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EXTENDED HERMITE-HADAMARD(H-H) AND FEJER'S INEQUALITIES BASED ON GEOMETRICALLY-s-CONVEX FUNCTIONS IN THIRD AND FOURTH SENSE

SABIR YASIN*, MASNITA MISIRAN, ZURNI OMAR, RABIA LUQMAN

ABSTRACT. In this paper, geometrically convex and s-convex functions in third and fourth sense are merged to form (g, s)-convex function. Characterizations of (g, s)-convex function, algebraic and functional properties are presented. In addition, novel functions based on the integral of (g, s)-convex functions in the third sense are created, and inequality relations for these functions are explored and examined under particular conditions. Further, there are also some relationships between (g, s)-convex function and previously defined functions. The (g, s)-convex function and its derivatives will then be used to extend the well-known H-H and Fejer's type inequalities. In order to obtain the previously mentioned conclusions, several special cases from previous literature for extended H-H and Fejer's inequalities are also investigated. The relation between the average (mean) values and newly created H-H and Fejer's inequalities are also examined.

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1. Introduction

Convex functions connect the main branches of mathematics from applied and pure mathematics such as in analysis [1], geometry [2], and theory of optimization [3]. In this perspective, recent generalization, and findings regarding these convexities in one area of mathematics have a broad influence on many others, including other sciences e.g., microeconomics [4], equilibrium theory [5] and game theory [6]. Many researchers presented the generalizations and extensions of the convexity such as h-convex functions [7], s-convex functions [4]-[9]

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quasiconvexity functions [10], relative strongly exponentially convex functions [11], co-ordinated s-convex functions [12], geometrically convex functions [13] which are employed in microeconomics, stochastic analysis, signal processing, equilibrium theory, finance [29], and fractal theory [1]-[19].
The combination of convex functions is a novel technique in expanding or generalization of convex functions [20]. These combinations have made extensive contributions to mathematical inequalities and optimization. The inequalities made by combining different families of convex functions attracts many researchers. They presented various extensions, refinements, and generalizations based on these combined functions. For example, Sabir, Misiran and Omar extended the H-H and Fejer's types of inequalities by combining (h_1, h_2)-convex and s-convex functions [21]. Similarly, Micherda in [22] presented a new generalization of convex function called (k, h)-convexity, in which two functions on (0,1) are used. One function k defines the set for convexity and the other function h determines the type of convexity for the function.
The famously used definitions of convex function are well-defined as follows.

Definition 1.1. [22] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is convex using $a_1 \in [0, 1]$, if

$$f(a_1x + (1 - a_1)y) \leq a_1f(x) + (1 - a_1)f(y). \quad (1)$$

Hereunder are some new classes for convex functions based on Definition (1).

Definition 1.2. [23] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is called K_s^2 or s-convex function using $a_1 \in [0, 1]$ and $s \in (0, 1]$ if

$$f(a_1x + (1 - a_1)y) \leq (a_1)^s f(x) + ((1 - a_1))^s f(y). \quad (2)$$

Definition 1.3. [24] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is called K_s^3 or s-convex function using $a_1 \in [0, 1]$ and $s \in (0, 1]$ if

$$f(a_1x + (1 - a_1)y) \leq (a_1)^{(1/s)} f(x) + ((1 - a_1))^{(1/s)} f(y). \quad (3)$$

Definition 1.4. [25] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is called K_s^4 or s-convex function using $a_1 \in [0, 1]$ and $s \in (0, 1]$ with $(a_1)^s + ((1 - a_1))^s = 1$ if

$$f(a_1x + (1 - a_1)y) \leq (a_1)^{(1/s)} f(x) + ((1 - a_1))^{(1/s)} f(y). \quad (4)$$

Definition 1.5. [26] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is called g-convex function or multiplicatively convex function using $a_1 \in [0, 1]$ if

$$f(x^{(a_1)}y^{(1-a_1)}) \leq (f(x))^{(a_1)}(f(y))^{(1-a_1)}). \quad (5)$$

In [27], Yin and Wang combined Definition (2) and Definition (5) to form (g, s)-convex function in second sense as given below.

Definition 1.6. [27] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is called (g, s)-convex function or multiplicatively convex function using $s \in (0, 1]$ and $a_1 \in [0, 1]$ if

$$f(x^{(a_1)}y^{((1-a_1))})) \leq (f(x))^{((a_1)^s)}(f(y))^{((1-a_1)^s)}. \quad (6)$$

In this article, some novel integral inequalities of the Hermite-Hadamard and Fejer's type of inequalities using (g, s)-convex function in third and fourth sense will be found with the help of following lemma.

Lemma 1.7. [28] *For $\eta \geq 1$, $0 \leq a_1 \leq 1$ and $(1/s) \in (0,1)$, we get*
The newly established inequalities will inherit all the results of (g, s)-convex function established by Yin and Wang [27].

$$(\eta)^{((a_1)^{(1/s)})} \leq (\eta)^{((1/s)a_1+1-(1/s))}$$

and

$$\begin{aligned}\int_0^1 f^{{a_1}^s}(p) f^{(1-a_1)^s}(q) da_1 &\leq f^{(sa_1+1-s)}(p) f^{(s(1-a_1)+1-s)}(q)\\ &= [f(p)f(q)]^{1-s} L(f^{(s)}(p), f^{(s)}(q)).\end{aligned}$$

The newly established inequalities will inherit all the results of (g, s)-convex function established by Yin and Wang [27].

2. Preliminaries

In this section, a new definition is derived by combining K_s^3-convex, K_s^4-convex and g-convex functions.

Definition 2.1. A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $x, y \in \mathbb{I}$ is called (g, s)-convex function for $s \in (0, 1]$ and $a_1 \in [0, 1]$ if

$$f(x^{(a_1)} y^{((1-a_1))}) \leq (f(x))^{((a_1)^{1/s})} (f(y))^{((1-a_1)^{1/s})}. \tag{7}$$

Remark 2.1. If $1/s = 1$, then Definition (7) will be used or reduced to geometrically-convex function.
If using $1/s = s$ in Definition (7), then we will produce Definition (2).

3. Main results

In this section, we are generalizing Hermite-Hadamard (H-H) inequality on the basis of (g, s)-convex function in third sense.

Main results are here

Theorem 3.1. *Let $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be a (g,s)-convex function and integrable on $s \in (0, 1]$ then*

$$\begin{aligned}[f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}(\sqrt{pq})](&\leq \frac{1}{\ln q - \ln p} \int_p^q \frac{1}{x} f(x) dx\\ &\leq [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right), \end{aligned}\tag{8}$$

where the special mean $L(p, q)$ is defined as

$$L(p,q) = \begin{cases} \frac{1}{\ln q - \ln p}(q-p), & (p \neq q), q = p. \\ p, \end{cases} \tag{9}$$

Proof. Let f be a (g,s)-convex function and $x = p^{a_1}q^{(1-a_1)}$ for $a_1 \in [0,\ 1]$. By utilizing Definition (7), we have

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \left(\frac{1}{x}f(x)\right)dx = \int_0^1 f\left(p^{a_1}q^{(1-a_1)}\right)da_1$$
$$\leq \int_0^1 f^{{a_1}^{\left(\frac{1}{s}\right)}}(p)f^{(1-a_1)^{\left(\frac{1}{s}\right)}}(q)da_1. \tag{10}$$

By using Lemma (1.7), we get

$$\eta^{{a_1}^{\left(\frac{1}{s}\right)}} \leq \eta^{\left(\frac{1}{s}\right)a_1+1-\left(\frac{1}{s}\right)}. \tag{11}$$

As (g,s)-convex function satisfies the condition $f(x)\geq 1$ for $\left(\frac{1}{s}\right)\in(0,1)$. By using Inequality (11), we obtain

$$\int_0^1 f^{{a_1}^{\left(\frac{1}{s}\right)}}(p)f^{(1-a_1)^{\left(\frac{1}{s}\right)}}(q)da_1 \leq \int_0^1 f^{\left(\frac{1}{s}\right)a_1+1-\left(\frac{1}{s}\right)}(p)f^{\left(\frac{1}{s}\right)(1-a_1)+1-\left(\frac{1}{s}\right)}(q)da_1 \tag{12}$$
$$= [f(p)\times f(q)]^{1-\left(\frac{1}{s}\right)}L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right). \tag{13}$$

Since $\left(\sqrt{pq}\right) = \left(\sqrt{p^{a_1}q^{(1-a_1)}p^{(1-a_1)}q^{a_1}}\right)$, $\forall a_1\in[0,1]$ then by (g,s)-convex function, we find

$$f^{\left(\frac{1}{2}\right)^{\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \left[f\left(p^{a_1}q^{(1-a_1)}\right)f\left(p^{1-a_1)}q^{a_1}\right)\right]^{\left(\frac{1}{2}\right)} \leq \frac{f\left(p^{a_1}q^{(1-a_1)}\right)+f\left(p^{1-a_1)}q^{a_1}\right)}{2}. \tag{14}$$

Integrating Inequality (14) with respect to a_1 on both sides yields

$$\int_0^1 f^{\left(\frac{1}{2}\right)^{\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right)da_1 \leq \left(\frac{1}{2}\right)\int_0^1\left[f\left(p^{a_1}q^{(1-a_1)}\right)+f\left(p^{1-a_1)}q^{a_1}\right)\right]da_1 \tag{15}$$
$$= \left(\frac{1}{\ln q-\ln p}\right)\int_p^q\left(\frac{1}{x}f(x)\right)dx.$$

Now by comparing Inequality (13) and Inequality (15), we obtain

$$f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q-\ln p}\int_p^q\left(\frac{1}{x}f(x)\right)dx$$
$$\leq [f(p)\times f(q)]^{1-\left(\frac{1}{s}\right)}L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right),$$

which was required to prove. □

Theorem 3.2. *Let $f:\mathbb{I}=[p,q]\subseteq\mathbb{R}\to\mathbb{R}$ be a (g,s)-convex function and integrable on $s\in(0,1]$ then*

$$\left(\sqrt{pq}\right)f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q-\ln p}\int_p^q f(x)dx$$
$$\leq [f(q)\times f(p)]^{1-\left(\frac{1}{s}\right)}L\left(pf^{\left(\frac{1}{s}\right)}(p), qf^{\left(\frac{1}{s}\right)}(q)\right), \tag{16}$$

where the special mean $L(p,q)$ is defined in Equation (9).

Proof. Let f be a (g,s)-convex function and $x = p^{a_1}q^{(1-a_1)}$ for $a_1 \in [0,\ 1]$. By applying Definition (7) and Lemma (1.7), we obtain

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q f(x)dx = p^{a_1}q^{(1-a_1)}\int_0^1 f\left(p^{a_1}q^{(1-a_1)}\right)da_1 \tag{17}$$

$$\leq p^{a_1}q^{(1-a_1)}\int_0^1 f^{a_1\left(\frac{1}{s}\right)}(p)f^{(1-a_1)\left(\frac{1}{s}\right)}(q)da_1$$

$$\leq \int_0^1 p^{a_1}q^{(1-a_1)}f^{a_1\left(\frac{1}{s}\right)+1-\left(\frac{1}{s}\right)}(p)f^{(1-a_1)\left(\frac{1}{s}\right)+1-\left(\frac{1}{s}\right)}(q)da_1 \tag{18}$$

$$= [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} L\left(pf^{\left(\frac{1}{s}\right)}(p), qf^{\left(\frac{1}{s}\right)}(q)\right). \tag{19}$$

By multiplying $\left(\sqrt{pq}\right)$ on both sides of Inequality (14), we have

$$\left(\sqrt{pq}\right) f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s}\right)}}\left(\sqrt{pq}\right)$$

$$= \int_0^1 \sqrt{p^{a_1}q^{(1-a_1)}q^{a_1}p^{(1-a_1)}}\, f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s}\right)}}\left(\sqrt{p^{a_1}q^{(1-a_1)}p^{(1-a_1)}q^{a_1}}\right)da_1 \tag{20}$$

$$\leq \int_0^1 \sqrt{p^{a_1}q^{(1-a_1)}q^{a_1}p^{(1-a_1)}}\left[f\left(p^{a_1}q^{(1-a_1)}\right)f\left(p^{(1-a_1)}q^{a_1}\right)\right]^{\frac{1}{2}}da_1 \tag{21}$$

$$\leq \frac{1}{2}\int_0^1 \left[p^{a_1}q^{(1-a_1)}f\left(p^{a_1}q^{(1-a_1)}\right) + p^{(1-a_1)}q^{a_1}f\left(p^{(1-a_1)}q^{a_1}\right)\right]da_1, \tag{22}$$

$$= \left(\frac{1}{\ln q - \ln p}\right)\int_p^q f(x)dx. \tag{23}$$

Now by comparing Inequality (19) and Inequality (23), we obtain

$$\left(\sqrt{pq}\right) f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q - \ln p}\int_p^q f(x)dx \leq [f(p)\times f(q)]^{1-\left(\frac{1}{s}\right)}$$
$$\times L\left(pf^{\left(\frac{1}{s}\right)}(p), qf^{\left(\frac{1}{s}\right)}(q)\right),$$

where the special mean $L(p,q)$ is defined in Equation (9). □

Theorem 3.3. *Let $f, g : \mathbb{I} = [p,q] \subseteq \mathbb{R} \to \mathbb{R}$ be a (g,s_1)-convex function and (g,s_2)-convex functions respectively and integrable on $s_1, s_2 \in (0,1]$ then*

$$\left(\sqrt{pq}\right) f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right)\left(\sqrt{pq}\right) g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q - \ln p}\int_p^q \left(g(x)\times f(x)\right)dx$$

$$\leq [f(p)\times f(q)]^{1-\left(\frac{1}{s_1}\right)}[f(p)\times f(q)]^{1-\left(\frac{1}{s_2}\right)}\times$$
$$L\left(pf^{\left(\frac{1}{s_1}\right)}(p)g^{\left(\frac{1}{s_2}\right)}(p), qf^{\left(\frac{1}{s_1}\right)}(q)g^{\left(\frac{1}{s_2}\right)}(q)\right) \tag{24}$$

where the special mean $L(p,q)$ is defined in Equation (9).

Proof. Let f be a (g,s)-convex function and $x = p^{a_1}q^{(1-a_1)}$ for $a_1 \in [0,\ 1]$. After using Definition (7) and Lemma (1.7) results

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q [f(x)\times g(x)]\,dx$$
$$= p^{a_1}q^{(1-a_1)}\int_0^1 f\left(p^{a_1}q^{(1-a_1)}\right) g\left(p^{a_1}q^{(1-a_1)}\right)da_1. \tag{25}$$

Similarly, as Inequality (15), we have

$$(\sqrt{pq})\, f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}(\sqrt{pq}) \times (\sqrt{pq})\, g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_2}\right)}}(\sqrt{pq})$$

$$= \int_0^1 \left(p^{a_1}q^{(1-a_1)}p^{(1-a_1)}q^{a_1}\right)^{\frac{1}{2}}\left[f\left(p^{a_1}q^{(1-a_1)}\right)\right. \tag{26}$$

$$\left.f\left(p^{(1-a_1)}q^{a_1}\right) g\left(p^{a_1}q^{(1-a_1)}\right)\ \ g\left(p^{(1-a_1)}q^{a_1}\right)\right)\Big]^{\frac{1}{2}} \tag{27}$$

$$\leq \frac{1}{2}\int_0^1 \left[p^{a_1}q^{(1-a_1)} f\left(p^{a_1}q^{(1-a_1)}\right) g\left(p^{a_1}q^{(1-a_1)}\right)\right.$$
$$\left.+\, p^{(1-a_1)}q^{a_1} f\left(p^{(1-a_1)}q^{a_1}\right) g\left(p^{(1-a_1)}q^{a_1}\right)\right] da_1 \tag{28}$$
$$= \left(\frac{1}{\ln q - \ln p}\right)\int_p^q [f(x)\times g(x)]\,dx. \tag{29}$$

After comparing Inequality (26) and Inequality (29), we obtain our proof

$$(\sqrt{pq})\, f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}(\sqrt{pq})\,(\sqrt{pq})\, g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}(\sqrt{pq}) \leq \frac{1}{\ln q - \ln p}\int_p^q (g(x)\times f(x))\,dx$$

$$\leq [f(p)\times f(q)]^{1-\left(\frac{1}{s_1}\right)}\,[f(p)\times f(q)]^{1-\left(\frac{1}{s_2}\right)} \times$$
$$L\left(pf^{\left(\frac{1}{s_1}\right)}(p)g^{\left(\frac{1}{s_2}\right)}(p), qf^{\left(\frac{1}{s_1}\right)}(q)g^{\left(\frac{1}{s_2}\right)}(q)\right) \tag{30}$$

where the special mean $L(p,q)$ is defined in Equation (9). □

Theorem 3.4. *$f, g : \mathbb{I} = [p,q] \subseteq \mathbb{R} \to \mathbb{R}$ be a (g, s_1)-convex function and (g, s_2)-convex functions respectively and integrable on $s_1, s_2 \in (0,1]$ then*

$$f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}(\sqrt{pq})\, g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}(\sqrt{pq}) \leq \frac{1}{\ln q - \ln p}\int_p^q \frac{1}{x} f(x)g(x)dx$$

$$\leq [f(p)\times f(q)]^{1-\left(\frac{1}{s_1}\right)}\,[f(p)\times f(q)]^{1-\left(\frac{1}{s_2}\right)}\, L\left(f^{\left(\frac{1}{s_1}\right)}(p)\times g^{\left(\frac{1}{s_2}\right)}(p), f^{\left(\frac{1}{s_1}\right)}(q)\times g^{\left(\frac{1}{s_2}\right)}(q)\right),$$

where the special mean $L(p,q)$ is defined in Equation (9).

Proof. Let f, g be a (g, s)-convex function and $x = p^{a_1}q^{(1-a_1)}$ for $a_1 \in [0,\ 1]$. By using Definition (7) and utilizing Inequality (11) yields

$$\begin{aligned}
&\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \frac{1}{x} f(x)g(x)dx \\
&= \int_0^1 f\left(p^{a_1}q^{(1-a_1)}\right) g\left(p^{a_1}q^{(1-a_1)}\right) da_1 \qquad (31)\\
&\leq \int_0^1 f^{a_1\left(\frac{1}{s_1}\right)}(p) f^{(1-a_1)\left(\frac{1}{s_1}\right)}(q) g^{a_1\left(\frac{1}{s_2}\right)}(p) g^{(1-a_1)\left(\frac{1}{s_2}\right)}(q) da_1 \\
&\leq \int_0^1 f^{a_1\left(\frac{1}{s_1}\right)+1-\left(\frac{1}{s_1}\right)}(p) f^{a_1\left(\frac{1}{s_2}\right)+1-\left(\frac{1}{s_2}\right)}(p) g^{(1-a_1)\left(\frac{1}{s_1}\right)+1-\left(\frac{1}{s_1}\right)}(q) da_1 \\
&\quad \times g^{(1-a_1)\left(\frac{1}{s_2}\right)+1-\left(\frac{1}{s_2}\right)}(q) da_1 \\
&= \left[f(p)f(q)\right]^{1-\left(\frac{1}{s_1}\right)} \left[g(p)\times g(q)\right]^{1-\left(\frac{1}{s_2}\right)} \\
&\quad \times L\left(f^{\left(\frac{1}{s_1}\right)}(p)\times g^{\left(\frac{1}{s_2}\right)}(p), f^{\left(\frac{1}{s_1}\right)}(q) g^{\left(\frac{1}{s_2}\right)}(q)\right). \qquad (32)
\end{aligned}$$

Similarly, as Inequality (14) and integrating with respect to a_1, we have

$$f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}(\sqrt{pq})\, g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_2}\right)}}(\sqrt{pq}) \qquad (33)$$

$$= \int_0^1 \left(f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}\left(\sqrt{p^{a_1}q^{(1-a_1)}p^{(1-a_1)}q^{a_1}}\right) da_1 \qquad (34)$$

$$\times\ g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}\left(\sqrt{p^{a_1}q^{(1-a_1)}p^{(1-a_1)}q^{a_1}}\right)\right) \qquad (35)$$

$$\leq \frac{1}{2}\int_0^1 \left[f\left(p^{a_1}q^{(1-a_1)}\right) g\left(p^{a_1}q^{(1-a_1)}\right) + f\left(p^{(1-a_1)}q^{a_1}\right) g\left(p^{(1-a_1)}q^{a_1}\right)\right] da_1 \qquad (36)$$

$$= \left(\frac{1}{\ln q - \ln p}\right)\int_p^q \frac{1}{x} f(x)g(x)dx. \qquad (37)$$

Now by comparing Inequality (33) and Inequality (36), we obtain

$$\begin{aligned}
&f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}(\sqrt{pq})\, g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}(\sqrt{pq}) \\
&\leq \frac{1}{\ln q - \ln p}\int_p^q \frac{1}{x} f(x)g(x)dx \\
&\leq \left[f(p)\times f(q)\right]^{1-\left(\frac{1}{s_1}\right)} \left[f(p)\times f(q)\right]^{1-\left(\frac{1}{s_2}\right)} \\
&\qquad \times L\left(f^{\left(\frac{1}{s_1}\right)}(p)\times g^{\left(\frac{1}{s_2}\right)}(p), f^{\left(\frac{1}{s_1}\right)}(q)\times g^{\left(\frac{1}{s_2}\right)}(q)\right),
\end{aligned}$$

where the special mean $L(p, q)$ is defined in Equation (9). □

Remark 3.1. If we set $\frac{1}{s} = 1$ in Theorem (3.1), (3.2), (3.3) and (3.4), the results will be reduced to geometrically convex function.
If $\frac{1}{s} = s$ in Theorem (3.1), (3.2), (3.3) and (3.4), then the results for K_s^2 will be produced.

4. Conclusion

This paper adds a significant member in the family of convex functions using the combination of s-convex and g-convex functions in Yin and Wang (2018) for the third and fourth senses. The H-H and Fejer's type of inequalities are enhanced or generalized by utilizing newly made (g, s)-convex function. H-H and Fejer's type of inequalities are employed to yield new results. The relations between the newly formed inequalities and the mathematical means (averages) are examined as an application. Further, the product of two (g, s)-convex functions is also obtained.

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Sabir is a Ph.D. at Universiti Utara, Malaysia. His research interests are functional analysis, mathematical inequalities, and functional analysis.

Department of Mathematics and Statistics, School of Quantitative Sciences, Universiti Utara, Malaysia, 06010 UUM Sintok, Kedah, Malaysia.
e-mail: sabiryasin77@gmail.com

Masnita Misiran is currently working as an Associate professor at University Utara, Malaysia. Her research interests are computational mathematics, mathematical inequalities, and functional analysis.

Department of Mathematics and Statistics, School of Quantitative Sciences, Universiti Utara, Malaysia, 06010 UUM Sintok, Kedah, Malaysia.

Centre for Testing, Measurement and Appraisal, Universiti Utara Malaysia, 06010 UUM Sintok, Kedah, Malaysia.
e-mail: masnita@uum.edu.my

Zurni Omar is currently working as professor at University Utara, Malaysia. His research interests are Applied mathematics, mathematical inequalities, and functional analysis.

Department of Mathematics and Statistics, School of Quantitative Sciences, Universiti Utara, Malaysia, 06010 UUM Sintok, Kedah, Malaysia.
e-mail: zurni@uum.edu.my

Rabia Luqman is currently working as Assistant professor at COMSATS University, Pakistan (vehari campus). Her research interests are Finance, Financial mathematics and functional analysis.

Department of Management, COMSATS University, Islamabad. Vehari Campus.
e-mail: rabia@ciitvehari.edu.pk