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ON HOMOGENEOUS SHEAR FLOWS WITH BOTTOM CROSS SECTION

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Abstract. We consider inviscid, incompressible homogeneous shear flows of variable cross section known as extended Rayleigh problem. For this extended Rayleigh problem, we derived instability region which intersect with semi-circle instability region under some condition. Also we derived condition for stability , upper bound for amplification factor and growth rate of an unstable mode.

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1. Introduction

The extended Rayleigh problem deals with incompressible, inviscid, homogeneous shear flows with bottom cross section. This problem was introduced by [4], derivation and mathematical analysis was given in [2] and [9]. This is a special case of extended Taylor-Goldstein problem. [2] focused on stratified shear flows with bottom cross section and this is an extension of standard Taylor-Goldstein problem. When density remains constant then extended Taylor-Goldstein problem reduces to problem of study. General results have been proved.

The following results are already known for problem.

(1) Instability regions that intersects with semi-circle region under some conditions (cf. [7]).
(2) The neutral modes are bounded (cf. [7]).
(3) Howard's conjecture is derived. (cf. [7]).
(4) Sufficient criterion for the stability is derived (cf. [3], [8]).
(5) Short wave stability condition is derived (cf. [3], [8]).

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(6) Instability region that intersects with semi-circle region under some condition is derived (cf. [5]).
(7) Instability region that intersects with semi-circle is derived (cf. [6]).
(8) Bounds for amplification factor is derived (cf. [6]).

The instability region derived by [7] depends on two conditions.The instability region derived by [5] depends on conditions like $U_{0\,\mathrm{min}} > 0$ (or) $\left[bD\left(\frac{D(U_0)}{b}\right)\right] > 0$. [6] followed the previous work and relaxed the conditions like of [7], [5] and derived an instability region that intersect with semi-circle region. [8] derived condition for stability and short waves stability for special class of topography namely breadth function remains constant. [3] derived short wave stability condition and stability condition for a class of topography.
In this paper, we obtained a parabolic instability region which intersects with semi-circle region under certain condition. New parabolic instability region depends on shear function, breadth function minimum, maximum of velocity profile and vorticity function. Also we obtained condition for short wave stability, bounds for amplification factor and estimate for growth rate. Unlike [3], the new result does not depends on any conditions.

2. Extended Rayleigh Problem

$$D\left[\frac{D\left(b\phi\right)}{b}\right]-\left[k^{2}+\frac{\left[bD\left(\frac{D(U_0)}{b}\right)\right]}{U_{0}-c}\right]\phi=0, \tag{1}$$

with boundary conditions

$$\phi(0)=0=\phi(D). \tag{2}$$

Here $k>0$ wave number, U_0 is the basic velocity profile, $b(z)$ is the breadth function, $c=c_r+ic_i$ is the phase velocity.
For an unstable mode $c_i>0$, define $\phi=(U_0-c)^{\frac{1}{2}}\,\psi$ then (1), (2) becomes

$$D\left[\left(U_{0}-c\right)\frac{D\left(b\psi\right)}{b}\right]-\frac{1}{2}\left[bD\left(\frac{D\left(U_{0}\right)}{b}\right)\right]\psi-k^{2}\left(U_{0}-c\right)\psi-\frac{\frac{\left[D(U_0)\right]^{2}}{4}}{\left(U_{0}-c\right)}\psi=0, \tag{3}$$

with boundary conditions

$$\psi(0)=0=\psi(D), \tag{4}$$

3. Parabolic Instability Region

Theorem 3.1. *For an unstable mode with $c_i>0$ and an arbitrary real number $U_{0m}=\frac{U_{0\,\mathrm{min}}+U_{0\,\mathrm{max}}}{2}$, we have the following*

$$c_i^{2}\leq\lambda\left[c_r+\frac{U_{0\,\mathrm{max}}}{4}-\frac{U_{0\,\mathrm{min}}}{4}\right],$$

where

$$\lambda = \frac{\left[D\left(U_0\right)\right]^2{}_{\max}}{\left|3U_{0\,\min} + U_{0\,\max}\right|\left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2\right] + \left|bD\left(\frac{D(U_0)}{b}\right)\right|_{\min}}.$$

Proof: Multiplying (3) by $(b\psi^*)$, integrating , applying (4),comparing real and imaginary parts we get

$$\int_0^D \left(U_0 - c_r\right)\left[\frac{\left|D(b\psi)\right|^2}{b} + k^2 b\left|\psi\right|^2\right]dz + \frac{1}{2}\int_0^D bD\left(\frac{D\left(U_0\right)}{b}\right)b\left|\psi\right|^2 dz + \int_0^D \frac{\left[D\left(U_0\right)\right]^2}{4\left|U_0 - c\right|^2}\left(U_0 - c_r\right)b\left|\psi\right|^2 dz = 0, \qquad (5)$$

$$-c_i\int_0^D \left[\frac{\left|D(b\psi)\right|^2}{b} + k^2 b\left|\psi\right|^2\right]dz + c_i\int_0^D \frac{\left[D\left(U_0\right)\right]^2}{4\left|U_0 - c\right|^2}b\left|\psi\right|^2 dz = 0. \qquad (6)$$

Multiplying (6) by $\frac{(c_r + U_{0m})}{c_i}$ and subtracting from (5), we get

$$\int_0^D \left(U_0 + U_{0m}\right)\left[\frac{\left|D(b\psi)\right|^2}{b} + k^2 b\left|\psi\right|^2\right]dz + \frac{1}{2}\int_0^D bD\left(\frac{D\left(U_0\right)}{b}\right)b\left|\psi\right|^2 dz + \int_0^D \frac{\left[D\left(U_0\right)\right]^2}{4\left|U_0 - c\right|^2}\left(U_0 - 2c_r - U_{0m}\right)b\left|\psi\right|^2 dz = 0.$$

Applying Rayleigh-Ritz inequality, we get

$$\int_0^D \left[\left(U_0 + U_{0m}\right)\left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2\right] + \frac{1}{2}bD\left(\frac{D\left(U_0\right)}{b}\right)\right]b\left|\psi\right|^2 dz + \int_0^D \left[\frac{\left[D\left(U_0\right)\right]^2}{4\left|U_0 - c\right|^2}\left(U_0 - 2c_r - U_{0m}\right)\right]b\left|\psi\right|^2 dz \le 0;$$

i.e.,

$$\int_0^D \left[\left[\left(U_0 + U_{0m}\right)\left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2\right] + \frac{1}{2}bD\left(\frac{D\left(U_0\right)}{b}\right)\right]\left|U_0 - c\right|^2\right]\frac{b\left|\psi\right|^2}{\left|U_0 - c\right|^2}dz + \int_0^D \left[\frac{\left[D\left(U_0\right)\right]^2}{4}\left(U_0 - 2c_r - U_{0m}\right)\right]\frac{b\left|\psi\right|^2}{\left|U_0 - c\right|^2}dz \le 0.$$

Since $\left|U_0 - c\right|^2 \ge {c_i}^2$ and $U_{0m} = \frac{U_{0\,\min} + U_{0\,\max}}{2}$, we have

$$\left[\left(U_{0\,\min} + U_{0m}\right)\left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2\right] + \frac{1}{2}bD\left(\frac{D\left(U_0\right)}{b}\right)\right]{c_i}^2 \le \frac{\left[D\left(U_0\right)\right]^2}{4}\left(2c_r - U_{0\,\min} + \frac{U_{0\min} + U_{0\max}}{2}\right);$$

i.e.,

$$c_i^2 \leq \lambda \left[c_r + \frac{U_{0\,\max}}{4} - \frac{U_{0\,\min}}{4} \right], \tag{7}$$

where

$$\lambda = \frac{\left[D\left(U_0\right)\right]^2{}_{\max}}{\left|3U_{0\,\min} + U_{0\,\max}\right| \left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2\right] + \left|bD\left(\frac{D(U_0)}{b}\right)\right|_{\min}}$$

Theorem 3.2. *If $\lambda < \lambda_c$, where*

$$\lambda_c = \frac{(3U_{0\,\max} + U_{0\,\min}) - \sqrt{5U_{0\,\max}^2 - 3U_{0\,\min}^2 + 14U_{0\,\max}U_{0\,\min}}}{2}.$$

then the parabola $c_i^2 \leq \lambda \left[c_r + \frac{U_{0\,\max}}{4} - \frac{U_{0\,\min}}{4}\right]$ intersect the semi-circle

$$\left[c_r - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right]^2 + {c_i}^2 \leq \left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]^2.$$

Proof: Semi-circle region (cf. [2]) is given by

$$\left[c_r - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right]^2 + {c_i}^2 \leq \left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]^2. \tag{8}$$

Substitute (7) in (8), we have

$${c_r}^2 + c_r\left[\lambda - U_{0\,\max} - U_{0\,\min}\right] + \left[U_{0\,\max}U_{0\,\min} + \frac{\lambda U_{0\,\max}}{4} - \frac{\lambda U_{0\,\min}}{4}\right] \leq 0.$$

For real roots, the discriminant parts is greater than or equal to zero, we have

$$\lambda^2 - (3U_{0\,\max} - U_{0\,\min})\,\lambda + \left[U_{0\,\max} - U_{0\,\min}\right]^2 \geq 0.$$

Solving for λ, we get

$$\lambda = \frac{(3U_{0\,\max} + U_{0\,\min}) \pm \sqrt{5U_{0\,\max}^2 - 3U_{0\,\min}^2 + 14U_{0\,\max}U_{0\,\min}}}{2},$$

If

$$\lambda = \frac{(3U_{0\,\max} + U_{0\,\min}) + \sqrt{5U_{0\,\max}^2 - 3U_{0\,\min}^2 + 14U_{0\,\max}U_{0\,\min}}}{2},$$

will lead to $c_r < U_0$ and hence, we have

$$\lambda_c = \frac{(3U_{0\,\max} + U_{0\,\min}) - \sqrt{5U_{0\,\max}^2 - 3U_{0\,\min}^2 + 14U_{0\,\max}U_{0\,\min}}}{2}.$$

If $\lambda < \lambda_c$ then the parabola given in (7) intersect with semi-circle given in (8).

Example 1. Let us consider the flow $U_0 = z + \frac{1}{2}$, $1 \leq z \leq 2$, $b = e^{T_0 z}$, $T = T_0(\text{constant})$.

In this case $U_{0\,\min} = \frac{1}{2}$, $U_{0\,\max} = \frac{3}{2}$, $\lambda < \lambda_c$ for different values of k and T_0.

$$\lambda = \frac{1}{5\left(\frac{\pi^2}{e^{T_0}} + k^2\right) + T_0},$$

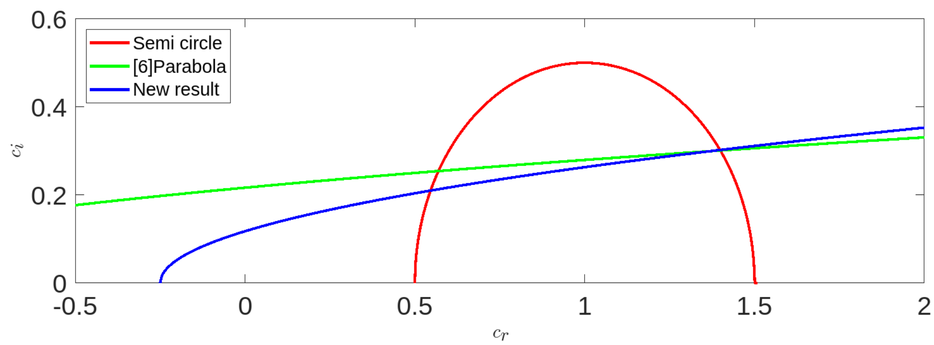

FIGURE 1. c_r *vs* c_i (Intersection of parabola with semi circle)

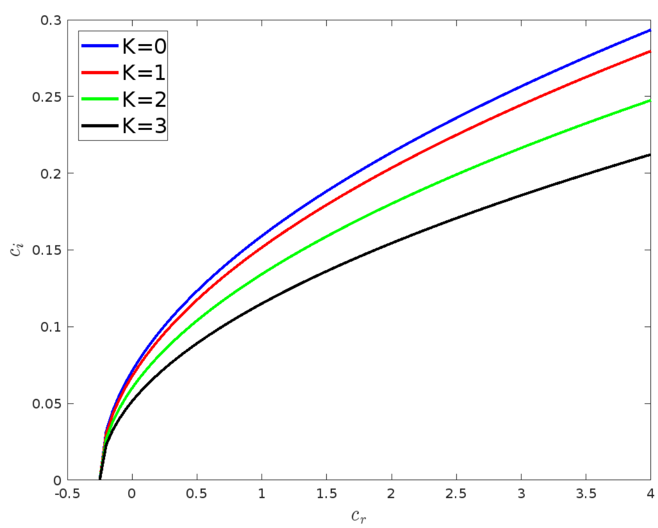

FIGURE 2. c_r *vs* c_i (parabolic instability regions for distinct values of k and $T_0 = 0$)

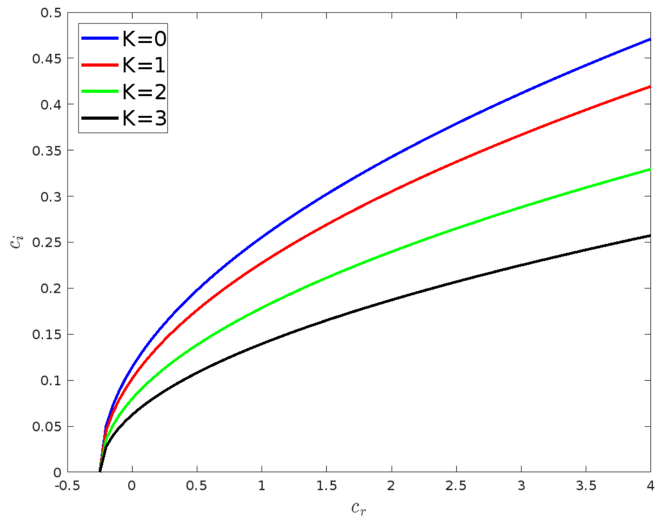

FIGURE 3. c_r vs c_i (parabolic instability regions for distinct values of k and $T_0 = 1$)

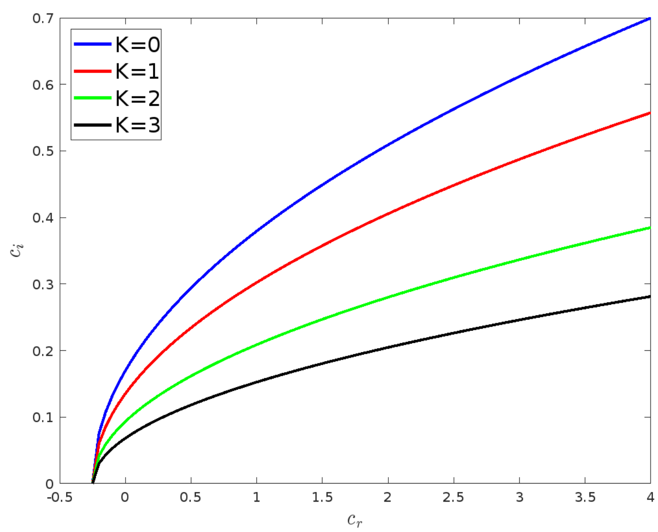

FIGURE 4. c_r vs c_i (parabolic instability regions for distinct values of k and $T_0 = 2$)

Example 2. Let us consider the flow $U_0 = sinz$, $1 \leq z \leq 2$, $b = e^{T_0 z}$, $T = T_0$(constant) . In this case $U_{0\,\min} = 0.8415$, $U_{0\,\max} = 1$, $\lambda < \lambda_c$ for different values of k and T_0.

$$\lambda = \frac{(0.5403023)^2}{(3.5245)\left[\frac{\pi^2}{e^{T_0}} + k^2\right] + T_0},$$

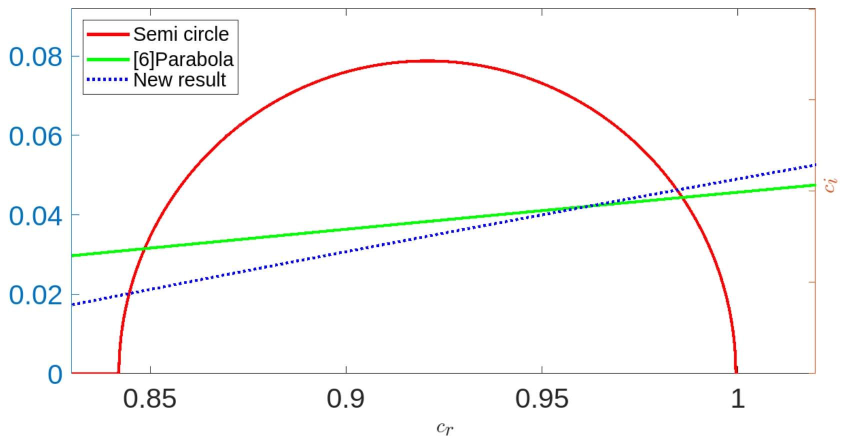

FIGURE 5. c_r *vs* c_i (Intersection of parabola with semi circle)

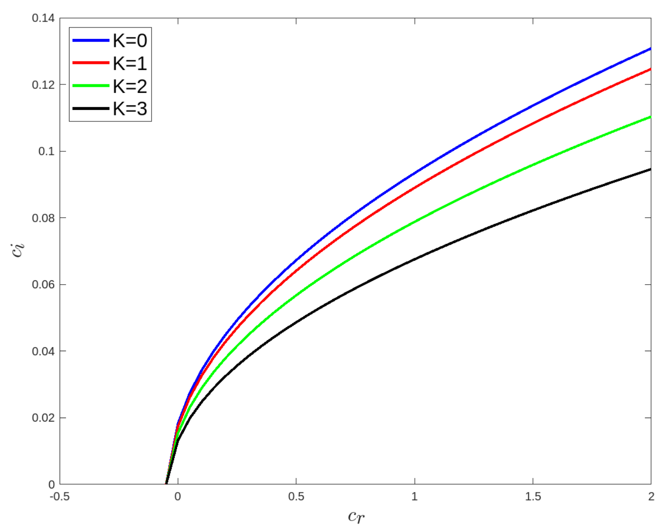

FIGURE 6. c_r *vs* c_i (parabolic instability regions for distinct values of k and $T_0 = 0$)

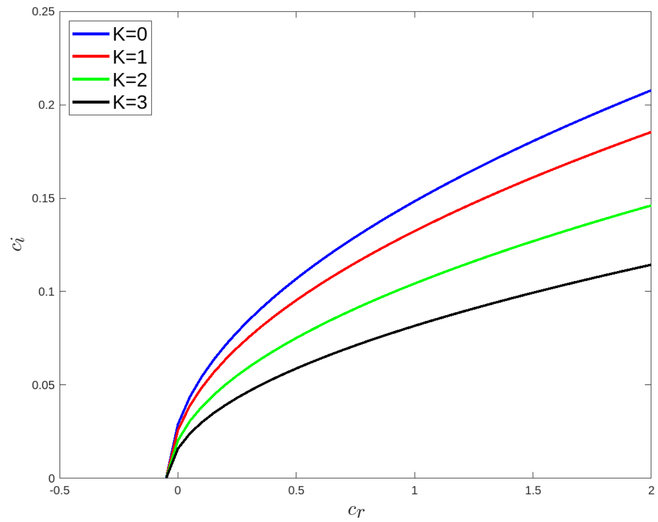

FIGURE 7. c_r *vs* c_i (parabolic instability regions for distinct values of k and $T_0 = 1$)

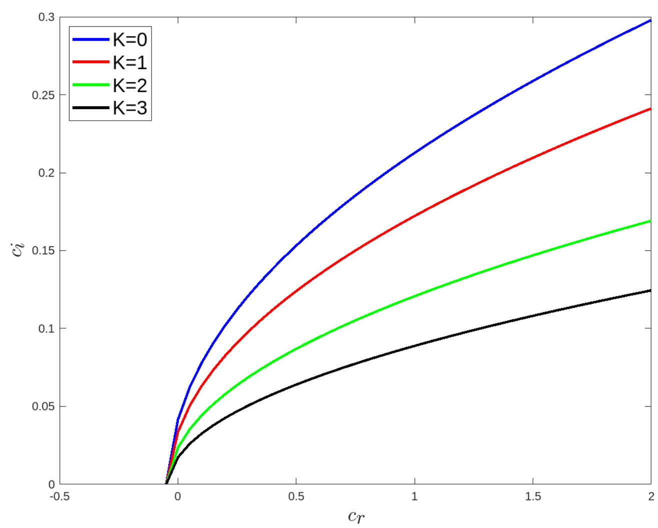

FIGURE 8. c_r *vs* c_i (parabolic instability regions for distinct values of k and $T_0 = 2$)

4. Bounds for Growth Rate

Theorem 4.1. *If* $c_i > 0$ *then estimate for* kc_i *is*

$$k^2 {c_i}^2 \leq \frac{\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \left| bD\left(\frac{D(U_0)}{b}\right)\right|_{\max} (U_0 - U_{0s}) + \left| bD\left(\frac{D(U_0)}{b}\right)^2\right|_{\max}}{k^2}.$$

Proof: Multiplying (1) by $bD\left(\frac{D(b\phi^*)}{b}\right)$, integrating and applying (2), we get

$$\int_0^D b\left|D\left[\frac{D(b\phi)}{b}\right]\right|^2 dz-\int_0^D\left[k^2+\frac{bD\left(\frac{D(U_0)}{b}\right)}{U_0-c}\right]b\phi D\left[\frac{D(b\phi^*)}{b}\right]dz=0. \quad (9)$$

From (1) taking complex conjugate, we have

$$D\left[\frac{D(b\phi^*)}{b}\right]=\left[k^2+\frac{bD\left(\frac{D(U_0)}{b}\right)}{U_0-c^*}\right]\phi^*. \quad (10)$$

Substituting (10) in (9), comparing real parts, we get

$$\int_0^D b\left|D\left[\frac{|D(b\phi)|^2}{b}\right]\right|^2 dz-k^4\int_0^D b\,|\phi|^2\,dz$$
$$-2k^2\int_0^D\frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0-c|^2}(U_0-c_r)\,b\,|\phi|^2\,dz$$
$$-\int_0^D\frac{\left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{|U_0-c|^2}b\,|\phi|^2\,dz=0. \quad (11)$$

Multiplying (1) by $(b\phi^*)$, integrating , applying (2), comparing real and imaginary parts we have

$$\int_0^D\left[\frac{|D(b\phi)|^2}{b}+k^2b\,|\phi|^2\right]dz+\int_0^D\frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0-c|^2}(U_0-c_r)\,b\,|\phi|^2\,dz=0, \quad (12)$$

$$c_i\int_0^D\frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0-c|^2}b\,|\phi|^2\,dz=0. \quad (13)$$

Multiplying (12) by $2k^2$and adding with (11), we get

$$\int_0^D b\left|D\left[\frac{D(b\phi)}{b}\right]\right|^2 dz+k^4\int_0^D b\,|\phi|^2\,dz$$
$$-\int_0^D\frac{\left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{|U_0-c|^2}b\,|\phi|^2\,dz+2k^2\int_0^D\frac{|D(b\phi)|^2}{b}dz=0. \quad (14)$$

Multiplying (13) by $\frac{(c_r-U_{0s})}{c_i}$ and adding with (12), we get

$$\int_0^D\left[\frac{|D(b\phi)|^2}{b}+k^2b\,|\phi|^2\right]dz+\int_0^D\frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0-c|^2}(U_0-U_{0s})\,b\,|\phi|^2\,dz=0. \quad (15)$$

Multiplying (15) by $\frac{\pi^2 b_{\min}}{D^2 b_{\max}}$ and subtracting from (14), we get

$$\int_0^D \left[b\left| D\left[\frac{D(b\phi)}{b} \right] \right|^2 dz - \frac{\pi^2 b_{\min}}{D^2 b_{\max}} \int_0^D \frac{|D(b\phi)|^2}{b} dz \right]$$
$$+k^2 \left[\int_0^D \frac{|D(b\phi)|^2}{b} dz - \frac{\pi^2 b_{\min}}{D^2 b_{\max}} \int_0^D b\,|\phi|^2\, dz \right]$$
$$+k^2 \int_0^D \frac{|D(b\phi)|^2}{b} dz + k^4 \int_0^D b\,|\phi|^2\, dz - \int_0^D \frac{\left[bD\left(\frac{D(U_0)}{b} \right) \right]^2}{|U_0 - c|^2} b\,|\phi|^2\, dz$$
$$-\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \int_0^D \frac{bD\left(\frac{D(U_0)}{b} \right)}{|U_0 - c|^2} (U_0 - U_{0s})\, b\,|\phi|^2\, dz = 0.$$

In the above equation first bracket, second bracket terms and third term are positive, hence dropping the terms, we get

$$k^4 \int_0^D b\,|\phi|^2\, dz$$
$$-\int_0^D \left[\frac{\frac{\pi^2 b_{\min}}{D^2 b_{\max}} bD\left(\frac{D(U_0)}{b} \right)(U_0 - U_{0s}) + \left[bD\left(\frac{D(U_0)}{b} \right) \right]^2}{|U_0 - c|^2} \right] b\,|\phi|^2\, dz \le 0.$$

Since $\frac{1}{|U_0-c|^2} \le \frac{1}{{c_i}^2}$, we get

$$k^2 {c_i}^2 \le \frac{\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \left| bD\left(\frac{D(U_0)}{b} \right) \right|_{\max} (U_0 - U_{0s}) + \left| bD\left(\frac{D(U_0)}{b} \right)^2 \right|_{\max}}{k^2};$$

i.e.,

$$k^2 {c_i}^2 \le \frac{\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \left| bD\left(\frac{D(U_0)}{b} \right) \right|_{\max} (U_0 - U_{0s}) + \left| bD\left(\frac{D(U_0)}{b} \right)^2 \right|_{\max}}{k^2}. \tag{16}$$

Theorem 4.2. *Howard's conjecture $kc_i \to 0$ as $k \to \infty$.*

Proof: From (16), it follows.

Theorem 4.3. *If $K(z) = \frac{-bD\left(\frac{D(U_0)}{b}\right)}{(U_0 - U_{0s})} \ge 0$ then criterion for stability is that*

$$0 \le K(z) \le \frac{\pi^2 b_{\min}}{D^2 b_{\max}}.$$

Proof: From (16), we have

$$k^2 {c_i}^2 \le \frac{\left[bD\left(\frac{D(U_0)}{b} \right) \right]^2}{K(z)} \frac{\left[K(z) - \frac{\pi^2 b_{\min}}{D^2 b_{\max}} \right]}{k^2}.$$

It follows from the above equation $c_i = 0$ for $K(z) \le \frac{\pi^2 b_{\min}}{D^2 b_{\max}}$.
Hence the flow is stable.

Theorem 4.4. *For $c_i > 0$, it is necessary that*

$$k^2 < \left[\frac{\left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{-bD\left(\frac{D(U_0)}{b}\right)(U_0 - U_{0s})} \right]_{z=z_p}.$$

Proof: Substituting (10) in (9), comparing real and imaginary parts we get

$$\int_0^D b\left|D\left[\frac{D(b\phi)}{b}\right]\right|^2 dz + k^2 \int_0^D \frac{|D(b\phi)|^2}{b} dz$$
$$-k^2 \int_0^D \frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0 - c|^2}(U_0 - c_r)\, b\,|\phi|^2\, dz$$
$$-\int_0^D \frac{\left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{|U_0 - c|^2} b\,|\phi|^2\, dz = 0, \tag{17}$$

$$k^2 c_i \int_0^D \frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0 - c|^2} b\,|\phi|^2\, dz = 0. \tag{18}$$

Multiplying (18) by $\frac{(c_r - U_{0s})}{c_i}$ and subtracting from (17), we get

$$\int_0^D b\left|D\left[\frac{D(b\phi)}{b}\right]\right|^2 dz + k^2 \int_0^D \frac{|D(b\phi)|^2}{b} dz$$
$$-k^2 \int_0^D \frac{bD\left(\frac{D(U_0)}{b}\right)}{|U_0 - c|^2}(U_0 - U_{0s})\, b\,|\phi|^2\, dz$$
$$-\int_0^D \frac{\left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{|U_0 - c|^2} b\,|\phi|^2\, dz = 0.$$

$$\int_0^D b\left|D\left[\frac{D(b\phi)}{b}\right]\right|^2 dz + k^2 \int_0^D \frac{|D(b\phi)|^2}{b} dz$$
$$-\int_0^D \frac{k^2 bD\left(\frac{D(U_0)}{b}\right)(U_0 - U_{0s}) + \left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{|U_0 - c|^2} b\,|\phi|^2\, dz = 0. \tag{19}$$

From the above equation it follows that

$$k^2 bD\left(\frac{D(U_0)}{b}\right)(U_0 - U_{0s}) + \left[bD\left(\frac{D(U_0)}{b}\right)\right]^2 > 0;$$

i.e.,

$$k^2 < \left[\frac{\left[bD \left(\frac{D(U_0)}{b} \right) \right]^2}{-bD \left(\frac{D(U_0)}{b} \right) (U_0 - U_{0s})} \right]_{z=z_p} .$$

Remark:
As $\lambda = \frac{2\pi}{k}$, small wave number corresponds to large wave length.
Let the critical value be

$$k_c{}^2 = \underset{z\Sigma[0,D]}{Max} \left[\frac{\left[bD \left(\frac{D(U_0)}{b} \right) \right]^2}{-bD \left(\frac{D(U_0)}{b} \right) (U_0 - U_{0s})} \right] . \tag{20}$$

It follows that $k > k_c$ implies the flow is stable provided critical value of k_c should be finite.

Example:

(1) Let $U_0\,(z) = \sin\left(z^2\right)$, $b = z$ in $0 \leq z \leq 2\pi$,U_0 changes its sign at $z_s = \sqrt{\pi}$. Using (20),we get $k > k_c = 4\pi$, which implies stability of the mode.
(2) Let$U_0\,(z) = e^z \sin z$, $b = e^{2z}$ in $0 \leq z \leq 2\pi$ U_0 changes its sign at $z_s = \pi$. Using (20), we get $k > k_c = 2$, which implies stability of the mode.

The new stability condition does not depend on any condition like $T^{'} \leq 0$ or $c_r = U_{0s}$, as in [3]. When $b = 1$ or $T = 0$, this result reduces to [1].

5. Bounds for Amplification Factor

Theorem 5.1. *The upper bound for amplification factor c_i is given by*

$$c_i \leq \sqrt{\frac{\left[k^4 \left[\frac{U_{0\max} - U_{0\min}}{2} \right]^2 + 2k^2 \left[bD \left(\frac{D(U_0)}{b} \right) \right] (U_0 - U_{0s}) + \left[bD \left(\frac{D(U_0)}{b} \right) \right]^2 \right]_{\max}}{\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2}}}.$$

Proof: Multiplying (13) by $\left(\frac{c_r - U_{0s}}{c_i} \right) \left(2k^2 \right)$ and subtracting from (11), we get

$$\int_0^D b \left| D \left[\frac{D(b\phi)}{b} \right] \right|^2 dz$$

$$- \int_0^D \left[k^4 + \frac{2k^2 bD \left(\frac{D(U_0)}{b} \right) (U_0 - U_{0s}) + \left[bD \left(\frac{D(U_0)}{b} \right) \right]^2}{\left| U_0 - c \right|^2} \right] b \left| \phi \right|^2 dz = 0.$$

Applying Rayleigh-Ritz inequality , $\frac{1}{|U_0 - c|^2} \leq \frac{1}{c_i{}^2}$, we get

$$\frac{\pi^4 b^2{}_{\min}}{D^4 b^2{}_{\max}} \int_0^D b \left| \phi \right|^2 dz \leq$$

$$\int_0^D \frac{k^4 c_i^2 + 2k^2 bD\left(\frac{D(U_0)}{b}\right)(U_0 - U_{0s}) + \left[bD\left(\frac{D(U_0)}{b}\right)\right]^2}{{c_i}^2} b\,|\phi|^2\, dz.$$

Since $c_i \leq \frac{U_{0\max} - U_{0\min}}{2}$, we have

$$c_i \leq \sqrt{\frac{\left[k^4\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2 + 2k^2\left[bD\left(\frac{D(U_0)}{b}\right)\right](U_0 - U_{0s}) + \left[bD\left(\frac{D(U_0)}{b}\right)\right]^2\right]_{\max}}{\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2}}}.$$

6. Conclusion

We study stability of incompressible inviscid homogeneous shear flows with bottom cross section. We derived a parabolic instability region which intersects with semi-circle region under some criterion. The parabolic instability region depends on parameters like breadth function, shear function, vorticity function which is important to decide stability or otherwise. Also we obtained condition for stability, bounds for growth rate and amplification factor .We proved Howard's conjecture also.

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