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SEMI-ANALYTICAL SOLUTIONS TO HOLLING-TANNER MODEL USING BOTH DIFFERENTIAL TRANSFORM METHOD AND ADOMIAN DECOMPOSITION METHOD

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Abstract. This paper summarizes some research findings that show how the differential transform method (DTM) is used to resolve the Holling-Tanner model. To confirm the application, effectiveness, and correctness of the approach, a comparison between the differential transform method (DTM) and the Adomian decomposition method (ADM) is carried out, and an accurate solution representation in truncated series is discovered. The approximate solution obtain using both techniques and comparison demonstrates same outcome which remains a preferred numerical method for resolving a system of nonlinear differential equations.

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1. Introduction

The prey-predator model is a system of equations that describes an undetermined number of competitive model which is dynamic in nature. The simple prey-predator model is among the most popular models, being used to demonstrate linear and non-linear control system such as the relationship between different species in an habitat and love equation [1, 2]. In the concerned field of science and technology, numerous significant physical phenomenons are frequently modeled by the non-linear differential equations. There are several nonlinear differential equations that can't be solved analytically and don't have an exact solution, but they can have their solutions approximated numerically. The approximate solutions to linear and nonlinear real-world problems have been determined by researchers using a variety of numerical methods. Examples of methods used include the differential transform method [3, 4, 5, 6], multi-stage

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differential transform method [6, 9, 10], Adomian decomposition and Laplace Domitian decomposition method [7, 8], Runge-Kutta methods family [11, 12]. Throughout time, the simple Lotka-Volterra has developed some limitations. To address this limitation, a logistic growth component was developed to describe the quadratic models, namely the feeding saturation effect. Yet, contrary to what may be expected, the quadratic term's inclusion aids in understanding whether the connection in the model is one of commensalism or parasitism. Among this models with such ability is the Holling-Tanner model [13, 14], helps to understand the dynamics and its characteristics. One of the application of the Holling-Tanner model to epidemiology is the influence of transmissible disease, under the assumption that it spreads among the prey species only[15]. The mathematical features and dependability of this model, which describes true ecological models including symbiosis, parasitism, mutualism, and commensalism, are well recognized. To mention a few, Canadian lynx and snowshoe hare [16], sparrow and sparrowhawk [17]and mite and spider-mite [18] interactions.

The Holling-Tanner model will be explored in this work utilizing the differential transform method (DTM) and the Adomian decomposition method (ADM) to obtain numerical approximations to the nonlinear differential equations. Differential transform method (DTM) was first introduced [19] to solve problems in electric circuit. DTM (as known) has applications to investigate ODE and PDE problems [20], Zika virus vector population [21], predator-prey models [22]. To find a solution to initial value problems with linear and nonlinear functions, the differential transform approach has been presented. This approach uses Taylor's series expansion as a semi-numerical analytic tool; neither linearization nor intensive computational labor are used in this process.

The differential transform approach differs from the high order Taylor's series method in that it provides a polynomial of various degrees as the form of the solution. The benefit of using the differential transform approach is the ability to approximatively solve a closed form nonlinear system of ordinary differential equations that is impossible to solve exactly but can be approximated using a series solution. To do this, a semi-analytic technique such as DTM and ADM would be used to investigate the series solution.

This paper is structured as follows: Section 2 details on the formulation of governing systems of equations, which describes the interaction between the species, section 4.1 & 4.2 discussed the methodologies. Under section 5, the numerical simulations governing the system of nonlinear differential equations and numerical approximation are investigated using Mathematica® and Maple®. In section 6, the discussion of the numerical solutions was done and the conclusions of the study are stated.

The goal of this study, in connection to the Holling-Tanner model, is to conduct a systematic investigation of the numerical comparisons of the approximations solutions between the differential transform method and Adomian decomposition method.

2. Governing systems of Equation

Definition 1. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined as nonlinear such that it does not satisfy superposition principle that is

$$f(x_1 + x_2 + \ldots) \neq f(x_1) + f(x_2) + \ldots \tag{1}$$

The Equation 1, becomes a nonlinear expression by definition. A system of nonlinear equation contains a set of equations such that

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ f_3(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0, \end{aligned} \tag{2}$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and each f_i is nonlinear real function such that $i = 1, 2, 3, \ldots, n.$. In the problem considered, the nonlinear Holling-Tanner model is a system of first order differential equations. The model does not have an exact solution and to obtain an approximate solution, we employed the use of numerical techniques such as DTM and ADM to compute the systematic analysis using Mathematica® and Maple®.

3. Holling-Tanner Model

The governing systems of equations is considered with maximum possible number of parameters. Consider the Holling-Tanner model [14] as a special case of prey-predator model, described by the following system of equations:

$$\dot{x} = b_1 x - b_2 x^2 - b_3 \frac{x \cdot y}{b_4 + x}, \tag{3}$$

$$\dot{y} = b_5 y - b_6 \frac{y^2}{x}, \tag{4}$$

$$t = 0, \quad x(0) = x_0, \quad y(0) = y_0, \tag{5}$$

where $x = x(t)$, $y = y(t)$, $\dot{x} = \frac{d}{dt}x(t)$, $\dot{y} = \frac{d}{dt}y(t)$, t be time and $b_1, \cdots, b_6$ are positive parameters, $x(t)$ is used to represent the size of the prey population at time t, $y(t)$ represent the predator population at time t. Initial conditions for this system are $t = 0, x(t = 0) = x_0 > 0$ and $y(t = 0) = y_0 > 0$. The Holling-Tanner model conceptualize the prey equation from the Lotka-Volterra model, but for the predator it assumes a logistic-like term, in which the predator carrying capacity is directly proportional to prey density.

4. Methodology

The numerical methods of solution are described in details in this section:

4.1. Differential Transform Method. Consider the following definitions:

Definition 2. Arbitrary function $f(x)$ is expressed and expanded in Taylor's series at point $x = 0$ in Equation 6

$$f(x) = \sum_{n=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k f}{dx^k} \right]_{x=0}. \tag{6}$$

Definition 3. The differential transform of $f(x)$ is defined in Equation 7

$$F(x) = \frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=0} \tag{7}$$

Definition 4. The differential inverse transform of $F(x)$ with respect to Equation 7 is represented in Equation 8

$$f(x) = \sum_{n=0}^{\infty} x^k F(k) \tag{8}$$

It can be denoted that Equation 8, $f(x)$ can be written as a finite series representation and represented in Equation 9

$$f(x) \approx \sum_{n=0}^{n} (x - x_0)^k F(k) \tag{9}$$

where n is convergence of natural values.
If $X(k)$ and $Y(k)$ are the differential transformed functions of $x(t)$ and $y(t)$ with time t, the following properties shows as follows:

(1) If $X(k) = D[x(t)]$, $Y(k) = D[y(t)]$, with constants r_1, r_2 are independents of time t and k, we obtain $D[r_1 x(t) \pm r_2 y(t)] = r_1 X(k) \pm r_2 Y(k)$.
(2) By convolution, if $x(t) = D^{-1}[X(k)]$ and $y(t) = D^{-1}[Y(k)]$, and we assume $v(t) = x(t)y(t)$, then $D[z(t)] = D[x(t)y(t)] = X(k) \circledast Y(k) = \sum_{k_1=0}^{r} X(k)Y(k-r)$

The differential transform method has properties that engages with inverse transform to obtain series solution for numerical approximation. This properties are studied according to [23],[24], the theorems are formulated from Equation 7 and Equation 9 and represented in the Theorem below:

Theorem 5. If $y(t) = u(t) \pm v(t)$, then $Y(k) = U(k) \pm V(k)$.

Theorem 6. If $y(t) = \frac{du(t)}{dt}$, then $Y(k) = (k+1)U(k+1)$.

Theorem 7. If $y(t) = \frac{d^2u(t)}{dt^2}$, then $Y(k) = (k+1)(k+2)U(k+2)$.

Theorem 8. If $y(t) = \alpha u(t)$, then $Y(k) = \alpha U(k)$.

Theorem 9. If $y(t) = \frac{d^n u(t)}{dt^n}$, then $Y(k) = \frac{(k+n)!}{k!} \cdot U(k+n) \Rightarrow (k+1)(k+2)\ldots(k+n)U(k+n)$.

Theorem 10. If $y(t) = u(t)v(t)$, then $Y(k) = \sum_{n=k_1}^{k} U(k_1)V(k-k_1)$.

Theorem 11. If $y(t) = t^n$, then $Y(k) = \delta(k-n)$ when $\delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$.

Theorem 12. If $y(t) = e^{\lambda t}$, then $Y(k) = \frac{\lambda^k}{k!}$

Theorem 13. If $y(t) = (1+t)^k$, then $Y(k) = \frac{m(m-1)\ldots(m-n+1)}{k!}$.

Theorem 14. If $y(t) = \sin(\gamma t + \alpha)$, then $Y(k) = \frac{\omega^k}{k!} \sin\left(\frac{\pi n}{2} + \omega\right)$.

Theorem 15. If $y(t) = \cos(\gamma t + \alpha)$, then $Y(k) = \frac{\omega^k}{k!} \cos\left(\frac{\pi n}{2} + \omega\right)$.

The proofs are available in [24],[25]. The theorems above is used in transforming either a linear or nonlinear system ordinary differential equations and to obtain series solution to system of ODE's that has no exact solution. In this paper, we are considering the Holling-Tanner model which comprises of nonlinear terms and requires huge effort or impossible effort to obtain the closed form solution. To obtain the closed form of the Holling-Tanner model is unobtainable, hence the reason we employ the differential transform method to obtain an approximate solution been expressed into series solution.

4.2. Adomian Decomposition Method (ADM). In mathematical biology problems are modeled as system of nonlinear differential equation. To obtain the analytical solution of system on differential equations is not an easy task. In early 1980, George Adomian introduced a semi-analytical method for approximating nonlinear differential equation both for partial differential equations and ordinary differential equations. The method was named Adomian decomposition method. The solutions to these differential equations are obtained in a form of series solutions.
Given

$$Ly + Ny = p \tag{10}$$

where $L = \frac{d^n}{dt^n}$ is the order of derivative function which is integrable , N is a nonlinear operator, y is a dependent variable and p is integrable non homogeneous function.

$$L^{-1}Ly = L^{-1}p - L^{-1}Ny. \tag{11}$$

L^{-1} represent integration and with initials conditions, $L^{-1}Ly$ will give an equation for y including these conditions resulting in

$$y(t) = g(t) - L^{-1}Ny. \tag{12}$$

where $g(t)$ represent the function from the integrating p and using the initial conditions.
The resulting function is expressed in a series form:

$$y(t) = \sum_{i=0}^{\infty} y_i. \tag{13}$$

Assuming $y_0 = g(t)$ and the following terms are determined by repetitive formula. Nonlinear terms are expressed as series of polynomials, A_n. The nonlinear terms are expressed as

$$Ny = \sum_{i=0}^{\infty} A_i. \tag{14}$$

Finding the Adomian polynomial, a parameter value,γ, is introduced into expression.

$$A_i = \frac{1}{i!}\frac{d^i}{d\gamma^i} N \left[\sum_{i=0}^{\infty} \gamma^i y_i\right]_{\gamma=0} \tag{15}$$

thus

$$y(t) = g(t) - L^{-1}\sum_{0}^{\infty} A_i. \tag{16}$$

5. Series Approximation

This section gives the results, computational analysis, and implementation of the algorithm in Mathematica® with its inbuilt functions, the exact solution can't be obtained for the model considered which necessitate the reason for the numerical approximate solution. The series solution is obtained using the differential transform method and Adomian decomposition method.

5.1. Differential Transform Method (DTM). In section 4.1, we discussed about the properties and theorems of differential transform method to obtain numerical approximation to nonlinear systems of ODE whose closed form solution is unobtainable. An autonomous systems is assumed which is independent of t
Consider the an autonomous system of ODE in Equation 17

$$\begin{aligned} Y' &= BY, \\ Y(0) &= Y_0. \end{aligned} \tag{17}$$

where B is $m \times m$ matrix.

Theorem 16. Let $Y(k)$ be the k^{th} differential transform of $y(t)$ and B^k be the k^{th} of matrix B.

The solution to the system is obtained from Equation 17 to be represented as follows:

$$y(t) = \sum_{n=0}^{\infty} Y(k)t^k = \sum_{n=0}^{\infty} Y(0)t^k \tag{18}$$

Proof. Engaging Theorem 2, to the Equation 17, the transform procedure is obtained as follows:

$$Y' = B \cdot Y,$$

then

$$(k+1)Y(k+1) = B \cdot Y(k); \forall K = 0, 1, 2, \dots$$
$$Y(k+1) = \frac{B \cdot Y(k)}{(k+1)} \tag{19}$$

Equation 19 is used to obtain the parameters $Y(1), Y(2), Y(3), \dots$
when $k = 0, k = 1, k = 2$ in Table 1,

TABLE 1. Values of $Y(k)$, when $k = 0, 1, 2$

k=0	k=1	k=2
$Y(1) = \frac{B \cdot Y(0)}{1}$	$Y(2) = \frac{B}{2} \cdot \frac{B \cdot Y(K)}{1}$	$Y(3) = \frac{B}{3} \cdot \frac{B}{2} \cdot \frac{B \cdot Y(K)}{1}$

By assuming the procedure for $k = j$ in Table 1, we obtain Equation 20 as

$$X(j) = \frac{B^j}{j!} X(0), \tag{20}$$

□

where B^j is the power of matrix and Equation 18 is obtained. We shall present the numerical analysis of differential transform method to the Holling-Tanner model. Consider the Equation 3 - Equation 5

$$\dot{x} = b_1 x - b_2 x^2 - b_3 \frac{x \cdot y}{b_4 + x},$$
$$\dot{y} = b_5 y - b_6 \frac{y^2}{x},$$
$$t = 0, \quad x(0) = 10, \quad y(0) = 5,$$

with the parameters $b_1 = 0.2, b_2 = 0.01, b_3 = 0.05, b_4 = 1.0, b_5 = 0.0623$ and $b_6 = 0.0223$, with respect to Equation 7, we introduce the application of Theorems and the operations of differential transform to the Holling-Tanner model, we have the following,

$$X(k+1) = \frac{1}{(k+1)}\left[0.2 \cdot X(k) - 0.01 \cdot \sum_{k_1=0}^{k} X(k_1)X(k-k_1)\right] - \left[0.05 \cdot \frac{\sum_{k_1=0}^{k} X(k_1)Y(k-k_1)}{1.0 + X(k)},\right] \tag{21}$$

$$Y(k+1) = \frac{1}{(k+1)}\left[0.062 \cdot Y(k)\right] - \left[0.0223 \cdot \frac{\sum_{k_1=0}^{k} Y(k_1)Y(k-k_1)}{X(k)},\right] \tag{22}$$

where $X(k)$ and $Y(k)$ are the differential transform of $x(t)$ and $y(t)$ and the initial conditions are expressed as $X(0) = 10$ and $Y(0) = 5$ With respect to the Equation 8 using the differential inverse transform, the approximate series solution to Holling-Tanner model is obtained as

$$x(t) = \sum_{n=0}^{k} X(k)t^k, \tag{23}$$

$$y(t) = \sum_{n=0}^{k} Y(k)t^k, \tag{24}$$

For computation, Maple® was used to generate both the series solutions for the prey $x(t)$ and predator $y(t)$ population. Likewise the values of parameters $X(1), Y(1), X(2), Y(2), X(3), Y(3), X(4), Y(4)$ are obtained. From Equation 21, substitute the value(s) of $k = 1, 2, 3$ to obtain the series solution and expressed as follows

$$X(1) = -\frac{r_1}{x_0 + 1.0} \tag{25}$$

$$Y(1) = \frac{r_1'}{x_0} \tag{26}$$

$$X(2) = -\frac{r_2}{(x_0 + 1.0)^2} \tag{27}$$

$$Y(2) = \frac{r_2'}{x_0^2} \tag{28}$$

$$X(3) = -\frac{r_3 + l_3}{(x_0 + 1.0)^3} \tag{29}$$

$$Y(3) = \frac{r_3'}{x_0^3} \tag{30}$$

$$X(4) = -\frac{(r_4 - l_4)}{(x_0 + 1.0)^4} \tag{31}$$

$$Y(4) = \frac{r_4' + l_4'}{x_0^4} \tag{32}$$

Let the following be represented as

$$\begin{aligned}
r_1 =& 0.01000000000 x_0 \left(x_0^2 - 19.0x_0 + 5.0y_0 - 20.0\right)\\
r_1' =& 0.0001000000000 y_0 \left(620.0x_0 - 223.0y_0\right)\\
r_2 =& 0.005000000000(2.0x_0^3x_1 - 16.0x_0^2x_1\\
&+ 5.0x_0^2y_1 - 38.0x_0x_1 + 5.0x_0y_1 + 5.0x_1y_0 - 20.0x_1)\\
r_2' =& 0.00005000000000 \left(620.0x_0^2y_1 - 446.0x_0y_0y_1 + 223.0x_1y_0^2\right)\\
r_3 =& 0.003333333333(2.0x_0^4x_2 + x_0^3x_1^2 - 14.0x_0^3x_2\\
&+ 5.0x_0^3y_2 + 3.0x_0^2x_1^2 - 54.0x_0^2x_2 +\\
&10.0x_0^2y_2 + 3.0x_0x_1^2 + 5.0x_1x_0y_1)\\
l_3 =& 5.0x_0x_2y_0 - 5.0x_1^2y_0 - 58.0x_2x_0 + 5.0x_0y_2 + x_1^2 + 5.0x_1y_1\\
&+ 5.0x_2y_0 - 20.0x_2\\
r_3' =& 0.00003333333333(620.0x_0^3y_2 - 446.0x_0^2y_0y_2 - 223.0x_0^2y_1^2\\
&+ 446.0x_0x_1y_0y_1 + 223.0x_0x_2y_0^2 - 223.0x_1^2y_0^2)\\
r_4 =& 0.002500000000(-112.0x_3x_0^2 - 78.0x_3x_0 + 2.0x_1x_2\\
&+ 8.0x_1x_2x_0 - 10.0x_0x_1x_2y_0 + 2.0x_0^4x_1x_2 +\\
&8.0x_0^3x_1x_2 + 12.0x_0^2x_1x_2 - 10.0x_1x_2y_0 + 10.0x_1x_0y_2\\
&- 20.0x_3 + 5.0x_0^2x_1y_2 + 5.0x_0^2x_2y_1 + 5.0x_0^2x_3y_0)\\
l_4 =& (5.0x_0x_1^2y_1 + 10.0x_0x_2y_1 + 10.0x_0x_3y_0 + 2.0x_0^5x_3 - 12.0x_0^4x_3\\
&+ 5.0x_0^4y_3 - 68.0x_0^3x_3 + 15.0x_0^3y_3 + 5.0x_1^3y_0 + 15.0x_0^2y_3\\
&- 5.0x_1^2y_1 + 5.0x_0y_3 + 5.0x_1y_2 + 5.0x_2y_1 + 5.0x_3y_0)\\
r_4' =& 0.00002500000000(620.0x_0^4y_3 - 446.0x_0^3y_0y_3 - 446.0x_0^3y_1y_2\\
&+ 446.0x_0^2x_1y_0y_2 + 223.0x_0^2x_1y_1^2)\\
l_4' =& (446.0x_0^2x_2y_0y_1 + 223.0x_0^2x_3y_0^2 - 446.0x_0x_1^2y_0y_1\\
&- 446.0x_0x_1x_2y_0^2 + 223.0x_1^3y_0^2)
\end{aligned}$$

From Equation 25 - Equation 32, the corresponding values obtained are as follows:

$$\begin{aligned} X(1) =& 0.7727272727, \quad && Y(1) = 0.2542500000, \\ X(2) =& -0.006576681065, \quad && Y(2) = 0.007200839775, \\ X(3) =& -0.002084609146, \quad && Y(3) = 0.00007009529776, \\ X(4) =& 0.00002426688503, \quad && Y(4) = 8.730936000 * 10^{-8}. \end{aligned} \tag{33}$$

Therefore, from Equation 9, the series solution to Equation 3 - Equation 5 using the differential transform method is obtained as

$$\begin{aligned} x(t) =& 10 + 0.7727272727t - 0.006576681065(t)^2 \\ & - 0.002084609146(t)^3 + 0.00002426688503(t)^4 \end{aligned} \tag{34}$$

$$\begin{aligned} y(t) =& 5 + 0.2542500000t + 0.007200839775(t)^2 \\ & + 0.00007009529776(t)^3 + 8.730936000 \times 10^{-8}(t)^4 \end{aligned} \tag{35}$$

5.2. Adomian decomposition method (ADM). This section provides the Adomian decomposition method procedure to the initial value problem by integrating Equation 3 and Equation 4 which yields

$$x(t) = x_0 + b_1 \int_0^t x dt - b_2 \int_0^t x^2 dt - b_3 \int_0^t \left(\frac{xy}{b_4 + x} \right) dt \tag{36}$$

$$y(t) = y_0 + b_5 \int_0^t y dt - b_6 \int_0^t \frac{y^2}{x} dt \tag{37}$$

Applying the ADM algorithm with the representation in Equation 38

$$x = \sum_{n=0}^{\infty} x_n, \quad y = \sum_{n=0}^{\infty} y_n \tag{38}$$

The nonlinear terms are evaluated as

$$\begin{aligned} x^2 =& \sum_{n=0}^{\infty} H_n \left(x_0, \ldots, x_n \right), \\ \frac{y^2}{x} =& g_n \left(x_0, \ldots, x_n, y_0, \ldots, y_n \right) \\ \frac{xy}{b_4 + x} =& P_n \left(x_0, \ldots, x_n, y_0, \ldots, y_n \right) \end{aligned} \tag{39}$$

$$H_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\sum_{n=0}^{\infty} (\lambda^n x_n)^2 \right]_{\lambda=0} \tag{40}$$

$$G_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\frac{\sum_{n=0}^{\infty}(\lambda^n y_n)^2}{\sum_{n=0}^{\infty}(\lambda^n x_n)}\right]_{\lambda=0} \tag{41}$$

$$P_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\frac{\sum_{n=0}^{\infty}(x_n\lambda^n)^2\sum_{n=0}^{\infty}(y_n\lambda^n)}{\sum_{n=0}^{\infty}(\lambda^n x_n)}\right]_{\lambda=0} \tag{42}$$

The nonlinear terms G_n, H_n, and P_n are the Adomian polynomials that replaced Equation 38 to Equation 42 into Equation 36 and Equation 37

$$\sum_{n=0}^{\infty} x(t) = x_0 + b_1\int_0^t \sum_{n=0}^{\infty} x_n dt - b_2\int_0^t \sum_{n=0}^{\infty} H_n dt - b_3\int_0^t \sum_{n=0}^{\infty} P_n dt \tag{43}$$

$$\sum_{n=0}^{\infty} y(t) = y_0 + b_5\int_0^t \sum_{n=0}^{\infty} y_n dt - b_6\int_0^t \sum_{n=0}^{\infty} g_n dt \tag{44}$$

The Adomian Polynomials

$$H_0 = x_0^2 \tag{45}$$

$$\tag{46}$$

$$H_1 = 2x_0x_1 \tag{47}$$

$$H_2 = x_1^2 + 2x_0^2 \tag{48}$$

$$H_3 = 2x_1x_2 + 2x_0x_3 \tag{49}$$

$$G_0 = \frac{y_0^2}{x_0} \tag{50}$$

$$G_1 = \frac{2x_0y_0y_1 - y_0^2x_1}{x_0^2} \tag{51}$$

$$G_2 = \frac{x_1^2y_0^2 - 2x_0x_1y_0y_1 - x_0x_2y_0^2 + 2x_0^2y_0y_2}{x_0^3} \tag{52}$$

$$G_3 = \frac{r - j}{x_0^4} \tag{53}$$

Let

$$r = -x_1^2y_0^2 + 2x_0x_1x_2y_0^2 - x_0^2x_3y_0^2 + 2x_0x_1^2y_0x_1 - 2x_0^2x_2y_0y_1$$

$$j = x_0^2 x_1 y_1^2 - 2x_0^2 x_1 y_0 y_2 + 2x_0^3 y_1 y_0 + 2x_0^3 y_0 y_3$$

as represented in Equation 53

$$P_0 = \frac{x_0 y_0}{b_4 + x_0} \tag{54}$$

$$P_1 = \frac{x_0 y_0}{b_4 + x_0} \tag{55}$$

$$\begin{aligned} P_2 = & \frac{x_0^3 y_2 + b_4^2 x_2 y_0 + b_4^2 x_0 y_2 - b_4 x_1^2 y_0}{(b_4 + x_0)^3} \\ & + \frac{b_4 x_0 x_1 y_1 + b_4 x_0 x_2 y_0 + b_4 2 x_0^3 y_2}{(b_4 + x_0)^3} \end{aligned} \tag{56}$$

$$P_3 = \frac{d + v}{(b_4 + x_0)^4} \tag{57}$$

Let

$$\begin{aligned} d = & b_4 x_1^3 y_0 - 2b_4^2 x_1 x_2 y_0 - 2b_4 x_0 x_1 x_2 y_0 + b_4^3 x_3 y_0 + 2b_4^2 x_0 x_3 y_0 \\ & + b_4 x_0^2 x_3 y_0 - b_4^2 x_1^2 y_1 - b_4 x_0 x_1^2 y_1 + b_4^3 x_2 y_1 \end{aligned}$$

$$\begin{aligned} v = & 2b_4^2 x_0 x_2 y_1 + b_4 x_0^2 x_2 y_1 + b_4^3 x_1 y_2 + 2b_4^2 x_0 x_1 y_2 + b_4 x_0^2 x_1 y_2 \\ & + b_4^3 x_0 y_3 + 3b_4^2 x_0^2 y_3 + 3b_4 x_0^3 y_3 + x_0^4 y_3 \end{aligned}$$

as represented in Equation 57.
Taking $n = 0$ and $n = 1$, we obtain:

$$x_1 = b_1 \int_0^t x_0 dt - b_2 \int_0^t H_0 dt - b_3 \int_0^t P_0 dt \tag{58}$$

$$y_1 = b_5 \int_0^t y_0 dt - b_6 \int_0^t g_0 dt \tag{59}$$

and

$$x_2 = b_1 \int_0^t x_1 dt - b_2 \int_0^t H_1 dt - b_3 \int_0^t P_1 dt \tag{60}$$

$$y_2 = b_5 \int_0^t y_1 dt - b_6 \int_0^t g_2 dt \tag{61}$$

By Equation 43 - Equation 44 and Equation 58 - Equation 61, we obtain the representation of recursive formulas as expressed in Equation 61 - Equation 62

$$x_{n+1} = b_1 \int_0^t x_n dt - b_2 \int_0^t H_n dt - b_3 \int_0^t P_n dt \tag{62}$$

$$y_{n+1} = b_5 \int_0^t y_n dt - b_6 \int_0^t g_n dt \tag{63}$$

where the initial value problem in Equation 3 - Equation 5, the values for the $b-parameters$ are assumed to have the following values $b_1 = 0.2, b_2 = 0.01, b_3 = 0.05, b_4 = 1.0, b_5 = 0.0623$ and $b_6 = 0.0223$ with initial condition as $x_0 = 10$, and $y_0 = 5$ the following series solution is obtained

$$x(t) = 10 + 0.772727t - 0.00657668(t)^2 - 0.00208461(t)^3 + 0.0000242669(t)^4 \tag{64}$$

$$y(t) = 5 + 0.25425t + 0.00720084(t)^2 + 0.0000700953(t)^3 + 8.73094 \times 10^{-8}(t)^4 \tag{65}$$

6. Discussion and Conclusion

The paper highlights the numerical solutions of holling-tanner model where a well established method i.e differential transform method and Adomian decomposition method is introduced. We obtain a numerical approximation of Holling-Tanner model whose closed form is un-obtainable by employing two semi-analytical methods; differential transform method and Adomian decomposition method. An approximate solutions was obtained in a series solution, and it's concluded that numerical computation and approximations executed by both methods are accurate and efficient procedure to solve nonlinear systems of ordinary differential equations with no exact solution. Applying this semi-analytic methods to the Holing-Tanner model have shown that the methods considered gives good results, show efficiency in accuracy and convergence. Conclusively the DTM and ADM investigation shows that the techniques are reliable, effective for nonlinear problems as the same series representations was obtained in both cases.

Conflicts of interest : The authors declare no conflict of interest.

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