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ON NONNIL-EXACT SEQUENCES AND NONNIL-COMMUTATIVE DIAGRAMS

Wei Zhao and Dechuan Zhou

Abstract. In this paper, we investigate the nonnil-exact sequences and nonnil-commutative diagrams and show that they behave in a way similar to the classical ones in Abelian categories.

1. Introduction

Throughout this paper, it is assumed that all rings are commutative and associative with non-zero identity and all modules are unitary. If R is a ring, then $Nil(R)$ denotes the set of nilpotent elements of R, and $Z(R)$ denotes the set of zero-divisors of R. A ring with $Nil(R)$ being divided prime is called a ϕ-ring. In this paper, if the nilradical $Nil(R)$ of a ring R is prime, then R is called a PN-*ring*. If $Z(R) = Nil(R)$, then R is called a ZN-*ring*. We recommend [1–20] for the study of the ring-theoretic characterizations on ϕ-rings, and [21–28] for the study of the module-theoretic characterizations on ϕ-rings.

In order to extend the homological methods to commutative rings with the nilradical as a prime ideal, the authors in [26] introduced nonnil-exact sequences and nonnil-commutative diagrams. Since commutative diagrams play an important role in homological theory, we investigate nonnil-commutative diagrams in order to lay the foundation for future work on homological theories over PN-rings. We show that nonnil-commutative diagrams behave in a way similar to the classical ones. For example, Five Lemma, Snake Lemma and 3×3 Lemma show the same pattern in PN-rings. To this end, we introduce some necessary concepts and symbols.

Let R be a PN-ring and M an R-module. We set

$$NN(R) = \{I \mid I \text{ is a nonnil ideal of } R\},$$

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and

$$\mathrm{Ntor}(M) = \{x \in M \mid Ix = 0 \text{ for some } I \in NN(R)\}.$$

If $\mathrm{Ntor}(M) = M$, M is called a *nonnil-torsion R-module*, and if $\mathrm{Ntor}(M) = 0$, M is called a *nonnil-torsion-free R-module.* Let $f : A \to B$ be a homomorphism of R-modules. Set

$$\mathrm{NKer}(f) = \{a \in A \mid sf(a) = 0 \text{ for some } s \in R\backslash Nil(R)\},$$

$$\mathrm{NIm}(f) = \{b \in B \mid sb = sf(a) \text{ for some } a \in A \text{ and } s \in R\backslash Nil(R)\}.$$

Since $Nil(R)$ is a prime ideal, $\mathrm{NKer}(f)$ is a submodule of A and $\mathrm{NIm}(f)$ is a submodule of B. We set $\mathrm{NCoker}(f) = B/\mathrm{NIm}(f)$.

Recall from [26] that the submodule $\mathrm{NKer}(f)$ of A is called the *nonnil-kernel* of f and the submodule $\mathrm{NIm}(f)$ of B is called the *nonnil-image* of f. The homomorphism $f : A \to B$ is called a *nonnil-monomorphism* if $\mathrm{NKer}(f) = \mathrm{Ntor}(A)$ and it is called a *nonnil-epimorphism* if $\mathrm{NIm}(f) = B$. A sequence of R-modules and homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is called a *nonnil-complex* (resp., *a nonnil-exact sequence*) if $\mathrm{NIm}(f) \subseteq \mathrm{NKer}(g)$ (resp., $\mathrm{NIm}(f) = \mathrm{NKer}(g)$). Every R-module M has a *free nonnil-resolution*, that is, there exists a nonnil-exact sequence

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0,$$

where each F_i is free.

Let $f : A \to B$, $g : B \to D$, $h : A \to C$ and $k : C \to D$ be homomorphisms of R-modules. Then the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

is said to be *nonnil-commutative* if $\mathrm{NIm}(gf - kh) = \mathrm{Ntor}(D)$, that is, there exists $s_a \in R\backslash Nil(R)$ such that $s_a gf(a) = s_a kh(a)$ for any $a \in A$, where the choice of s_a depends on a.

An R-module homomorphism $f : A \to B$ is called a *nonnil-isomorphism* if there exists a homomorphism $g : B \to A$ such that $\mathrm{NIm}(\mathbf{1}_A - gf) = \mathrm{Ntor}(A)$ and $\mathrm{NIm}(\mathbf{1}_B - fg) = \mathrm{Ntor}(B)$, that is, the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathbf{1}_A}\downarrow & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle \mathbf{1}_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

is nonnil-commutative. If there exists a nonnil-isomorphism $f : A \to B$, we say that A, B are nonnil-isomorphic, denoted by $A \overset{N}{\simeq} B$. An R-module homomorphism $f : A \to B$ is called a *weakly nonnil-isomorphism* if $\mathrm{NKer}(f) = \mathrm{Ntor}(A)$

and $\mathrm{NIm}(f) = B$, in this case, we say that A is *weakly nonnil-isomorphic* to B by f, denoted by $f : A \overset{WN}{\rightrightarrows} B$.

In this paper, R always denotes a PN-ring.

2. Main results

Theorem 2.1 (Five Lemma in PN-rings)**.** *Consider the following nonnil-commutative diagram with nonnil-exact rows*:

$$\begin{array}{ccccccccc} D & \xrightarrow{h} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{k} & E \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mu} \\ D' & \xrightarrow{h'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{k'} & E' \end{array}$$

(a) *If α and γ are nonnil-monomorphisms and δ is a nonnil-epimorphism, then β is a nonnil-monomorphism.*
(b) *If α and γ are nonnil-epimorphisms and μ is a nonnil-monomorphism, then β is a nonnil-epimorphism.*
(c) *If δ is a nonnil-epimorphism, μ is a nonnil-monomorphism and α, γ are weakly nonnil-isomorphisms, then β is a weakly nonnil-isomorphism.*
(d) *If $\delta, \alpha, \gamma, \mu$ are weakly nonnil-isomorphisms, then β is a weakly nonnil-isomorphism.*

Proof. (a) Let $b \in \mathrm{NKer}(\beta)$. Then there is some $s \in R\backslash Nil(R)$ such that $s\beta(b) = 0$. Since the diagram is nonnil-commutative, we have that $s_1 s\gamma g(b) = s_1 s g'\beta(b) = 0$ for some $s_1 \in R\backslash Nil(R)$. Thus

$$g(b) \in \mathrm{NKer}(\gamma).$$

Since $\mathrm{NKer}(\gamma) = \mathrm{Ntor}(C)$, there is some $s_2 \in R\backslash Nil(R)$ such that $s_2 g(b) = 0$. Thus

$$b \in \mathrm{NKer}(g).$$

Since $\mathrm{NKer}(g) = \mathrm{NIm}(f)$, there are some $a \in A$ and $s_3 \in R\backslash Nil(R)$ such that $s_3 b = s_3 f(a)$. Hence $ss_3s_4 f'\alpha(a) = ss_3s_4\beta f(a) = ss_3s_4\beta(b) = 0$ for some $s_4 \in R\backslash Nil(R)$. Thus

$$\alpha(a) \in \mathrm{NKer}(f').$$

Since $\mathrm{NKer}(f') = \mathrm{NIm}(h')$, there are some $d' \in D'$ and $s_5 \in R\backslash Nil(R)$ such that $s_5\alpha(a) = s_5 h'(d')$. Since δ is a nonnil-epimorphism, there are some $d \in D$ and $s_6 \in R\backslash Nil(R)$ such that $s_6 d' = s_6\delta(d)$. So we have that $s_5s_6s_7\alpha(a) = s_5s_6s_7h'(d') = s_5s_6s_7h'\delta(d) = s_5s_6s_7\alpha h(d)$ for some $s_7 \in R\backslash Nil(R)$. Hence

$$a - h(d) \in \mathrm{NKer}(\alpha).$$

Thus there exists some $s_8 \in R\backslash Nil(R)$ such that $s_8 a = s_8 h(d)$. Since $s_9s_8s_3b = s_9s_8s_3f(a) = s_9s_8s_3fh(d) = 0$ for some $s_9 \in R\backslash Nil(R)$,

$$b \in \mathrm{Ntor}(B).$$

Therefore,

$$\mathrm{NKer}(\beta) = \mathrm{Ntor}(B),$$

which implies that β is a nonnil-monomorphism.

(b) Let $b' \in B'$. Since γ is a nonnil-epimorphism, there are some $c \in C$ and $s \in R\backslash Nil(R)$ such that

$$s\gamma(c) = sg'(b').$$

Nonnil-commutativity of the right square gives $s_1s_2s\mu k(c) = s_1s_2sk'\gamma(c) = s_1s_2sk'g'(b') = 0$ for some $s_1, s_2 \in R\backslash Nil(R)$. Since μ is a nonnil-monomorphism, $s_3k(c) = 0$ for some $s_3 \in R\backslash Nil(R)$. Because of the nonnil-exactness of the top row, there are some $b \in B$ and $s_4 \in R\backslash Nil(R)$ such that

$$s_4g(b) = s_4c.$$

Hence $ss_4s_5g'(b') = ss_4s_5\gamma(c) = ss_4s_5\gamma g(b) = ss_4s_5g'\beta(b)$ for some $s_5 \in R\backslash Nil(R)$. Hence

$$b' - \beta(b) \in \mathrm{NKer}(g').$$

Since $\mathrm{NKer}(g') = \mathrm{NIm}(f')$, $s_6(b' - \beta(b)) = s_6f'(a')$ for some $a' \in A'$, $s_6 \in R\backslash Nil(R)$ by the nonnil-exactness of the bottom row. Since α is a nonnil-epimorphism, there are some $a \in A$ and $s_7 \in R\backslash Nil(R)$ with $s_7\alpha(a) = s_7a'$. Hence $s_6s_7s_8(b' - \beta(b)) = s_6s_7s_8f'(a') = s_6s_7s_8f'\alpha(a) = s_6s_7s_8\beta f(a)$ for some $s_8 \in R\backslash Nil(R)$. Thus $s_6s_7s_8b' = s_6s_7s_8\beta(b + f(a))$. Therefore

$$b' \in \mathrm{NIm}(\beta),$$

which implies that β is a nonnil-epimorphism.

It is easy to see that (c) follows from (a) and (b), while (d) follows from (c). □

Theorem 2.2 (Snake Lemma in PN-rings)**.** *Consider the following nonnil-commutative diagram with nonnil-exact rows*:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

(a) *There is a nonnil-exact sequence*

$$\mathrm{NKer}(\alpha) \to \mathrm{NKer}(\beta) \to \mathrm{NKer}(\gamma) \to \mathrm{NCoker}(\alpha) \to \mathrm{NCoker}(\beta) \to \mathrm{NCoker}(\gamma).$$

(b) *If f is a nonnil-monomorphism, then*

$$0 \to \mathrm{NKer}(\alpha) \to \mathrm{NKer}(\beta) \to \mathrm{NKer}(\gamma) \to \mathrm{NCoker}(\alpha) \to \mathrm{NCoker}(\beta) \to \mathrm{NCoker}(\gamma)$$

is nonnil-exact.

(c) *If g' is a nonnil-epimorphism, then*

$$\mathrm{NKer}(\alpha) \to \mathrm{NKer}(\beta) \to \mathrm{NKer}(\gamma) \to \mathrm{NCoker}(\alpha) \to \mathrm{NCoker}(\beta) \to \mathrm{NCoker}(\gamma) \to 0$$

is nonnil-exact.

Proof. (a) The proof is completed by the following three steps.

(1) Let $a \in \mathrm{NKer}(\alpha)$. Then there exists $s \in R\backslash Nil(R)$ such that $s\alpha(a) = 0$. Because of nonnil-commutativity, there exists some $s_1 \in R\backslash Nil(R)$ such that $ss_1\beta f(a) = ss_1 f'\alpha(a) = 0$. Hence $f(a) \in \mathrm{NKer}(\beta)$. Define $f_1 : \mathrm{NKer}(\alpha) \to \mathrm{NKer}(\beta)$ by

$$f_1(a) = f(a).$$

Then f_1 is well-defined. Similarly, we get a homomorphism $g_1 : \mathrm{NKer}(\beta) \to \mathrm{NKer}(\gamma)$ by $g_1(b) = g(b)$ for $b \in B$.

For $a' \in A'$, define

$$f_2(a' + \mathrm{NIm}(\alpha)) = f'(a') + \mathrm{NIm}(\beta).$$

It is easy to check that $f_2 : A'/\mathrm{NIm}(\alpha) \to B'/\mathrm{NIm}(\beta)$ is well-defined. In fact, if $a' \in \mathrm{NIm}(\alpha)$, then $sa' = s\alpha(a)$ for some $a \in A$ and $s \in R\backslash Nil(R)$. So there exists $s_1 \in R\backslash Nil(R)$ such that $ss_1 f'(a') = ss_1 f'\alpha(a) = ss_1\beta f(a)$. Thus $f'(a') \in \mathrm{NIm}(\beta)$. Similarly, we get a homomorphism $g_2 : B'/\mathrm{NIm}(\beta) \to C'/\mathrm{NIm}(\gamma)$ by $g_2(b' + \mathrm{NIm}(\beta)) = g'(b') + \mathrm{NIm}(\gamma)$ for $b' \in B'$.

Let $c \in \mathrm{NKer}(\gamma)$. Since g is a nonnil-epimorphism, there exist $b \in B$ and $s, s_1 \in R\backslash Nil(R)$ such that $s_1\gamma(c) = 0$ and $sg(b) = sc$. Thus $ss_1s_2g'\beta(b) = ss_1s_2\gamma g(b) = ss_1s_2\gamma(c) = 0$ for some $s_2 \in R\backslash Nil(R)$. Hence $\beta(b) \in \mathrm{NKer}(g') = \mathrm{NIm}(f')$. In this sense, there exist some $a' \in A'$ and $s_3 \in R\backslash Nil(R)$ such that $s_3 f'(a') = s_3\beta(b)$. Define

$$\delta(c) = a' + \mathrm{NIm}(\alpha).$$

If $sg(b) = 0$, then there are some $a \in A$ and $s_4 \in R\backslash Nil(R)$ such that $s_4 f(a) = s_4 b$ by the fact that $\mathrm{NIm}(f) = \mathrm{NKer}(g)$. Since $s_4s_5\beta(b) = s_4s_5\beta f(a) = s_4s_5 f'\alpha(a)$ for some $s_5 \in R\backslash Nil(R)$, we have $s_6 a' = s_6\alpha(a)$ for some $s_6 \in R\backslash Nil(R)$ and $a' \in \mathrm{NIm}(\alpha)$. Therefore, δ is well-defined.

(2) Next, we show that the following sequence is a nonnil-complex of R-modules and homomorphisms:

$$\mathrm{NKer}(\alpha) \xrightarrow{f_1} \mathrm{NKer}(\beta) \xrightarrow{g_1} \mathrm{NKer}(\gamma) \xrightarrow{\delta} \mathrm{NCoker}(\alpha) \xrightarrow{f_2} \mathrm{NCoker}(\beta) \xrightarrow{g_2} \mathrm{NCoker}(\gamma).$$

For any $a \in \mathrm{NKer}(\alpha)$, we have that $s_1 g_1 f_1(a) = s_1 g f(a) = 0$ for some $s_1 \in R\backslash Nil(R)$. Similarly, $s_2 g_2 f_2(a' + \mathrm{NIm}(\alpha)) = s_2 g' f'(a') + \mathrm{NIm}(\gamma) = 0$ for some $s_2 \in R\backslash Nil(R)$.

Let $b \in \mathrm{NKer}(\beta)$. Then there exists some $s \in R\backslash Nil(R)$ such that $s\beta(b) = 0$. Hence $\beta(b) \in \mathrm{NKer}(g') = \mathrm{NIm}(f')$. Thus there are some $a' \in A'$ and $s_3 \in R\backslash Nil(R)$ such that $s_3 f'(a') = s_3\beta(b) = 0$. We have that $a' \in \mathrm{NKer}(f') = \mathrm{Ntor}(A')$. So

$$\delta g_1(b) = a' + \mathrm{NIm}(\alpha) = 0.$$

Let $c \in \mathrm{NKer}(\gamma)$. Then there exists some $s \in R\backslash Nil(R)$ such that $s\gamma(c) = 0$ and $sc = sg(b)$. Because there exist some $s_1, s_2 \in R\backslash Nil(R)$ such that $s_1s_2g'\beta(b) = s_1s_2\gamma g(b) = s_1s_2\gamma(c) = 0$, $\beta(b) \in \mathrm{NKer}(g') = \mathrm{NIm}(f')$. Hence we have that $s_3\beta(b) = s_3 f'(a')$, where $s_3 \in R\backslash Nil(R)$. Thus

$$f_2\delta(c) = f_2(a' + \mathrm{NIm}(\alpha)) = f'(a') + \mathrm{NIm}(\beta) = 0.$$

(3) Notice we have that

$$\begin{aligned}\mathrm{NKer}(g_1) &= \{b \in \mathrm{NKer}(\beta) \mid sg_1(b) = sg(b) = 0, s \in R\backslash Nil(R)\}\\ &= \mathrm{NKer}(\beta) \cap \mathrm{NKer}(g)\end{aligned}$$

and

$$\mathrm{NIm}(f_1) = \{b \in \mathrm{NKer}(\beta) \mid sb = sf_1(a) = sf(a), a \in \mathrm{NKer}(\alpha), s \in R\backslash Nil(R)\}.$$

In verifying that $\mathrm{NKer}(g_1) = \mathrm{NIm}(f_1)$, we show only that $\mathrm{NKer}(g_1) \subseteq \mathrm{NIm}(f_1)$.

Let $b \in \mathrm{NKer}(g_1)$. Then $b \in \mathrm{NKer}(\beta)$ and $b \in \mathrm{NKer}(g) = \mathrm{NIm}(f)$. Thus there exist some $s \in R\backslash Nil(R)$ and $a \in \mathrm{NKer}(\alpha)$ such that $sb = sf(a)$. Hence $ss_1 f'\alpha(a) = ss_1\beta f(a) = ss_1\beta(b)$ for some $s_1 \in R\backslash Nil(R)$. So $ss_1s_2 f'\alpha(a) = ss_1s_2\beta(b) = 0$ for some $s_2 \in R\backslash Nil(R)$. Thus

$$\alpha(a) \in \mathrm{NKer}(f') = \mathrm{Ntor}(A').$$

Since there exists some $s_3 \in R\backslash Nil(R)$ such that $s_3\alpha(a) = 0$. In this sense, $b \in \mathrm{NIm}(f_1)$. Therefore $\mathrm{NIm}(f_1) = \mathrm{NKer}(g_1)$.

Notice that

$$\mathrm{NIm}(g_1) = \{c \in \mathrm{NKer}(\gamma) \mid sc = sg_1(b) = sg(b), b \in \mathrm{NKer}(\beta), s \in R\backslash Nil(R)\}$$

and

$$\begin{aligned}\mathrm{NKer}(\delta) = \{c \in \mathrm{NKer}(\gamma) \mid sc = sg(b), s_1\beta(b) = s_1 f'(a'),\\ a' \in \mathrm{NIm}(\alpha), s_1 \in R\backslash Nil(R)\}.\end{aligned}$$

Let $c \in \mathrm{NKer}(\delta)$. Then $c \in \mathrm{NKer}(\gamma)$ and there exist some $a' \in \mathrm{NIm}(\alpha)$ and $s_1 \in R\backslash Nil(R)$ such that $sc = sg(b_1)$, $s_1\beta(b_1) = s_1 f'(a')$. So there exists some $s_2 \in R\backslash Nil(R)$ such that $s_2a' = s_2\alpha(a)$. Thus we have that $s_1s_2\beta(b_1) = s_1s_2 f'(a') = s_1s_2 f'\alpha(a) = s_1s_2\beta f(a)$. Hence $b_1 - f(a) \in \mathrm{NKer}(\beta)$. Set

$$b = b_1 - f(a).$$

We have that $b \in \mathrm{NKer}(\beta)$ and $sc = sg_1(b) = sg(b)$. Thus $c \in \mathrm{NIm}(g_1)$, which implies that $\mathrm{NIm}(g_1) = \mathrm{NKer}(\delta)$.

Notice that

$$\begin{aligned}\mathrm{NKer}(f_2) &= \{a' + \mathrm{NIm}(\alpha) \mid sf_2(a' + \mathrm{NIm}(\alpha)) = 0, a' \in A', s \in R\backslash Nil(R)\}\\ &= \{a' + \mathrm{NIm}(\alpha) \mid sf'(a') \in \mathrm{NIm}(\beta), a' \in A', s \in R\backslash Nil(R)\}\\ &= \{a' + \mathrm{NIm}(\alpha) \mid s_1sf'(a') = s_1\beta(b), a' \in A', b \in B, s, s_1 \in R\backslash Nil(R)\}.\end{aligned}$$

If $s_1sf'(a') = s_1\beta(b)$, then $s's_1sg'\beta(b) = s's_1sg'f'(a') = 0$ for some $s' \in R\backslash Nil(R)$. So we have that $\beta(b) \in \mathrm{NKer}(g') = \mathrm{NIm}(f')$. Thus $s\beta(b) = sf'(a'_1)$ for some $s \in R\backslash Nil(R)$ and $a'_1 \in A'$. Therefore, we have

$$\mathrm{NKer}(f_2) = \{a' + \mathrm{NIm}(\alpha) \mid s\beta(b) = sf'(a'), b \in B, a' \in A', s \in R\backslash Nil(R)\}.$$

Set $c = g(b)$. We have $s\gamma(c) = s\gamma g(b) = sg'f'(a') = 0$. Hence $c \in \mathrm{NKer}(\gamma)$. Therefore $\mathrm{NKer}(f_2) = \mathrm{NIm}(\delta)$.

Similarly,

$$\mathrm{NKer}(g_2) = \{b' + \mathrm{NIm}(\beta) \mid s\gamma(c) = sg'(b'), c \in C, b' \in B', s \in R\backslash Nil(R)\}.$$

Notice that

$$\begin{aligned}\mathrm{NIm}(f_2) = \{b' + \mathrm{NIm}(\beta) \mid sb' = sf'(a') + s_1\beta(b),\\ a' \in A', b \in B, b' \in B', s, s_1 \in R\backslash Nil(R)\}.\end{aligned}$$

If $\overline{b'} \in \mathrm{NKer}(g_2)$, then there exist $c \in C$, $b' \in B'$ and $s \in R\backslash Nil(R)$ such that $s\gamma(c) = sg'(b')$. Because $c \in C = \mathrm{NIm}(g)$, we have that $s_1 c = s_1 g(b)$ for some $b \in B$, $s_1 \in R\backslash Nil(R)$. Thus

$$s_1 s g'(b') = s_1 s\gamma(c) = s_1 s\gamma g(b) = s_1 s g'\beta(b).$$

So

$$b' - \beta(b) \in \mathrm{NKer}(g') = \mathrm{NIm}(f').$$

Hence $sb' - s\beta(b) = sf'(a')$ for some $a' \in A'$, $s \in R\backslash Nil(R)$. We have $\overline{b'} \in \mathrm{NIm}(f_2)$, which implies that $\mathrm{NIm}(f_2) = \mathrm{NKer}(g_2)$.

(b) Consider that $\mathrm{NKer}(f_1) = \mathrm{NKer}(\alpha) \cap \mathrm{NKer}(f) = \mathrm{Ntor}(A)$.

(c) Consider that $\mathrm{NIm}(g_2) \supseteq (\mathrm{NIm}(g') + \mathrm{NIm}(\gamma))/\mathrm{NIm}(\gamma) = C'/\mathrm{NIm}(\gamma)$. □

Corollary 2.3. *Consider the following nonnil-commutative diagram with nonnil-exact rows*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0. \end{array}$$

(a) *If α is a nonnil-epimorphism, then the sequence*

$$0 \to \mathrm{NKer}(\alpha) \to \mathrm{NKer}(\beta) \to \mathrm{NKer}(\gamma) \to 0$$

is nonnil-exact.

(b) *If γ is a nonnil-monomorphism, then the sequence*

$$0 \to \mathrm{NCoker}(\alpha) \to \mathrm{NCoker}(\beta) \to \mathrm{NCoker}(\gamma) \to 0$$

is nonnil-exact.

Proof. (a) It holds by Theorem 2.2.

(b) Suppose that γ is a nonnil-monomorphism. For $a' + \mathrm{NIm}(\alpha) \in \mathrm{NKer}(f_2)$, $a' \in A'$, there exists $s_1 \in R\backslash Nil(R)$ such that $s_1 f'(a') = s_1\beta(b)$, where $b \in B$. Since the diagram is nonnil-commutative, we have some $s_2 \in R\backslash Nil(R)$ such that $s_2\gamma g(b) = s_2 g'\beta(b) = s_2 g'f'(a') = 0$. Thus

$$g(b) \in \mathrm{NKer}(\gamma) = \mathrm{Ntor}(C).$$

Hence $s_3 g(b) = 0$ for some $s_3 \in R\backslash Nil(R)$. So $b \in \mathrm{NKer}(g) = \mathrm{NIm}(f)$, we have some $s_4 \in R\backslash Nil(R)$, $a \in A$ such that $s_4 b = s_4 f(a)$. It is clear that $s_1 s_4 f'(a') = s_1 s_4 \beta(b) = s_1 s_4 \beta f(a) = s_1 s_4 f' \alpha(a)$. So we have that

$$a' - \alpha(a) \in \mathrm{NKer}(f') = \mathrm{Ntor}(A').$$

Hence there exists some $s_5 \in R\backslash Nil(R)$ such that $s_5 a' = s_5 \alpha(a)$ and $a' \in \mathrm{NIm}(\alpha)$. Therefore $a' + \mathrm{NIm}(\alpha) = 0$, which implies that $0 \to \mathrm{NCoker}(\alpha) \to \mathrm{NCoker}(\beta) \to \mathrm{NCoker}(\gamma) \to 0$ is nonnil-exact. □

Lemma 2.4. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a nonnil-exact sequence of R-modules. If A, C are nonnil-torsion, then B is also nonnil-torsion.*

Proof. For any $b \in B$, since $g(b) \in C$ is nonnil-torsion, $sg(b) = 0$ for some $s \in R\backslash Nil(R)$. Hence $b \in \mathrm{NKer}(g) = \mathrm{NIm}(f)$. So there are some $a \in A$ and $s_1 \in R\backslash Nil(R)$ such that

$$s_1 b = s_1 f(a).$$

Since $a \in A$ is nonnil-torsion, there is some $s_2 \in R\backslash Nil(R)$ such that $s_2 a = 0$. Thus

$$s_1 s_2 b = s_1 s_2 f(a) = 0.$$

Therefore b is nonnil-torsion, which implies that B is a nonnil-torsion R-module. □

Corollary 2.5. *Consider the following nonnil-commutative diagram with nonnil-exact rows*:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

(a) *If α and γ are nonnil-monomorphisms, then β is a nonnil-monomorphism.*
(b) *If α and γ are nonnil-epimorphisms, then β is a nonnil-epimorphism.*
(c) *If α and γ are weakly nonnil-isomorphisms, then β is a weakly nonnil-isomorphism.*

Proof. (a) If α and γ are nonnil-monomorphisms, then $\mathrm{NKer}(\alpha)$ and $\mathrm{NKer}(\gamma)$ are nonnil-torsion. So $\mathrm{NKer}(\beta)$ is also nonnil-torsion by Theorem 2.2 and Lemma 2.4. Thus β is a nonnil-monomorphism.

(b) If α and γ are nonnil-epimorphisms, then $\mathrm{NCoker}(\alpha) = \mathrm{NCoker}(\gamma) = 0$. Hence $A' = \mathrm{NIm}(\alpha)$ and $C' = \mathrm{NIm}(\gamma)$. For any $b' \in B'$, we have that

$$g_2(\overline{b'}) = \overline{g'(b')} = \overline{0}.$$

Thus $g'(b') \in \mathrm{NIm}(\gamma)$. Hence $sg'(b') = s\gamma(c)$ for some $c \in C$, $s \in R\backslash Nil(R)$. Suppose that $s_1 c = s_1 g(b)$, $b \in B$, $s_1 \in R\backslash Nil(R)$. We have $ss_1 s_2 g'(b') = ss_1 s_2 \gamma g(b) = ss_1 s_2 g' \beta(b)$ for some $s_2 \in R\backslash Nil(R)$. Hence

$$b' - \beta(b) \in \mathrm{NKer}(g') = \mathrm{NIm}(f').$$

In this sense, there are some $a' \in A'$ and $s_3 \in R\backslash Nil(R)$ such that

$$s_3 b' - s_3\beta(b) = s_3 f'(a').$$

Because $a' \in A' = \mathrm{NIm}(\alpha)$, we have that $s_4 a' = s_4\alpha(a)$ for some $a \in A$, $s_4 \in R\backslash Nil(R)$. Hence

$$s_3 s_4 b' = s_3 s_4 \beta(b) + s_3 s_4 \beta f(a).$$

This is to say that $b' \in \mathrm{NIm}(\beta)$. Therefore $\mathrm{NCoker}(\beta) = B'/\mathrm{NIm}(\beta) = 0$, which implies that β is a nonnil-epimorphism.

(c) follows from (a) and (b). □

Theorem 2.6 (3 × 3 Lemma in PN-rings)**.** *Consider the following nonnil-commutative diagram*:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\beta_1} & & \downarrow{\scriptstyle\gamma_1} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all columns (*resp., rows*) *are nonnil-exact.*

(a) *If the first and the second rows* (*resp., columns*) *are nonnil-exact, then the third row* (*resp., column*) *is nonnil-exact.*
(b) *If the second and the third rows* (*resp., columns*) *are nonnil-exact, then the first row* (*resp., column*) *is nonnil-exact.*

Proof. (a) Suppose that the first and the second rows (resp., columns) are nonnil-exact. For any $a_2 \in \mathrm{NKer}(f_2)$, there is some $s_1 \in R\backslash Nil(R)$ such that $s_1 f_2(a_2) = 0$. Since $a_2 \in A_2 = \mathrm{NIm}(\alpha)$, there are some $a \in A$ and $s_2 \in R\backslash Nil(R)$ such that $s_2 a_2 = s_2\alpha(a)$. Thus we have some $s_3 \in R\backslash Nil(R)$ such that $s_1 s_2 s_3 \beta f(a) = s_1 s_2 s_3 f_2 \alpha(a) = s_1 s_2 s_3 f_2(a_2) = 0$ by the nonnil-commutativity. Hence

$$f(a) \in \mathrm{NKer}(\beta).$$

Since $\mathrm{NKer}(\beta) = \mathrm{NIm}(\beta_1)$, there are some $b_1 \in B_1$ and $s_4 \in R\backslash Nil(R)$ such that $s_4 f(a) = s_4\beta_1(b_1)$. Because the diagram is nonnil-commutative, we have some $s_6 \in R\backslash Nil(R)$ such that $s_6\gamma_1 g_1(b_1) = s_6 g\beta_1(b_1)$. Hence $s_4 s_5 s_6 \gamma_1 g_1(b_1) = s_4 s_5 s_6 g\beta_1(b_1) = s_4 s_5 s_6 g f(a) = 0$ for some $s_5 \in R\backslash Nil(R)$. Thus

$$g_1(b_1) \in \mathrm{NKer}(\gamma_1).$$

Since $\mathrm{NKer}(\gamma_1) = \mathrm{Ntor}(C_1)$, there is some $s_7 \in R\backslash Nil(R)$ such that $s_7 g_1(b_1) = 0$. Hence

$$b_1 \in \mathrm{NKer}(g_1).$$

Since $\mathrm{NKer}(g_1) = \mathrm{NIm}(f_1)$, there are some $a_1 \in A_1$ and $s_8 \in R\backslash Nil(R)$ such that $s_8 b_1 = s_8 f_1(a_1)$. Hence $s_4 s_8 s_9 f(a) = s_4 s_8 s_9 \beta_1 f_1(a_1) = s_4 s_8 s_9 f \alpha_1(a_1)$ for some $s_9 \in R\backslash Nil(R)$. Thus we have

$$a - \alpha_1(a_1) \in \mathrm{NKer}(f).$$

Since $\mathrm{NKer}(f) = \mathrm{Ntor}(A)$, $s_{10}(a - \alpha_1(a_1)) = 0$ for some $s_{10} \in R\backslash Nil(R)$. Thus we have that $s_{10} a = s_{10} \alpha_1(a_1)$. Hence $s_{11} s_2 s_{10} a_2 = s_{11} s_2 s_{10} \alpha \alpha_1(a_1) = 0$ for some $s_{11} \in R\backslash Nil(R)$. Therefore $a_2 \in \mathrm{Ntor}(A_2)$, which implies that

$$\mathrm{NKer}(f_2) = \mathrm{Ntor}(A_2).$$

For any $b_2 \in \mathrm{NKer}(g_2)$, there is some $s_1 \in R\backslash Nil(R)$ such that $s_1 g_2(b_2) = 0$. Since $b_2 \in B_2 = \mathrm{NIm}(\beta)$, there are some $b \in B$ and $s_2 \in R\backslash Nil(R)$ such that $s_2 b_2 = s_2 \beta(b)$. By the nonnil-commutativity, there is some $s_3 \in R\backslash Nil(R)$ such that $s_1 s_2 s_3 \gamma g(b) = s_1 s_2 s_3 g_2 \beta(b) = s_1 s_2 s_3 g_2(b_2) = 0$. Hence

$$g(b) \in \mathrm{NKer}(\gamma).$$

Since $\mathrm{NKer}(\gamma) = \mathrm{NIm}(\gamma_1)$, there are some $c_1 \in C_1$ and $s_4 \in R\backslash Nil(R)$ such that $s_4 g(b) = s_4 \gamma_1(c_1)$. Since $c_1 \in C_1 = \mathrm{NIm}(g_1)$, there are some $b_1 \in B_1$ and $s_5 \in R\backslash Nil(R)$ such that $s_5 c_1 = s_5 g_1(b_1)$. Thus $s_4 s_5 s_6 g(b) = s_4 s_5 s_6 \gamma_1 g_1(b_1) = s_4 s_5 s_6 g \beta_1(b_1)$ for some $s_6 \in R\backslash Nil(R)$. So

$$b - \beta_1(b_1) \in \mathrm{NKer}(g).$$

Since $\mathrm{NKer}(g) = \mathrm{NIm}(f)$, there are some $a \in A$ and $s_7 \in R\backslash Nil(R)$ such that $s_7(b - \beta_1(b_1)) = s_7 f(a)$. Hence $s_7 \beta(b - \beta_1(b_1)) = s_7 \beta f(a)$. So $s_7 s_8 \beta(b) = s_7 s_8 \beta f(a) = s_7 s_8 f_2 \alpha(a)$ for some $s_8 \in R\backslash Nil(R)$. Thus we have that $s_2 s_7 b_2 = s_2 s_7 \beta(b) = s_2 s_7 f_2 \alpha(a)$. Therefore $b_2 \in \mathrm{NIm}(f_2)$, which implies that

$$\mathrm{NKer}(g_2) \subseteq \mathrm{NIm}(f_2).$$

Conversely, for any $a_2 \in A_2 = \mathrm{NIm}(\alpha)$, there are some $a \in A$ and $s \in R\backslash Nil(R)$ such that $s a_2 = s\alpha(a)$. Hence

$$s' s g_2 f_2(a_2) = s' s g_2 f_2(\alpha(a)) = s' s g_2 \beta f(a) = s' s \gamma g f(a) = 0$$

for some $s' \in R\backslash Nil(R)$. Thus $f_2(a_2) \in \mathrm{NKer}(g_2)$, which implies that $\mathrm{Im}(f_2) \subseteq \mathrm{NKer}(g_2)$. We have that $\mathrm{NIm}(f_2) = \mathrm{Im}(f_2) + \mathrm{Ntor}(B_2) \subseteq \mathrm{NKer}(g_2)$. Therefore

$$\mathrm{NKer}(g_2) = \mathrm{NIm}(f_2).$$

For any $c_2 \in C_2 = \mathrm{NIm}(\gamma)$, there are some $c \in C$ and $s \in R\backslash Nil(R)$ such that $s c_2 = s\gamma(c)$. Since $c \in C = \mathrm{NIm}(g)$, we have that $s_1 c = s_1 g(b)$ for some $b \in B$, $s_1 \in R\backslash Nil(R)$. Thus

$$s s_1 c_2 = s s_1 \gamma g(b) = s s_1 g_2 \beta(b),$$

that is, $c_2 \in \mathrm{NIm}(g_2)$. So

$$\mathrm{NIm}(g_2) = C_2.$$

Therefore, the third row is also nonnil-exact.

(b) Similar to (a), we can show (b) by diagram-chases. □

Using known nonnil-commutative diagrams, we can transfer related properties between modules. However, a given diagram is often not nonnil-commutative, and so we have to adapt the nonnil-commutative diagram to complete the job. An R-module M is said to be NN-*torsion-free* if it is nonnil-isomorphic to some nonnil-torsion-free module.

Theorem 2.7. *Consider the following nonnil-commutative diagram of R-modules and homomorphisms with nonnil-exact rows*:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

where the square on the left is nonnil-commutative and C' is NN-*torsion-free. Then there is $\gamma : C \to C'$ such that the square on the right is nonnil-commutative.*

Proof. Suppose that $C' \overset{N}{\simeq} D$, D is a nonnil-torsion-free module and the following diagram

$$\begin{array}{ccc} C' & \xrightarrow{\mu} & D \\ {\scriptstyle\mathbf{1}_{C'}}\downarrow & \overset{\nu}{\swarrow} & \downarrow{\scriptstyle\mathbf{1}_D} \\ C' & \xrightarrow[\mu]{} & D \end{array}$$

is nonnil-commutative. Let $c \in C$. Since g is a nonnil-epimorphism, there are $b \in B$ and $s \in R\backslash Nil(R)$ such that $sg(b) = sc$. Define

$$\gamma(c) = \nu\mu g'\beta(b).$$

If $sg(b) = sc = 0$, then there exist $a \in A$ and $s_1 \in R\backslash Nil(R)$ such that $s_1 f(a) = s_1 b$. Thus

$$s_2 s_1 \mu g'\beta(b) = s_2 s_1 \mu g'\beta f(a) = s_2 s_1 \mu g' f'\alpha(a) = 0$$

for some $s_2 \in R\backslash Nil(R)$. Since D is a nonnil-torsion-free module, $\mu g'\beta(b) = 0$. Hence $\gamma(c) = \nu\mu g'\beta(b) = 0$. Then γ is a well-defined homomorphism. It is easy to see that the square on the right is nonnil-commutative. □

If the sequence $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ is only a nonnil-complex, then the result in Theorem 2.7 also holds.

Theorem 2.8. *Consider the following nonnil-commutative diagram of R-modules and homomorphisms with nonnil-exact rows*:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

where the square on the right is nonnil-commutative and A' is NN-*torsion-free. Then there is $\alpha : A \to A'$ such that the square on the left is nonnil-commutative.*

Proof. Suppose that $A' \overset{N}{\simeq} L$, L is a nonnil-torsion-free module and the following diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & L \\ {\scriptstyle \mathbf{1}_{A'}}\downarrow & \swarrow{\scriptstyle\nu} & \downarrow{\scriptstyle \mathbf{1}_L} \\ A' & \xrightarrow[\mu]{} & L \end{array}$$

is nonnil-commutative. Let $a \in A$. Then $s_1 g'\beta f(a) = s_1\gamma g f(a) = 0$ for some $s_1 \in R\backslash Nil(R)$. Thus there are $a' \in A'$ and $s_2 \in R\backslash Nil(R)$ such that $s_2 f'(a') = s_2\beta f(a)$. Define

$$\alpha(a) = \nu\mu(a').$$

Since L is a nonnil-torsion-free module, we have $\alpha(a) = \nu\mu(a') = 0$ if $a = 0$. Therefore, α is a well-defined homomorphism. It is easy to see that the square on the left is nonnil-commutative. □

If the sequence $A \overset{f}{\to} B \overset{g}{\to} C$ is only a nonnil-complex, then the result in Theorem 2.8 also holds.

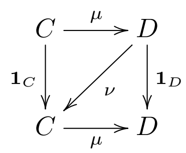
Theorem 2.9. *The following diagram with a nonnil-exact row*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & & & & & \\ & & E & & & & & & \end{array}$$

where C is NN-*torsion-free, can be completed to the following nonnil-commutative diagram with nonnil-exact rows*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle\beta} & & \Vert & & \\ 0 & \longrightarrow & E & \xrightarrow{\alpha} & M & \xrightarrow{\gamma} & C & \longrightarrow & 0 \end{array}$$

Proof. Suppose that $C \overset{N}{\simeq} D$, D is a nonnil-torsion-free module and the following diagram



is nonnil-commutative. Define $N = \{\,(b, -e) \mid sb = sf(a),\, se = sh(a),\, a \in A,\, s \in R\backslash Nil(R)\}$ and $M = (B \oplus E)/N$. For $b \in B$ and $e \in E$, define

$$\alpha(e) = (0, e) + N, \qquad \beta(b) = (b, 0) + N.$$

Then $\alpha h = \beta f$. Therefore, the square on the left is commutative and thus it is nonnil-commutative.

Define $\gamma : M \to C$ by $\gamma((b, e) + N) = \nu\mu g(b)$. If $(b, e) \in N$, then there are $a \in A$ and $s_1 \in R\backslash Nil(R)$ such that $s_1 f(a) = s_1 b$ and $s_1 h(a) = -s_1 e$. So $s_1 s_2 g(b) = s_1 s_2 g f(a) = 0$ for some $s_2 \in R\backslash Nil(R)$. Since D is a nonnil-torsion-free module, we have $\mu g(b) = 0$. Thus

$$\gamma((b, e) + N) = \nu\mu g(b) = 0.$$

Therefore, γ is a well-defined map.

For any $b \in B$, we have $g(b) \in C$ and there exists an element $s_3 \in R\backslash Nil(R)$ such that $s_3 \nu\mu g(b) = s_3 g(b)$. So

$$s_3 \gamma\beta(b) = s_3 \nu\mu g(b) = s_3 g(b).$$

Therefore, the square on the right is nonnil-commutative.

Since g is a nonnil-epimorphism, γ is also a nonnil-epimorphism.

If $\alpha(e) = 0$, then $(0, e) \in N$. So there exist $a \in A$ and $s_4 \in R\backslash Nil(R)$ such that $s_4 f(a) = 0$ and $s_4 h(a) = -s_4 e$. Since f is a nonnil-monomorphism, $s_5 a = 0$ for some $s_5 \in R\backslash Nil(R)$. Thus $-s_5 s_4 e = s_5 s_4 h(a) = 0$ and hence $e \in \mathrm{Ntor}(E)$. Therefore, α is a nonnil-monomorphism.

It is easy to see that $\gamma\alpha = 0$. Thus $\mathrm{NIm}(\alpha) \subseteq \mathrm{NKer}(\gamma)$.

If $s_6 \gamma((b, e) + N) = s_6 \nu\mu g(b) = s_6 g(b) = 0$ for some $s_6 \in R\backslash Nil(R)$, then there are $a \in A$ and $s_7 \in R\backslash Nil(R)$ such that $s_7 f(a) = s_7 b$. So

$$\begin{aligned} s_7(b, e) + N &= s_7(f(a), e) + N \\ &= s_7[(f(a), -h(a)) + (0, e + h(a))] + N \\ &= s_7(0, e + h(a)) + N \\ &= s_7 \alpha(e + h(a)). \end{aligned}$$

Therefore, $\mathrm{NKer}(\gamma) = \mathrm{NIm}(\alpha)$, that is, the bottom row is nonnil-exact. □

Theorem 2.10. *The following diagram with a nonnil-exact row*:

$$\begin{array}{ccccccccc} & & & & & & E & & \\ & & & & & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

can be completed to the following nonnil-commutative diagram with nonnil-exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & L & \xrightarrow{\gamma} & E & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Proof. Set

$$L = \{ (b, e) \in B \oplus E \mid sg(b) = sh(e),\ s \in R\backslash Nil(R)\}$$

and define

$$\beta(b, e) = b,\ \gamma(b, e) = e,\ b \in B,\ e \in E.$$

Then trivially, for $(b, e) \in B \oplus E$, there exists $s \in R\backslash Nil(R)$ such that $sg\beta(b, e) = sh\gamma(b, e)$. Define $\alpha : A \to L$ by $\alpha(a) = (f(a), 0)$. Then $\beta\alpha = f$. Since f is a nonnil-monomorphism, α is a nonnil-monomorphism.

For $e \in E$, since g is a nonnil-epimorphism, we choose $b \in B$ and $s_1 \in R\backslash Nil(R)$ such that $s_1 g(b) = s_1 h(e)$. Then $(b, e) \in L$ and $\gamma(b, e) = e$. Therefore, γ is an epimorphism.

For $a \in A$, we have $\gamma\alpha(a) = \gamma(f(a), 0) = \gamma(0) = 0$. And for $(b, e) \in \mathrm{NKer}(\gamma)$, we have some $s_2 \in R\backslash Nil(R)$ such that $s_2\gamma(b, e) = s_2 e = 0$. There exists some $s_3 \in R\backslash Nil(R)$ such that $s_2 s_3 g(b) = s_2 s_3 h(e) = 0$. Thus

$$b \in \mathrm{NKer}(g).$$

Since $\mathrm{NKer}(g) = \mathrm{NIm}(f)$, there exist $a \in A$ and $s_4 \in R\backslash Nil(R)$ such that $s_4 b = s_4 f(a)$. So we have

$$s_2 s_4(b, e) = s_2 s_4(f(a), e) = s_2 s_4(f(a), 0) = s_2 s_4\alpha(a).$$

Therefore, the top row is nonnil-exact. □

Theorem 2.11. *Consider the following diagram* Γ *of modules and homomorphisms*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

and the sequence Δ *of modules and homomorphisms*

$$0 \to A \xrightarrow{\sigma} B \oplus C \xrightarrow{\rho} D \to 0,$$

where $\sigma(a) = (\alpha(a), -\beta(a))$ *and* $\rho(b, c) = f(b) + g(c)$ *for* $a \in A$, $b \in B$, $c \in C$.

(a) *The homomorphism* ρ *is a nonnil-epimorphism if and only if*

$$D = \mathrm{NIm}(f) + \mathrm{NIm}(g).$$

(b) *The homomorphism* σ *is a nonnil-monomorphism if and only if*

$$\mathrm{NKer}(\alpha) \cap \mathrm{NKer}(\beta) = \mathrm{Ntor}(A).$$

(c) *The sequence* Δ *is a nonnil-complex if and only if the diagram* Γ *is nonnil-commutative.*

(d) $\mathrm{NKer}(\rho) \subseteq \mathrm{NIm}(\sigma)$ *if and only if there exist* $a \in A$ *and* $s \in R\backslash Nil(R)$ *such that* $sb = s\alpha(a)$ *and* $sc = s\beta(a)$ *if* $s'f(b) = s'g(c)$ *for some* $s' \in R\backslash Nil(R)$.

Proof. (a) If ρ is a nonnil-epimorphism, then there exist $b \in B$, $c \in C$ and $s_1 \in R\backslash Nil(R)$ such that $s_1 d = s_1\rho(b,c) = s_1 f(b) + s_1 g(c)$ for any element $d \in D$. Thus

$$d = f(b) + g(c) + t \in \mathrm{NIm}(f) + \mathrm{NIm}(g),$$

where $t \in \mathrm{Ntor}(D)$. Therefore $D = \mathrm{NIm}(f) + \mathrm{NIm}(g)$.

Conversely, suppose $D = \mathrm{NIm}(f) + \mathrm{NIm}(g)$. For any $d \in D$, there exist $d_1 \in \mathrm{NIm}(f)$ and $d_2 \in \mathrm{NIm}(g)$ such that $d = d_1 + d_2$. Hence there exist $b \in B$, $c \in C$ and $s_2, s_3 \in R\backslash Nil(R)$ such that $s_2 d_1 = s_2 f(b)$ and $s_3 d_2 = s_3 g(c)$. Thus we have

$$s_2 s_3 d = s_2 s_3 (f(b) + g(c)) = s_2 s_3 \rho(b, c).$$

So $d \in \mathrm{NIm}(\rho)$. Therefore, the homomorphism ρ is a nonnil-epimorphism.

(b) It is clear that $\mathrm{Ntor}(A) \subseteq \mathrm{NKer}(\alpha) \cap \mathrm{NKer}(\beta)$. If $a \in \mathrm{NKer}(\alpha) \cap \mathrm{NKer}(\beta)$, then there exist $s_4, s_5 \in R\backslash Nil(R)$ such that $s_4\alpha(a) = 0$, $s_5\beta(a) = 0$. So

$$s_4 s_5 \sigma(a) = s_4 s_5 (\alpha(a), -\beta(a)) = 0.$$

Since σ is a nonnil-monomorphism, we have $a \in \mathrm{Ntor}(A)$. Therefore, $\mathrm{NKer}(\alpha) \cap \mathrm{NKer}(\beta) = \mathrm{Ntor}(A)$.

Conversely, if $s_6\sigma(a) = s_6(\alpha(a), -\beta(a)) = 0$ for some $s_6 \in R\backslash Nil(R)$, then $a \in \mathrm{NKer}(\alpha) \cap \mathrm{NKer}(\beta) = \mathrm{Ntor}(A)$. Therefore, σ is a nonnil-monomorphism.

(c) The sequence Δ is a nonnil-complex if and only if $\mathrm{NIm}(\sigma) \subseteq \mathrm{NKer}(\rho)$, if and only if there exists $s_7 \in R\backslash Nil(R)$ such that

$$s_7 \rho\sigma(a) = s_7 (f\alpha(a) - g\beta(a)) = 0,$$

if and only if the diagram Γ is nonnil-commutative.

(d) Suppose that $\mathrm{NKer}(\rho) \subseteq \mathrm{NIm}(\sigma)$. If $s'f(b) = s'g(c)$ for some $s' \in R\backslash Nil(R)$, then $s'\rho(b,c) = 0$. Hence $(b,c) \in \mathrm{NKer}(\rho) \subseteq \mathrm{NIm}(\sigma)$. Thus there exist an element $a \in A$ and $s \in R\backslash Nil(R)$ such that $s(b,c) = s(\alpha(a), \beta(a))$.

Conversely, if $(b,c) \in \mathrm{NKer}(\rho)$, that is, $s'f(b) = s'g(-c)$ for some $s' \in R\backslash Nil(R)$, then there exist $a \in A$ and $s \in R\backslash Nil(R)$ such that $sb = s\alpha(a)$ and $-sc = s\beta(a)$. Thus $s(b,c) = s\sigma(a)$. Therefore $(b,c) \in \mathrm{NIm}(\sigma)$, which implies that $\mathrm{NKer}(\rho) \subseteq \mathrm{NIm}(\sigma)$. □

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Wei Zhao
School of Mathematics
Aba Teachers University
Aba 623002, P. R. China
Email address: zw9c248@163.com

Dechuan Zhou
Department of Mathematics
Southwest University of Science and Technology
Mianyang 621010, P. R. China
Email address: zdechuan11119@163.com