neq 0$ and so $r_R(YR[x]) \neq 0$. As $\{f_1, \ldots, f_n\} \subseteq YR[x]$ we get $r_{R[x]}(\{f_1, \ldots, f_n\}) \neq 0$, being a contradiction. Hence $R[x]$ is strongly right AB.

($\Leftarrow$) Clearly, R is strongly right AB. To prove it is right McCoy, assume that $f(x)g(x) = 0$, where $f(x)$, $g(x)$ are nonzero elements of $R[x]$. As $R[x]$ is strongly right AB, we get $r_{R[x]}(f(x)R[x]) \neq 0$. By [16, Theorem 2.2], we get that $r_R(f) \neq 0$, i.e., R is a right McCoy ring. $\square$

Corollary 2.18. *Let R be a right zip π-CN-ring. Then $R[x]$ is strongly right AB.*

Proposition 2.19. *A ring R is right zip, right McCoy and strongly right AB if and only if $R[x]$ is strongly right AB right zip.*

Proof. ($\Rightarrow$) As R is a right McCoy right zip ring, then $R[x]$ is a right zip ring. Also $R[x]$ is strongly right AB by Theorem 2.17.

($\Leftarrow$) Since $R[x]$ is a right zip ring, we get R is a right zip ring and using Theorem 2.17, it implies that R is a right McCoy strongly right AB ring. $\square$

According to Fields [12, Theorem 5], if R is a commutative Noetherian ring with identity in which $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a shortest primary representation of (0) with $\sqrt{Q_i} = P_i$, then $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a zero-divisor in $R[[x]]$ if and only if there is a non-zero element $r \in R$ which satisfies $rf(x) = 0$. In the following theorem, we extend this result to noncommutative rings.

Theorem 2.20. *Let R be a π-CN-ring. If $Nil(R)$ is Noetherian as a right R-module, then R is a power series-wise McCoy ring.*

Proof. We prove the right McCoy property, the left case is similar. First notice that $Nil(R)$ is a nilpotent ideal of the ring R and then there exists an integer t such that $(Nil(R))^t = 0$ but $(Nil(R))^{t-1} \neq 0$. Now, let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero elements in $R[[x]]$ such that $f(x)g(x) = 0$. We can assume that $b_0 \neq 0$. If $f(x) \in Nil(R[[x]])$, there clearly $f(x)c = 0$, where $0 \neq c \in (Nil(R))^{t-1}$. So assume $f(x) \notin Nil(R)[[x]]$. We get two cases:

Case 1: $g(x) \notin Nil(R[[x]])$. As $Nil(R)$ is Noetherian as a right R-module, it is a nilpotent ideal of R and so $Nil(R)[[x]] = Nil(R[[x]])$. Hence $g(x) \notin Nil(R)[[x]]$ and so there exists a minimal prime ideal P of R such that $g(x) \notin P[[x]]$. We claim that $f(x) \in P[[x]]$. Otherwise, there exist integers $a_m \in C_{f(x)}$, $b_n \in C_{g(x)}$ such that $a_m \notin P$ and $b_n \notin P$, but $a_i \in P$, $b_j \in P$ for every $i < m$, $j < n$. From $f(x)g(x) = 0$ we get

$$a_{m+n}b_0 + \cdots + a_m b_n + \cdots + a_0 b_{m+n} = 0.$$

Thus

$$-a_m b_n = a_{m+n}b_0 + \cdots + a_{m-1}b_n + a_m b_{n-1} + \cdots + a_0 b_{m+n} \in P.$$

As in 2-primal rings, any minimal prime ideal is a completely prime ideal, we must have either $a_m \in P$ or $b_n \in P$, being a contradiction. So $f(x) \in P[[x]]$. Consider $0 \neq q \in \bigcap Q$, where Q is a prime ideal of R with $Q \neq P$. Then there exists an integer k minimal with respect that $(Pq)^k = 0$. If $q(Pq)^{k-1} \neq 0$, then $f(x)c_1 = 0$, where $0 \neq c_1 \in q(Pq)^{k-1}$. If $q(Pq)^{k-1} = 0$, then $(Pq)^{k-1}q = 0$, because $((Pq)^{k-1})^2 = 0$ and so $(Pq)^{k-1} \in \mathrm{Cent}(R)$. If $q(Pq)^{k-2}q \neq 0$, then from $(Pq)^{k-1}q = 0$ we get $f(x)c_2 = 0$, where $0 \neq c_2 \in q(Pq)^{k-2}q$. If $q(Pq)^{k-2}q = 0$, then $(Pq)^{k-2}q^2 = 0$, because $((Pq)^{k-2}q)^2 = 0$ and so $(Pq)^{k-2}q \in \mathrm{Cent}(R)$. Continuing in this way there exists a nonzero element $r \in R$ such that $f(x)r = 0$.

Case 2: $g(x) \in Nil(R[[x]])$. As $Nil(R[[x]]) = Nil(R)[[x]]$ we get $C_{g(x)} \subseteq Nil(R)$. As $Nil(R)$ is a nilpotent ideal of R, there exists an integer s such that $g(x)(Nil(R))^s = 0 \neq g(x)(Nil(R))^{s-1}$. Put $h(x) = g(x)d$, where $d \in (Nil(R))^{s-1}$ such that $h(x) = g(x)d \neq 0$. Clearly $f(x)h(x) = 0$ and any element in $C_{h(x)}$ has the nilpotancy index 2 and so $h(x) \in \mathrm{Cent}(R[[x]])$. Put $h(x) = \sum_{j=0}^{\infty} c_j x^j$. As $f(x)h(x) = 0$ we get $a_i c_j \in Nil(R)$ for every $i, j \in \{0, 1, 2, \ldots\}$. Hence

$$a_0c_1R \subseteq a_0c_1R + a_1c_0R \subseteq a_0c_1R + a_1c_0R + a_2c_0R \subseteq \cdots .$$

As $Nil(R)$ is Noetherian as right R-module, there are integers m, n such that

$$(*_1) \qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i c_j R = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i c_j R.$$

Again as $Nil(R)$ is Noetherian as a right R-module, there exists an integer k such that the right ideal of R generated by the set $\{c_0, c_1, \ldots, c_k\}$ is equal to right ideal of R generated by $C_{h(x)}$. Hence

$$(*_2) \qquad l_R(h(x)) = l_R(\{c_0, c_1, \ldots, c_k\}).$$

Note that for every $i, j \in \{0, 1, 2, \ldots\}$, since $c_j^2 = 0$ we get $(a_i c_j)^2 = a_i^2 c_j^2 = 0$ and so $a_i c_j \in \mathrm{Cent}(R)$. Now from the equation $f(x)h(x) = 0$ we get

$$(1) \qquad a_0c_0 = 0,$$

$$(2) \qquad a_0c_1 + a_1c_0 = 0,$$

$$a_0c_2 + a_1c_1 + a_2c_0 = 0, \tag{3}$$
$$\vdots$$
$$a_0c_{m+n+k} + a_1c_{m+n+k-1} + \cdots + a_{m+n+k}c_0 = 0. \tag{4}$$

Multiplying the equation (2) on the right side by a_0 and using the fact that $c_j \in \mathrm{Cent}(R)$, $j = 0, 1, \ldots$, we get

$$0 = a_0c_1a_0 + a_1c_0a_0 = a_0^2c_1 + a_1a_0c_0 = a_0^2c_1.$$

Multiplying the equation (3) on the right side by a_0^2 and using the fact that $c_j \in \mathrm{Cent}(R)$, $j = 0, 1, \ldots$, we get

$$0 = a_0c_2a_0^2 + a_1c_1a_0^2 + a_2c_0a_0^2 = a_0^3c_2.$$

Continuing in this way there exists an integer l_0 such that $a_0^{l_0+1}(c_0+c_1x+\cdots+c_{m+n+k}x^{m+n+k}) = 0 \neq a_0^{l_0}(c_0 + c_1x + \cdots + c_{m+n}x^{m+n}) = (c_0 + c_1x + \cdots + c_{m+n+k}x^{m+n+k})a_0^{l_0}$. Since $h(x)$ is central in $R[[x]]$, we get $a_0^{l_0}h(x) = h(x)a_0^{l_0} \neq 0$. By $(*)$, $a_0h(x)a_0^{l_0} = a_0^{l_0+1}h(x) = 0$. From this and $f(x)h(x)a_0^{l_0} = 0$ we must have $x(a_1 + a_2x + \cdots)h(x)a_0^{l_0} = 0$ and hence

$$(a_1 + a_2x + \cdots)h(x)a_0^{l_0} = 0. \tag{$*_3$}$$

By similar arguments as above, there exists an integer l_1 such that $a_1^{l_1+1}h(x)a_0^{l_0} = 0 \neq a_1^{l_1}h(x)a_0^{l_0} = h(x)a_0^{l_0}a_1^{l_1}$. From this and $(*_3)$ we get $(a_2 + a_3x + \cdots)h(x)a_0^{l_0}a_1^{l_1} = 0$. Continuing in this way there are integers $l_0, l_1, \ldots, l_{m+n+k}$ such that

$$h(x)a_0^{l_0}a_1^{l_1}\cdots a_{m+n+k}^{l_{m+n+k}} \neq 0 = a_ih(x)a_0^{l_0}a_1^{l_1}\cdots a_{m+n+k}^{l_{m+n+k}},\ \forall\ 0 \leq i \leq m+n+k.$$

Put $r = a_0^{l_0}a_1^{l_1}\cdots a_{m+n+k}^{l_{m+n+k}}$. From this and using $a_ic_j \in \mathrm{Cent}(R)$, we get $a_ic_jRr = Ra_ic_jr = 0$ for every $0 \leq i, j \leq m+n+k$. So

$$\sum_{i=0}^{m+n+k}\sum_{j=0}^{m+n+k} a_ic_jRr = 0$$

and by $(*_1)$ we obtain that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_ic_jRr = 0.$$

Thus $f(x)c_jr = 0$ for every $j = 0, 1, \ldots$. As $h(x)r \neq 0$, there exists an integer p such that $c_pr \neq 0$, while $f(x)c_pr = 0$, which implying that R is a right power series McCoy ring. □

Corollary 2.21. *Let R be a right Noetherian π-CN-ring. Then R is a power series-wise McCoy ring.*

Let R be a ring and α denotes an endomorphism of R with $\alpha(1) = 1$. Let

$$T_n(R, \alpha) = \{(a_{ij})_{n\times n} \,|\, a_{ij} \in R,\ a_{ij} = 0 \ \text{ if } \ i > j\}.$$

For $(a_{ij}), (b_{ij}) \in T_n(R, \alpha)$ define,

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \quad \text{and} \quad (a_{ij}) * (b_{ij}) = (c_{ij}),$$

where $c_{ij} = 0$ for $i > j$ and $c_{ij} = \sum_{k=i}^{j} a_{ik}\alpha^{k-i}(b_{kj})$ for $i \leq j$. It can be easily checked that $T_n(R, \alpha)$ is a ring, called the skew triangular matrix ring over R. If α is the identity, $T_n(R, \alpha)$ is the triangular matrix ring $T_n(R)$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R, n, \alpha)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R, n, \alpha)$. We can denote $A = (a_{ij}) \in T(R, n, \alpha)$ by $(a_{11}, \ldots, a_{1n})$. Then $T(R, n, \alpha)$ is a ring with addition point-wise and multiplication given by:

$$(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) = (a_0b_0, a_0{*}b_1{+}a_1{*}b_0, \ldots, a_0{*}b_{n-1}{+}\cdots{+}a_{n-1}{*}b_0)$$

with $a_i * b_j = a_i\sigma^i(b_j)$ for each i and j. Also, there exists a ring isomorphism

$$\varphi : R[x; \alpha]/\langle x^n\rangle \longrightarrow T(R, n, \sigma)$$

such that $\varphi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n\rangle) = (a_0, \ldots, a_{n-1})$. Therefore,

$$T(R, n, \alpha) \cong R[x; \alpha]/\langle x^n\rangle,$$

where $\langle x^n\rangle$ is the ideal generated by x^n in $R[x; \alpha]$, (see [7]).

Example 2.22. Let R be a commutative reduced ring. If α is an automorphism of R such that $\alpha^2 = id$, then $T(R, 3, \alpha)$ is a π-CN-ring.

Proof. As R is a reduced ring,

$$Nil(T(R, 3, \alpha)) = \left\{ \begin{pmatrix} 0 & a_2 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix} \mid a_1, a_2 \in R \right\}.$$

As $\alpha^2 = id$, we get

$$\left\{ \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a_2 \in R \right\} \subseteq \mathrm{Cent}(T(R, 3, \alpha)).$$

Note that for any nonzero element $A = \left(\begin{smallmatrix} 0 & c_2 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in Nil(T(R, 3, \alpha))$, if $c_2 = 0$, then $A \in \mathrm{Cent}(T(R, 3, \alpha))$. If $a_2 \neq 0$, then $0 \neq A^2 \in \mathrm{Cent}(T(R, 3, \alpha))$, which implies that $T(R, 3, \alpha)$ is a π-CN-ring. $\square$

Corollary 2.23. *Let R be a commutative Noetherian reduced ring. If α is an automorphism of R such that $\alpha^2 = id$, then $T(R, 3, \alpha)$ is a power series-wise McCoy ring.*

Proof. Apply Theorem 2.21 and Example 2.22. $\square$

Example 2.24. Let R be a ring, $n \geq 4$. Then the following statements are equivalent:
(1) R is commutative and $\alpha = id$;
(2) $T(R, n, \alpha)$ is commutative;
(3) $T(R, n, \alpha)$ is a π-CN-ring;
(4) $R[x; \alpha]/\langle x^n \rangle$ is a π-CN-ring.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. (3) $\Rightarrow$ (1) Assume that $r \in R$. Then $(0, \ldots, 0, r)^2 = 0$. As $T(R, n, \sigma)$ is a π-CN-ring, we get

$$(0, \ldots, 0, r) \in \mathrm{Cent}(T(R, n, \sigma)),$$

and so $(0, \ldots, 0, 1)(s, 0, \ldots, 0) = (s, 0, \ldots, 0)(0, \ldots, 0, 1)$ for every $s \in R$. So we get $\sigma^{n-1}(s) = s$ for every $s \in R$. As $(0, \ldots, 0, 1, 0)^2 = 0$ and $T(R, n, \alpha)$ is a π-CN-ring, we get $(0, \ldots, 0, 1, 0) \in \mathrm{Cent}(T(R, n, \sigma))$, and so

$$(0, \ldots, 0, 1, 0)(s, 0, \ldots, 0) = (s, 0, \ldots, 0)(0, \ldots, 0, 1, 0).$$

This implies that $\alpha^{n-2}(s) = s$ for every $s \in R$. Since $\alpha^{n-2}(s) = \alpha^{n-1}(s) = s$ for every $s \in S$ we get $\alpha = id$. □

According to Krempa [22], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. By [13], for each ring R with an α rigid endomorphism of R, the ring $T(R, n, \alpha)$ is a reversible ring and hence is a McCoy ring which is not π-CN.

2.2. π-Duo rings

We present and analyse the notion of π-duo rings as a generalization of duo rings in the current research, which is driven by the foregoing discussion, and prove that π-duo rings are 2-primal. Our primary finding is that π-duo rings are McCoy and abelian at the same time. When R is a one-sided π-duo ring, we show that R is reduced only if R is nonsingular. Furthermore, we elaborate on the fact that a right π-duo ring R has the right property (A) provided that either of the following requirements are met:
- every prime ideal of R is maximal.
- R is right weakly continuous.
- R is right FI-extending with ACC on direct summands.

A ring R is defined as a duo ring by [11] if each one-sided ideal of R is a two-sided ideal. If all of right (resp., left) ideals of a ring are two-sided, the term right duo is preferred (resp., left duo). Recall that a ring R is called left π-duo if for every $r \in R$, there exists a natural number $n(r)$ (depending on r) such that the principal left ideal $Rr^{n(r)}$ is two-sided. Yao [29] proved that left π-duo rings are abelian, which extends the older result of Courter [8, Theorem 1.3] for left or right duo rings.

Definition 2.25. We say a ring R is left π-duo if for every $0 \neq a \in R$, there exists a positive integer k for which $0 \neq a^k R \subseteq Ra^k$. The definition of a right

π-duo ring is quite the same. If a ring is both left and right π-duo, it is called π-duo.

Duo rings are definitely π-duo, the examples provided below indicates that the classification of π-duo rings clearly encompasses duo rings.

Example 2.26. Assume that $A = \mathbb{Z}_2\langle u, v\rangle$ is the free algebra generated by noncommuting indeterminates x, y over $\mathbb{Z}_2$ and assume that

$$R := A/I, \text{ where } I \text{ is an ideal of } A \text{ generated by the set } \{u^3, v^3, uv, vu^2, v^2u\}.$$

We will show that R is a π-duo ring. To reach this conclusion, it is initially determined that any element of R is the form $\gamma = a_0 + a_1u + a_2v + a_3u^2 + a_4v^2 + a_5vu$, in which $a_i \in \mathbb{Z}_2$, $1 \le i \le 5$. Therefore, it is evident that is u^2, v^2 and vu represent the R center. Now consider the element $\gamma = a_0 + a_1u + a_2v + a_3u^2 + a_4v^2 + a_5vu$ of R. If $a_1 = a_2 = 0$, then clearly γ is in the center of R and there is nothing to prove. If $a_1 \neq 0$ or $a_2 \neq 0$, then $0 \neq \gamma^2 = a_0^2 + a_1^2u^2 + a_2^2v^2 + a_2a_1vu$, which implies that γ^2 is in the center of R. Therefore R is a π-duo ring.

Now consider $Rv = \{0, v, v^2, v + v^2\}$, $vR = \{0, v, v^2, vu, v + v^2, v + vu, v^2 + vu, v + v^2 + vu\}$, $Ru = \{0, u, u^2, vu, u + u^2, u + vu, u^2 + vu, u + u^2 + vu\}$ and $uR = \{0, u, u^2, u + u^2\}$. We have $vR \nsubseteq Rv$ and $Ru \nsubseteq uR$. This shows that R is not a duo ring.

Example 2.27. Consider the ring in Example 2.26, $A = \mathbb{Z}_2\langle u, v\rangle$ be the free algebra generated by noncommuting indeterminates x, y over $\mathbb{Z}_2$ and assume that

$$R = A/I, \text{ where } I \text{ is an ideal of } A \text{ generated by the set } \{u^3, v^3, uv, vu^2, v^2u\}.$$

We claim that $R[x]$ is a π-duo ring. For, assume that $f(x) = \gamma_0 + \gamma_1(x) + \cdots + \gamma_kx^k \in R[x]$. Then

$$f(x)^2 = \gamma_0^2 + (\gamma_1\gamma_0 + \gamma_0\gamma_1)x + (\gamma_0\gamma_2 + \gamma_1^2 + \gamma_2\gamma_0)x^2 + \cdots + \gamma_k^2x^{2k}.$$

We will show that for any $\gamma_i, \gamma_j \in R$ we get $\gamma_i\gamma_j + \gamma_j\gamma_i \in \mathrm{Cent}(R)$. Put $\gamma_i = a_{i0} + a_{i1}u + a_{i2}v + a_{i3}u^2 + a_{i4}v^2 + a_{i5}vu$ and $\gamma_j = a_{j0} + a_{j1}u + a_{j2}v + a_{j3}u^2 + a_{j4}v^2 + a_{j5}vu$. Then we get

$$\begin{aligned}
\gamma_i\gamma_j + \gamma_j\gamma_i &= (a_{i0}a_{j0} + a_{j0}a_{i0}) + (a_{i0}a_{j1} + a_{i1}a_{j0} + a_{j0}a_{i1} + a_{j1}a_{i0})u \\
&\quad + (a_{i0}a_{j2} + a_{i2}a_{j0} + a_{j0}a_{i2} + a_{j2}a_{i0})v \\
&\quad + (a_{i0}a_{j3} + a_{i3}a_{j0} + a_{j0}a_{i3} + a_{j3}a_{i0} + a_{i1}a_{j1} + a_{j1}a_{i1})u^2 \\
&\quad + (a_{i2}a_{j2} + a_{i0}a_{j4} + a_{i4}a_{j0} + a_{j_2}a_{i2} + a_{j0}a_{i4} + a_{j4}a_{i0})v^2 \\
&\quad + (a_{i0}a_{j5} + a_{i5}a_{j0} + a_{i2}a_{j1} + a_{j2}a_{i1} + a_{j0}a_{i5} + a_{j5}a_{i0})vu \\
&= 2a_{i0}a_{j0} + (2a_{i0}a_{j1} + 2a_{i1}a_{j0})u + (2a_{i0}a_{j2} + 2a_{i2}a_{j0})v \\
&\quad + (2a_{i0}a_{j3} + 2a_{i3}a_{j0} + 2a_{i1}a_{j1})u^2 \\
&\quad + (2a_{i2}a_{j2} + 2a_{i0}a_{j4} + 2a_{i4}a_{j0})v^2 \\
&\quad + (2a_{i0}a_{j5} + 2a_{i5}a_{j0} + a_{i2}a_{j1} + a_{j2}a_{i1})vu
\end{aligned}$$

$$= (a_{i2}a_{j1} + a_{j2}a_{i1})vu \in \mathrm{Cent}(R).$$

So $\gamma_i\gamma_j + \gamma_j\gamma_i \in \mathrm{Cent}(R)$ for $0 \leq i,j \leq k$. As $\gamma_i^2 = a_{i0}^2 + a_{i1}^2u^2 + a_{i2}^2v^2 + a_{i2}a_{i1}vu \in \mathrm{Cent}(R)$, therefore

$$f(x)^2 \in \mathrm{Cent}(R)[x] \subseteq \mathrm{Cent}(R[x]).$$

Recall from that a ring R is called left (right) *quasi-duo* if every maximal left (right) ideal of R is two-sided. By [30] it is immediately observed that every factor ring R/I of a left quasi-duo ring R is again left quasi-duo. It is obvious that if R is a commutative ring and $n \geq 2$, then the ring of upper triangular $n \times n$ matrices over R is left quasi-duo.

Proposition 2.28. *Every π-duo ring is quasi-duo.*

Proof. Assume that M is a maximal right ideal of R. Assume on the contrary that $RM \nsubseteq M$. Then there exists $r \in R$ such that $rM \nsubseteq M$. So we get $rM + M = R$. It follows that $rm + n = 1$ for some $m, n \in M$. As R is a π-duo ring, there exists an integer k such that $(mr)^kR = R(mr)^k$. Multiplying $rm+n = 1$ on the right side by $(rm)^k$ we get $r(mr)^km+n(rm)^k = (rm)^k$. As $(mr)^kR = R(mr)^k$, there exists $s \in R$ such that $r(mr)^k = (mr)^ks \in M$. So $r(mr)^km + n(rm)^k \in M$ which implies that $(rm)^k \in M$.

Now, multiplying $rm + n = 1$ on the right side by $(rm)^{k-1}$ we get $(rm)^k + n(rm)^{k-1} = (rm)^{k-1}$ implies that $(rm)^{k-1} \in M$. Continuing in this way yields that $rm \in M$, and so $rm + n \in M$, contradiction to $rm + n = 1 \in M$. Hence $RM \nsubseteq M$, i.e., R is a right quasi-duo. By a similar way we obtain that R is a left quasi-duo ring. □

Bass [2] proved that a left perfect ring R has the property that every left R-module has a maximal submodule and R contains no infinite set of orthogonal idempotents, and asked if the converse holds also. As a corollary of [30, Theorem 3.3], we get:

Corollary 2.29. *For a π-duo ring R, the following are equivalent*:

(1) *R is left perfect*;

(2) *R contains no infinite set of orthogonal idempotents and every left R-module has a maximal submodule.*

Theorem 2.30. *Let R be a π-duo ring. Then $W(R) = Nil_*(R) = L\text{-}rad(R) = Nil^*(R) = Nil(R)$.*

Proof. Since $W(R) \subseteq Nil_*(R) \subseteq L\text{-}rad(R) \subseteq Nil^*(R) \subseteq Nil(R)$, it is enough to show that $W(R) = Nil(R)$. Initially and with $b^2 = 0$ and R as a π-duo ring, the most likely outcomes will $Rb \subseteq bR$ and so $bRb \subseteq b^2R = 0$ thereby leading $(bR)^2 = (Rb)^2 = 0$. Following that and with $b^k = 0$ and $k \geq 3$, we will have

$$(*) \qquad Rb^{k-1}b = 0.$$

As $(b^{k-1})^2 = 0$, the likely outcomes will be $b^{k-1}R \subseteq Rb^{k-1}$ and $(*)$ leads to $b^{k-1}Rb = 0$, followed by

$$(**) \qquad bb^{k-2}RbR = 0.$$

The most likely outcome here will be $(b^{k-2}Rb)^2 = 0$, and R as π-duo leads to $Rb^{k-2}Rb \subseteq b^{k-2}RbR$. With $(**)$ we will obtain

$$(***) \qquad bRb^{k-2}RbR = bRbb^{k-3}RbR = 0.$$

Considering $(b^{k-3}Rb)^2 = 0$, the most likely outcome will be $Rb^{k-3}Rb \subseteq b^{k-3}RbR$ and $(***)$ gives us $0 = (bR)^2b^{k-3}RbR = (bR)^2b^{k-4}bRbR = (bR)^2b^{k-4}(bR)^2$. As $((bR)^2b^{k-4})^2 = 0$ the most likely outcome will be $(bR)^2b^{k-4}R = R(bR)^2b^{k-4}$ and $0 = (bR)^2b^{k-4}(bR)^2 = (bR)^2b^{k-4}R(bR)^2 = (bR)^2b^{k-5}(bR)^3$. Since we have $b^{k-6}(bR)^2bb^{k-6}(bR)^2b \subseteq b^{k-6}(bR)^2b^{k-5}(bR)^3 = 0$, we obtain $Rb^{k-6}(bR)^2b = b^{k-6}(bR)^3$ followed by $0 = (bR)^2b^{k-5}(bR)^3 = (bR)^2bb^{k-6}(bR)^3 = (bR)^3b^{k-6}(bR)^3$. If we proceed the same process, we get $(bR)^k = 0$, which renders $W(R) = Nil(R)$. □

Corollary 2.31. *If R is a π-duo ring, then $Nil(R)[x] = Nil(R[x])$.*

Proposition 2.32. *Every left π-duo ring is abelian.*

Proof. If we consider e an idempotent of R, for each $r \in R$, there exists $s \in R$ such that $er = se$ and so $er = ere$. In the presence of $(1-e)R \subseteq R(1-e)$, there exists $t \in R$ such that $(1-e)r = t(1-e)$, which will result in $(1-e)r = (1-e)r(1-e)$ for every $r \in R$. Here, the most likely outcome would be $0 = (1-e)re$, indicating $re = ere$ for every $r \in R$. So $er = ere = re$ for every $r \in R$. □

Proposition 2.33. *Right π-duo nonsingular rings are reduced.*

Proof. Considering $a \in R$ such that $a^2 = 0$, we have $(Ra)^2 = 0$. Following that, $r_R(aR)$ would be regarded as the essential right ideal of R. If $0 \neq b \in R$, the probable integer k will be associated with $0 \neq Rb^k \subseteq b^kR$ for the right π-duo condition. If $aRb^kR \neq 0$ will give us $aRb^kR \subseteq r_R(aR)$. If we consider arb^k to be a nonzero element of aRb^kR, the most likely outcome will be $arb^k = b^kc$, where $c \in R$, thereby leading to $0 \neq b^kcR \subseteq r_R(aR)$ and $r_R(aR)$ is an essential right ideal of R. However, the nonsingularity of R would give us $aR = 0$ and the subsequent $a = 0$, rendering R a reduced ring. □

Lemma 2.34. *Every minimal left ideal of a left π-duo ring is an ideal.*

Proof. If we consider L as minimal left ideal of R, we would reach $L = Ra$ for every non-zero element a of L. Since R is a left π-duo ring, there exists an integer k for which $0 \neq a^kR \subseteq Ra^k$. As $L = Ra^k$, there exists $r \in R$ such that $ra^k = a$. Hence $aR = ra^kR \subseteq rRa^k \subseteq Ra^k = L$, which implies that $LR \leq L$, as needed. □

Lemma 2.35. *Let R be a left π-duo ring. Then for each $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ with $f(x)g(x) = 0$, there exists an integer k such that $a_0^{(j+1)k} b_j = 0$ for all $0 \leq j \leq n$.*

Proof. We may assume that $a_0 \neq 0$. Hence there exists an integer k for which $a_0^k R \subseteq R a_0^k \neq 0$, as R is left π-duo. If $k \geq 2$, the $f(x)g(x) = 0$ equation may lead to the following outcomes:

$$a_0 b_0 = 0, \tag{1}$$

$$a_0 b_1 + a_1 b_0 = 0, \tag{2}$$

$$\vdots$$

$$a_0 b_t + a_1 b_{t-1} + \cdots + a_t b_0 = 0. \tag{3}$$

Multiplying equation (2) on the left side by a_0^k, we will obtain $a_0^{k+1} b_1 + a_0^k a_1 b_0 = 0$. In the presence of $a_0^k a_1 \in R a_0^k$, the most likely outcome will be $a_0^k a_1 b_0 = 0$ and the subsequent $a_0^{k+1} b_1 = 0$. In the case of $k \geq 2$, $2k \geq k+1$ the outcome will be $a_0^{2k} b_1 = 0$.

By induction assume that $a_0^{(j+1)k} b_j = 0$ for every $0 \leq j < t$. Especially $a_0^{tk} b_j = 0$ for every $0 \leq j \leq t-1$ and hence $a_0^{tk} R b_j \subseteq a_0^{(t-1)k} R a_0^k b_j \subseteq a_0^{(t-2)k} R a_0^{2k} b_j \subseteq \cdots \subseteq R a_0^{tk} b_j = 0$ for every $0 \leq j \leq t-1$. Multiplying equation (3) on the left side by a_0^{tk} we get $a_0^{tk} a_t b_0 + \cdots + a_0^{tk} a_1 b_{t-1} + a_0^{tk} a_0 b_t = 0$. It follows that $a_0^{tk} a_0 b_t = 0$, which implies that $a_0^{(t+1)k} b_t = 0$. This finishes our inductive step and so the proof is complete. □

Lemma 2.36. *Let R be a right π-duo ring. Then for each $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ with $f(x)g(x) = 0$, there exists an integer s such that $a_i b_0^{(i+1)s} = 0$ for all $0 \leq i \leq m$.*

Proof. It is similar to the one of Lemma 2.35. □

According to Camillo and Nielsen [5], duo rings are classified as McCoy, while they also state that McCoy rings are not abelian in all the cases. In this paper, proof is presented on the fact that the π-duo rings in Proposition 2.32 are abelian. Moreover, Nielsen demonstrated that reversible rings are McCoy [28, Theorem 2], elaborating on the subject with an example of a semi-commutative ring as an incorrect McCoy ring although semi-commutative rings are classified as abelian and 2-primal. In the following sections, π-duo rings have been shown to be McCoy as well in the primary results. Therefore, it could be inferred that π-duo rings are abelian and McCoy.

A ring R is said to be *strongly right McCoy* provided that $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the right ideal of R generated by the coefficients of $g(x)$, where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Strongly left McCoy rings are defined similarly. If a ring is both strongly left and strongly right McCoy, then the ring is called a strongly McCoy ring.

Theorem 2.37. *π-duo rings are strongly McCoy.*

Proof. We only proof the right case, the left case is similar. With the assumption that $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0$ and $g(x) \neq 0$. We will show, by induction on the degree of $f(x)$, that there is some non-zero element in $I_{g(x)}$, which annihilates $f(x)$ on the right, where $I_{g(x)}$ representing the right ideal obtained from coefficients of $g(x)$. Assume that $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$, as well as $b_0 \neq 0$. In the case of $\deg(f(x)) = 0$, $f(x)b_0 = 0$ will render the claim is trivial.

The next case is $\deg(f(x)) > 0$.

Case 1: If $a_0 g(x) = 0$, put $f_1(x) = (f(x) - a_0)/x$ which gives us $f_1(x)g(x) = 0$. But $\deg(f_1(x)) < \deg(f(x))$, and hence by induction there exists a non-zero element $c \in I_{g(x)}$ satisfying $f_1(x)c = 0$, whence $f(x)c = 0$.

Case 2: If $a_0 g(x) \neq 0$, then there exists a minimal integer $1 \leq v \leq n$ such that $a_0 b_v \neq 0$. As R is π-duo, there exists an integer s_v such that $0 \neq b_v^{s_v} R = R b_v^{s_v}$. Put $g_1(x) = g(x) b_v^{s_v - 1}$. Then $g_1(x) \neq 0$ and $f(x) g_1(x) = 0$. We claim that there exists $r \in R$ such that $g_1(x) r \neq 0$ and $a_0 b_v^{s_v} r = 0$. We may assume that $a_0 b_v^{s_v} \neq 0$.

By Lemma 2.35, there exists an integer k such that $a_0^k b_v^{s_v} = 0 \neq a_0^{k-1} b_v^{s_v}$. As $R b_v^{s_v} = b_v^{s_v} R$, there exists an element r_v such that $0 \neq a_0^{k-1} b_v^{s_v} = b_v^{s_v} r_v$. Put $g_1'(x) = g(x) b_v^{s_v - 1} r_v$. Then $g_1'(x) \neq 0$ and $I_{g_1(x)'} \subseteq I_{g(x)}$ and a_0 annihilates the first v coefficients of $g_1'(x)$, so after repeating this process a finite number of times we reduce to the previous case. $\square$

2.3. Property (A)

Lucas claims that a commutative ring R satisfies property (A) if any of the finitely generated ideals of R consisting entirely of zero divisors has a nonzero annihilator [25]. Numerous scholars have examined rings with property (A), including ([1,15,25,31] etc.), proposing a number of results that are considered significant in researching commutative rings with zero divisors. The concept of property (A) also encompasses non-commutative rings based on the investigations conducted by C. Y. Hong, N. K. Kim, Y. Lee, and S. J. Ryu [19].

Definition 2.38 (see [19])**.** *Right* (*left*) *property* (A) apply to any ring R provided that a nonzero $a \in R$ (resp. $b \in R$) is achieved for each finitely generated two-sided $I \subseteq Z_l(R)$ (resp. $I \subseteq Z_r(R)$), which will likely result $Ia = 0$ (resp. $bI = 0$), with $Z_l(R)$ (resp. $Z_r(R)$) representing the set of left (resp. right) of R zero-divisors. If a ring R possesses both left and right property (A), the presence of property (A) is confirmed.

A ring R is called *strongly right* (*resp. left*) *AB* if every non-zero right (resp. left) annihilator of R is bounded; R is called strongly AB if R is strongly right and strongly left AB.

Proposition 2.39. *Left π-duo rings are strongly right AB.*

Proof. Let R be a π-duo ring and assume that $X \subseteq R$ such that $r_R(X) \neq 0$. Then for every $0 \neq a \in r_R(X)$, there exists an integer k such that $0 \neq Ra^k \leq a^kR$. As $Xa^kR = 0$ we get $XRa^k = 0$ and so the nonzero ideal Ra^kR is contained in $r_R(X)$. □

Corollary 2.40. *If R is a π-duo ring, then $R[x]$ has property (A).*

Proof. It is an immediate consequence of [32, Theorem 3.7] and Lemmas 2.37, 2.39. □

A ring R is right FI-extending if every (ideal) right ideal of R is essential in a right ideal generated by an idempotent.

Theorem 2.41. *Let R be a right FI-extending ring with ACC on direct summands. If R is a π-duo ring, then R has right property (A).*

Proof. Assuming $I = \sum_{i=1}^{k} Ra_iR \subseteq Z_l(R)$, with R classified as right FI-extending, there exists a direct summand eR of R such that $I \leq_{es} eR$. In the case of $e \neq 1$, we will have $I(1-e) = 0$. Mean while, $e = 1$ will give us $I \leq_{es} R$. When R corresponds to the ACC on direct summands, the most likely outcome will be primitive idempotents f_j, $1 \leq j \leq t$, which are associated with $R = \sum_{j=1}^{t} Rf_j$ and result in $I = \sum_{j=1}^{t} If_j$, and the subsequent outcome will be $If_j \leq_{es} Rf_j$ in all cases of $1 \leq j \leq t$. With R classified as π-duo, the result will be $f_j \in \mathrm{Cent}(R)$ in all the cases of $1 \leq t$, thereby rendering Rf_j a right FI-extending and indecomposable ring in the all cases of $1 \leq j \leq t$, which is followed by uniform Rf_j as the left R-module in every possible case of $1 \leq j \leq t$.

We claim that there exists an integer $1 \leq v \leq t$ such that $If_v = \sum_{i=1}^{k} Ra_if_vR \subseteq Z_l(Rf_v)$. By contrary assume that $If_j \nsubseteq Z_l(Rf_j)$ for every $1 \leq j \leq t$. So for each j, there is a non-zero element $r_j \in If_i$ such that $r_js_j \neq 0$ for every non-zero $s_j \in Rf_j$. Assuming $r = r_1 + r_2 + \cdots + r_t$ would lead us to the nonzero element $b \in R$ and the subsequent $rb = 0$, in the case of $r \in I \subseteq Z_l(R)$. In the present of $b = b_1 + b_2 + \cdots + b_t$ we will have $b_j \in Rf_j$, as well as $1 \leq j \leq t$. From $rb = 0$ the most likely outcome will be $rbf_j = r_jb_j = 0$ for each $1 \leq j \leq t$. In the case of $r_j \notin Z_l(Rf_j)$, we get $b_j = 0$ for each j, which was identified a contradiction. Hence there exists $1 \leq p \leq t$ such that $If_p \subseteq Z_l(Rf_p)$. Now assume that $a_if_pb_if_p = 0$, where $0 \neq b_if_p$, $1 \leq i \leq k$. In addition Rf_p is classified as a right π-duo ring, which will result in integers s_i, $1 \leq i \leq k$ accompanied $0 \neq Rf_pb_i^{s_i} \subseteq b_i^{s_i}Rf_p$. Given the uniform Rf_p, which is classified as the left R-module, most likely outcome will be $\cap_{i=1}^{n} Rf_pb_i^{s_i} \neq 0$. In the case of $0 \neq c \in \cap_{i=1}^{n} Rf_pb_i^{s_i}$ followed by $Ic = 0$, the right property (A) of R is confirmed. □

Corollary 2.42. *Let R be a ring that is π-duo, semiperfect and extending. Then R has property (A).*

Proposition 2.43. *Let R be a ring that is π-duo and left extending with ACC on essential left ideals. Then R has right property (A).*

Proof. Our arguments are inspired by the techniques developed in [31, Proposition 2.5.]. $\square$

Theorem 2.44. *Let R be a ring that is π-duo and right weakly continuous. Then R has right property (A).*

Proof. Our arguments are inspired by the techniques developed in [31, Theorem 2.8]. $\square$

Theorem 2.45. *Let R be a one sided π-duo ring and assume that every prime ideal of R is maximal. Then R has property (A).*

Proof. Our arguments are inspired by the techniques developed in [31, Theorem 2.18]. $\square$

References

[1] F. Azarpanah, O. A. S. Karamzadeh, and R. A. Aliabad, *On ideals consisting entirely of zero divisors*, Comm. Algebra **28** (2000), no. 2, 1061–1073. https://doi.org/10.1080/00927870008826878

[2] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488. https://doi.org/10.2307/1993568

[3] G. F. Birkenmeier, H. E. Heatherly, and E. K. S. Lee, *Completely prime ideals and associated radicals*, in Ring theory (Granville, OH, 1992), 102–129, World Sci. Publ., River Edge, NJ, 1993.

[4] V. Camillo, C. Y. Hong, N. K. Kim, Y. Lee, and P. P. Nielsen, *Nilpotent ideals in polynomial and power series rings*, Proc. Amer. Math. Soc. **138** (2010), no. 5, 1607–1619. https://doi.org/10.1090/S0002-9939-10-10252-4

[5] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615. https://doi.org/10.1016/j.jpaa.2007.06.010

[6] F. Cedó, *Zip rings and Mal'cev domains*, Comm. Algebra **19** (1991), no. 7, 1983–1991. https://doi.org/10.1080/00927879108824242

[7] J. Chen, X. Yang, and Y. Zhou, *On strongly clean matrix and triangular matrix rings*, Comm. Algebra **34** (2006), no. 10, 3659–3674. https://doi.org/10.1080/00927870600860791

[8] R. C. Courter, *Finite-dimensional right duo algebras are duo*, Proc. Amer. Math. Soc. **84** (1982), no. 2, 157–161. https://doi.org/10.2307/2043655

[9] M. P. Drazin, *Rings with central idempotent or nilpotent elements*, Proc. Amer. Math. Soc. **27** (1971), no. 3, 427–433.

[10] C. Faith, *Annihilator ideals, associated primes and Kasch-McCoy commutative rings*, Comm. Algebra **19** (1991), no. 7, 1867–1892. https://doi.org/10.1080/00927879108824235

[11] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91. https://doi.org/10.2307/1993133

[12] D. E. Fields, *Zero divisors and nilpotent elements in power series rings*, Proc. Amer. Math. Soc. **27** (1971), 427–433. https://doi.org/10.2307/2036469

[13] M. Habibi, A. Moussavi, and A. Alhevaz, *The McCoy condition on Ore extensions*, Comm. Algebra **41** (2013), no. 1, 124–141. https://doi.org/10.1080/00927872.2011.623289

[14] E. Hashemi, *Extensions of zip rings*, Studia Sci. Math. Hungar. **47** (2010), no. 4, 522–528. https://doi.org/10.1556/SScMath.2009.1148

[15] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965), 110–130. https://doi.org/10.2307/1994260

[16] Y. Hirano, *On annihilator ideals of a polynomial ring over a noncommutative ring*, J. Pure Appl. Algebra **168** (2002), no. 1, 45–52. https://doi.org/10.1016/S0022-4049(01)00053-6

[17] Y. Hirano, D. V. Huynh, and J. K. Park, *On rings whose prime radical contains all nilpotent elements of index two*, Arch. Math. (Basel) **66** (1996), no. 5, 360–365. https://doi.org/10.1007/BF01781553

[18] C. Y. Hong, N. K. Kim, T. K. Kwak, and Y. Lee, *Extensions of zip rings*, J. Pure Appl. Algebra **195** (2005), no. 3, 231–242. https://doi.org/10.1016/j.jpaa.2004.08.025

[19] C. Y. Hong, N. K. Kim, Y. Lee, and S. J. Ryu, *Rings with Property* (A) *and their extensions*, J. Algebra **315** (2007), no. 2, 612–628. https://doi.org/10.1016/j.jalgebra.2007.01.042

[20] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761. https://doi.org/10.1081/AGB-120013179

[21] S. U. Hwang, N. K. Kim, and Y. Lee, *On rings whose right annihilators are bounded*, Glasg. Math. J. **51** (2009), no. 3, 539–559. https://doi.org/10.1017/S0017089509005163

[22] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. **3** (1996), no. 4, 289–300.

[23] T. Y. Lam, *A First Course in Noncommutative Rings*, second edition, Graduate Texts in Mathematics, 131, Springer, New York, 2001. https://doi.org/10.1007/978-1-4419-8616-0

[24] Z. Lei, J. Chen, and Z. Ying, *A question on McCoy rings*, Bull. Austral. Math. Soc. **76** (2007), no. 1, 137–141. https://doi.org/10.1017/S0004972700039526

[25] T. G. Lucas, *Two annihilator conditions*: *property* (A) *and* (A.C.), Comm. Algebra **14** (1986), no. 3, 557–580. https://doi.org/10.1080/00927878608823325

[26] G. Marks, *Duo rings and Ore extensions*, J. Algebra **280** (2004), no. 2, 463–471. https://doi.org/10.1016/j.jalgebra.2004.04.018

[27] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), 286–295. https://doi.org/10.2307/2303094

[28] P. P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141. https://doi.org/10.1016/j.jalgebra.2005.10.008

[29] W. M. Xue, *On weakly left duo rings*, Riv. Mat. Univ. Parma (4) **15** (1989), 211–217.

[30] H.-P. Yu, *On quasi-duo rings*, Glasgow Math. J. **37** (1995), no. 1, 21–31. https://doi.org/10.1017/S0017089500030342

[31] M. Zahiri, A. Moussavi, and R. Mohammadi, *On rings with annihilator condition*, Studia Sci. Math. Hungar. **54** (2017), no. 1, 82–96. https://doi.org/10.1556/012.2017.54.1.1355

[32] M. Zahiri, A. Moussavi, and R. Mohammadi, *On annihilator ideals in skew polynomial rings*, Bull. Iranian Math. Soc. **43** (2017), no. 5, 1017–1036.

[33] J. M. Zelmanowitz, *The finite intersection property on annihilator right ideals*, Proc. Amer. Math. Soc. **57** (1976), no. 2, 213–216. https://doi.org/10.2307/2041191

Rasul Mohammadi
Department of Pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
Email address: mohamadi.rasul@yahoo.com

Ahmad Moussavi
Department of Pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
Email address: moussavi.a@gmail.com, moussavi.a@modares.ac.ir

Masoome Zahiri
Department of Pure Mathematics
Faculty of Mathematical Sciences
Eqlid University
P.O.Box:7381943885, Eqlid, Iran
Email address: m.zahiri86@gmail.com