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MOMENTS OF ORDERED RANDOM VARIATES FOR TRANSMUTED POWER HAZARD DISTRIBUTION

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Abstract. This article studies some moments characteristics of generalized order statistics for transmuted power hazard distribution. It unifies the earlier results considered by several authors. The characterization results are also outlined at the end.

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1. Introduction

Generalized order statistics (*gos*) is defined as follows:
The random variables (r.v.) $X_{1:n,\tilde{m},k}, X_{2:n,\tilde{m},k}, \cdots, X_{n:n,\tilde{m},k}$ are n *gos* from an absolutely continuous distribution function (*cdf*) $F(.)$ with probability density function (*pdf*) $f(.)$, if their joint pdf takes the following form (see, [1]).

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i}f(x_i)\right)[\bar{F}(x_n)]^{k-1}f(x_n), \tag{1}$$

for $F^{-1}(0) < x_1 \le x_2 \le \cdots \le x_n < F^{-1}(1)$.
where $\gamma_j = k + (n-j)(m+1) > 0, \quad \forall \in j,\ 1 \le j \le n,\ m \ge -1$ and $k > 0$.
In view of (1), the pdf of r^{th} *gos* is given as in [1].

$$f_{X(r:n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!)}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}[F(x)], \quad -\infty < x < \infty. \tag{2}$$

The joint *pdf* of r^{th} and s^{th} ($1 \le r < s \le n$) is given as in [2].

$$f_{X(r:n,m,k),X(s:n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)g_m^{r-1}F(x)$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y), \quad -\infty < x < y < \infty, \tag{3}$$

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where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad h_m(x) = \begin{cases} -\frac{(1-x)^{m+1}}{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0, 1).$$

For a comprehensive study of *gos* and its variants. See [3, 4] for details.

2. Transmuted Power Hazard (TPH) Distribution

[5] introduced the Power Hazard (*PH*) distribution. The *cdf* and *pdf* of *PH* distribution are given, respectively by

$$F(x) = 1 - e^{-\frac{\alpha}{\lambda+1}x^{\lambda+1}}, \quad x > 0, \alpha > 0, \lambda > -1 \tag{4}$$

$$f(x) = \alpha x^{\lambda} e^{-\frac{\alpha}{\lambda+1}x^{\lambda+1}}, \quad x > 0, \alpha > 0, \lambda > -1 \tag{5}$$

This distribution represents an extension of several probability distribution previously existed in the literature, namely Rayleigh, exponential, Weibull, Linear hazard rate (*LHR*) distribution.

The *PH* distribution is very simple and got more recognition due to its hazard rate properties (increasing, constant and decreasing). These features enable this distribution to be used in several areas, namely, reliability, survival analysis, life testing and others.

The generalizations of existing distributions are used to get a more flexible and accurate model. Several techniques are available. One of them is a quadratic rank transmutation.

The *cdf* of *TPH* distribution is defined as,

$$F(x) = (1+\theta)\left(1 - e^{-\frac{\alpha}{\lambda+1}x^{\lambda+1}}\right) - \theta\left(1 - e^{-\frac{\alpha}{\lambda+1}x^{\lambda+1}}\right)^2, \quad -1 \leq \theta \leq 1. \tag{6}$$

The *pdf* of the *TPH* distribution is given as,

$$f(x) = \alpha x^{\lambda} e^{-\frac{\alpha}{\lambda+1}x^{\lambda+1}} \left[1 + \theta - 2\theta\left(1 - e^{-\frac{\alpha}{\lambda+1}x^{\lambda+1}}\right)\right] \tag{7}$$

where α, λ and θ are scale, shape, and transmuted parameters, respectively.

Putting $\theta = 0$ in (6), we get *PH* distribution. The result attained in this paper is valid for transmuted exponential, transmuted Rayleigh, transmuted Weibull and transmuted linear hazard distribution depending upon the parameters. For more information and basic properties of quadratic rank transmutation, one can refer [6].

The *pdf* and, *cdf* of *TPH* distribution are associated as

$$[1 - F(x)] = \left[\left(\frac{\lambda+1}{\alpha}\right) f(x) - \theta \sum_{i=0}^{\infty} (-2)^i \left(\frac{\alpha}{\lambda+1}\right)^i \frac{x^i}{i!}\right] \tag{8}$$

The application of moments of *gos* is thoroughly well-known in statistical science since its emergence. For instance, they are handy in modeling, inferences, decision making, reliability theory, and among others relying upon the specific

parameters. Several recurrence relations for moments of *gos* from many distributions have been reported by several authors. For example, see [7, 8, 9, 10, 11, 12, 13, 14] and many others are obtainable in the literature.
The moments properties of the *TPH* distribution have not been reported in the literature. This article represents the moments of *gos* from *TPH* distribution based on recurrence relations. Characterization results are also presented.

3. Single Moments

Theorem 3.1. *If $X \sim TPHD(\alpha, \lambda, \theta)$ and $p > 0$, then the single moments of r^{th} gos is given by $(1 \le r \le n)$,*

$$E[X^p_{r:n,m,k}] = E[X^p_{r-1:n,m,k}] + \left(\frac{\lambda+1}{\alpha}\right)\frac{p}{\gamma_r}E[X^{p-1}_{r:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times\frac{\gamma_{r(k-1)}C_{r-1}}{i!\gamma_r C_{r-1(k-1)}}\left\{E[X^{p+i}_{r:n,m,k-1}] - E[X^{p+i}_{r-1:n,m,k-1}]\right\} \quad (9)$$

where

$$\gamma_{p(k-1)} = (k-1) + (n-p)(m+1)$$

and

$$C_{r-1(k-1)} = \sum_{j=1}^{r}\gamma_{p(k-1)}.$$

Proof. From, [15], we have

$$E[X^p_{r:n,m,k}] - E[X^p_{r-1:n,m,k}] = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]dx. \quad (10)$$

The relation (10) can be expressed as

$$E[X^p_{r:n,m,k}] - E[X^p_{r-1:n,m,k}] = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)][\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]dx.$$

Now using (8) we have

$$E[X^p_{r:n,m,k}] - E[X^p_{r-1:n,m,k}] = \frac{pC_{r-1}}{\gamma_r(r-1)!}\times$$
$$\int_0^\infty x^{p-1}\left[\left(\frac{\lambda+1}{\gamma}\right)f(x) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{x^i}{i!}\right][\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]dx$$
$$= \left(\frac{\lambda+1}{\alpha}\right)\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx$$
$$-\theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{pC_{r-1}}{i!\gamma_r(r-1)!}\int_0^\infty x^{p+i-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]dx$$

or

$$= \left(\frac{\lambda+1}{\alpha}\right)\frac{pC_{r-1}}{\gamma_r(r-1)!}E[X^{p-1}_{r:n,m,k})] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times$$
$$\frac{p\gamma_{r(k-1)}C_{r-1}C_{r-1(k-1)}}{i!\gamma_{r(k-1)}\gamma_r jC_{r-1(k-1)}(r-1)!}\int_0^{\infty}x^{p+i-1}[\bar{F}(x)]^{\gamma_r(k-1)}g_m^{r-1}[F(x)]dx$$
$$= \frac{jC_{r-1}}{\alpha\gamma_r(r-1)!}E[X^{p-1}_{r:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{\gamma_{r(k-1)}C_{r-1}}{i!\gamma_r C_{r-1(k-1)}} \times$$
$$\frac{pC_{r-1(k-1)}}{\gamma_r C_{r-1(k-1)}(r-1)!}\int_0^{\infty}x^{p+i-1}[\bar{F}x)]^{\gamma_{r(k-1)}}g_m^{r-1}[F(x)]dx.$$

Now considering (8), we have

$$= \left(\frac{\lambda+1}{\alpha}\right)\frac{p}{\gamma_r}E[X^p_{r-1:n,m,k}]$$
$$-\theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{\gamma_{r(k-1)}C_{r-1}}{i!\gamma_r C_{r-1(k-1)}}\left\{E[X^{p+i}_{r:n,m,k-1}] - E[X^{p+i}_{r-1:n,m,k-1}]\right\}$$

or

$$\begin{aligned} & E[X^p_{r:n,m,k}] \\ = \quad & E[X^p_{r-1:n,m,k}] + \left(\frac{\lambda+1}{\alpha}\right)\frac{p}{\gamma_r}E[X^{p-1}_{r:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times \\ & \frac{\gamma_{r(k-1)}C_{r-1}}{i!\gamma_r C_{r-1(k-1)}}\{E[X^{p+i}_{r:n,m,k-1}] - E[X^{p+i}_{r-1:n,m,k-1}]\} \end{aligned} \tag{11}$$

Hence, the theorem.

□

Theorem 3.1 can be easily reduced for relationship of order statistics and k-upper record values by choosing $(m = 0, k = 1)$ and $m = -1$.

TABLE 1. Remarks based on single moments.

S. No.	α	λ	θ	Distribution	Author
1	–	0	–	Transmuted exponential	[16]
2	–	0	0	Exponential	[7]
3	–	–	0	Power hazard	[13]

4. Product Moments

Theorem 4.1. *If $X \sim TPHD(\alpha, \lambda, \theta)$, further suppose that $p, q > 0$ and $1 \leq r < s \leq n$, then,*

$$E[X^p_{r:n,m,k}, X^q_{s:n,m,k}] = E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] + \left(\frac{\lambda+1}{\alpha}\right)\frac{q}{\gamma_s} \times E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \frac{\gamma_{s(k-1)}C_{s-1}}{(i!\gamma_s C_{s-1(k-1)}} \times E[X^p_{r:n,m,k-1}, X^q_{s:n,m,k-1} - X^p_{r:n,m,k-1}, X^q_{s-1:n,m,k-1}] \tag{12}$$

Proof. From, [15], we have,

$$E[X^p_{r:n,m,k}, X^q_{s:n,m,k}] - E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s} dydx. \tag{13}$$

The above relation can be written as

$$E[X^p_{r:n,m,k}, X^q_{s:n,m,k}] - E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)][\bar{F}(y)]^{\gamma_s-1} dydx. \tag{14}$$

Now using (6) in (14), we have

$$= \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} \times \left[\left(\frac{\lambda+1}{\alpha}\right) f(y) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \frac{y^i}{i!}\right][\bar{F}(y)]^{\gamma_s-1} dydx.$$

$$= \left(\frac{\lambda+1}{\alpha}\right)\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \times \int_0^\infty\int_x^\infty x^p y^{q-1)} f(x)f(y)[\bar{F}(x)]^m g_m^{r-1}[F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} dydx - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times \frac{qC_{s-1}}{i!\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q+i-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} dydx$$

or

$$= \left(\frac{\lambda+1}{\alpha}\right)\frac{q}{\gamma_s}E[X^p_{r:n,m,k}, X^{q-1}_{s:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times$$
$$\frac{qC_{s-1}}{i!\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q+i-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$
$$\times[h_m(F(x)) - h_m(F(y))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}dydx$$

or

$$= \left(\frac{\lambda+1}{\alpha}\right)\frac{q}{\gamma_s}E[X^p_{r:n,m,k}, X^{q-1}_{s:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times$$
$$\frac{q\gamma_s(k-1)C_{s-1}C_{s-1(k-1)}}{(i!\gamma_s\gamma_s(k-1)C_{s-1(k-1)}(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q+i-1} f(x)$$
$$\times[\bar{F}(x)]^m g_m^{r-1}[F(x)][h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_{s(k-1)}}dydx$$

where

$$\gamma_{s(k-1)} = (k-1) + (n-s)(m+1)$$

and

$$C_{s-1(k-1)} = \sum_{q=1}^{s}\gamma_{q(k-1)}.$$

Now using (13) again, we have

$$E[X^p_{r:n,m,k}, X^q_{s:n,m,k}] - E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] =$$
$$\left(\frac{\lambda+1}{\alpha}\right)\frac{q}{\gamma_s}E[X^p_{r:n,m,k}, X^{q-1}_{s:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times$$
$$\frac{\gamma_{s(k-1)}C_{s-1}}{i!\gamma_s C_{s-1(k-1)}}\{E[X^p_{r:n,m,k-1}, X^q_{s:n,m,k-1}] - E[X^p_{r:n,m,k-1}, X^q_{s-1:n,m,k-1}]\}$$

Hence the theorem. □

Theorem 4.1 can be easily reduced for relationship of order statistics and k-upper record values by choosing $(m = 0, k = 1)$ and $m = -1$.

TABLE 2. Remarks based on product moments.

S. No.	α	λ	θ	Distribution	Author
1	–	0	–	Transmuted exponential	[16]
2	–	0	0	Exponential	[7]
3	–	–	0	Power hazard	[13]

5. Characterization

The characterizations of *TPH* distribution is derived in the following theorems.

Theorem 5.1. *Let $X \sim TPHD(\alpha, \lambda, \theta)$, then the necessary and sufficient condition for a r.v. X is that,*

$$E[X^p_{r:n,m,k}] = E[X^p_{r-1:n,m,k}] + \left(\frac{\lambda+1}{\alpha}\right)\frac{p}{\gamma_r}E[X^{p-1}_{r:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i \times\frac{\gamma_{r(k-1)}C_{r-1}}{i!\gamma_r C_{r-1(k-1)}}\{E[X^{p+i}_{r:n,m,k-1}] - E[X^{p+i}_{r-1:n,m,k-1}]\} \tag{15}$$

Proof. From (9), necessary part follows. If the relation in (15) is complied. Then the terms in (15), can be rearranged as,

$$\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]dx = \left(\frac{\lambda+1}{\alpha}\right)\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{pC_{r-1}}{i!\gamma_r(r-1)!}\int_0^\infty x^{p+i-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]dx$$

or

$$\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]dx = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)] \times\frac{p}{\gamma_r}\left\{\left(\frac{\lambda+1}{\alpha}\right)f(x) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\beta+1}\right)^i\frac{x^{p+i-1}}{i!}\right\}dx$$

or

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{p-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]\times \left[\bar{F}(x) - \left\{\left(\frac{\lambda+1}{\alpha}\right)f(x) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\beta+1}\right)^i\frac{x^i}{i!}\right\}\right]dx = 0.(16)$$

Using M*ü*ntz-Sz*á*sz generalized theorem (see, [17]) to (16), it gives.

$$\bar{F}(x) = \left(\frac{\lambda+1}{\alpha}\right)f(x) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{x^i}{i!}.$$

□

Theorem 5.2. *Under the condition illustrated in Theorem 4.1, then the following expression holds,*

$$E[X^p_{r:n,m,k}, X^q_{s:n,m,k}] = E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] + \left(\frac{\lambda+1}{\alpha}\right)\frac{q}{\gamma_s} \times$$
$$E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}] - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\left(\frac{\gamma_{s(k-1)}C_{s-1}}{i!\gamma_s C_{s-1(k-1)}}\right)\times$$
$$\left\{E[X^p_{r:n,m,k-1}, X^q_{s:n,m,k-1}] - E[X^p_{r:n,m,k-1}, X^q_{s-1:n,m,k-1}]\right\} \tag{17}$$

Proof. From (12), the necessary part follows. For the sufficient part, consider (13) as

$$E[X^p_{r:n,m,k}, X^q_{s:n,m,k}] - E[X^p_{r:n,m,k}, X^q_{s-1:n,m,k}]$$
$$= \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s}dydx$$

Now using above relation with (8), we have

$$\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s}dydx$$
$$= \frac{qC_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$
$$\times[h_m(F(x)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_{s-1}}\frac{i}{\gamma_s}\times$$
$$\left\{\left(\frac{\lambda+1}{\alpha}\right)f(y) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{y^i}{i!}\right\}dydx$$

or

$$\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1} f(x)[\bar{F}(x)]^m g_m^{r-1}F(x)$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}$$
$$\left[\bar{F}(y) - \left\{\left(\frac{\lambda+1}{\alpha}\right)f(y) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{y^i}{i!}\right\}\right]dydx = 0. \tag{18}$$

Applying Müntz-Szász generalized theorem to (18), we obtain,

$$\bar{F}(y) = \left[\left(\frac{\lambda+1}{\alpha}\right)f(y) - \theta\sum_{i=0}^{\infty}(-2)^i\left(\frac{\alpha}{\lambda+1}\right)^i\frac{y^i}{i!}\right].$$

□

6. Conclusion

Some moments properties of *gos* from *TPH* distribution is derived. The results outlined in this manuscript can be applied to evaluate the moments of any model of *gos*. The characterization results are also derived.

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